तमसो मा ज्योतिर्गमय

SANTINIKETAN
VISWA BHARATI
LIBRARY

510

H9

The various works of Dr. Hutton have ever been held in hig
estimation by a numerous class of instructors, and the intri
excellence of his Course of Mathematics has been un
acknowledged.

During the lapse of the last twenty or thirty yea
achievements in science have been varied and extensive.
the higher branches of Mathematics, several elegant and
theoretical researches have added considerably to the pr
stock of knowledge; while in the elementary branches m
new and valuable arithmetical processes have been discov
affording additional facilities to the practical computist.
proof of this necessary, we have only to refer to the va
discovery of M. Sturm for the separation of the real
imaginary roots of equations of all degrees—a subject on
energies of every order been hitherto unavailingly ex
but which has now yie the talents and industry of
ingenious and distinguished nental mathematician. Not
valuable and important have been the re the rese
of our countryman, the late W. Horner, of Bath, e
his beautiful, simple, and process for the evolutio
the roots of numerical equa which, combined with
theorem of Sturm, furnishes the dent with ample mea
the complete resolution of any l equation whateve

The Editor of this edition of on's Course has av
himself of these valuable discoveries; thinking that a w
on elementary mathematics would now be considered i
plete without them, he has not scrupled to devote a sh
tion of the work to the important subject of Equations.

In this edition several alterations have been made, and, it is hoped, many improvements have been introduced. The wh[ole] of the work has been thoroughly revised; every example has been recomputed; and the matter has also been subjected to a somewhat different arrangement. The plane, solid, and spherical Geometry, and also the Geometry of the Conic Sections, have been placed continuously; and the Differential and Integral Calculus have been made to precede Mechanics. This arrangement has enabled the Editor to introduce into Mechanics the language of the Calculus, without which little or no progress can be made in Dynamics. To enumerate the various changes [that] have been made in the work would be altogether unnecessary. The more prominent of these are—a new rule for the [extra]ction of the cube root; new and simple demonstrations of [the bin]omial and exponential theorems; the complete analy[tical] investigation of several important problems in trigonome[try and] surveying; the method of *least squares*; besides many [other] investigations and examples, which, it is hoped, will be [equal]ly acceptable and useful to the student.

[T]he subject of Mechanics is now divided into Statics and [Dyna]mics; and several additions have been made to this part [as wel]l as to the Integral Calculus; though, from the limits of [the w]ork, the Editor has not been able to introduce so much [of the]se interesting and highly useful branches of study as he [would] have desired.

[A] new, correct, and improved edition of Hutton's Course [has b]een a desideratum for several years; and while the pre[sent e]dition is intended to supply the deficiency, it has likewise [been ass]imilated to the course of instruction now pursued in the [Royal] Military Academy, over which the Author so ably pre[sided] for many years; and from the attention bestowed on the [comp]utations and investigations, and the exercise of a careful [supe]rvision of the work as it has emanated from the press, the [Edit]or trusts that the present edition will be found to be the [mo]st correct of any extant.

[Royal] Military Academy, Woolwich,
November, 1840.

CONTENTS.

CONTEN

CONTENTS.

A COURSE

OF

MATHEMATICS.

GENERAL PRINCIPLES.

1. Quantity, or Magnitude, is any thing that will admit of increase or decrease; or that is capable of any sort of calculation or mensuration: such as, numbers, lines, space, time, motion, weight.

2. Mathematics is the science which treats of all kinds of quantity whatever, that can be numbered or measured.—That part which treats of numbering, is called *Arithmetic*; and that which concerns measuring, or figured extension, is called *Geometry*.—These two, which are conversant about multitude and magnitude, and are the foundation of all the other parts, are called *Pure Abstract Mathematics*; because they investigate and demonstrate the pr[illegible]ties of abstract numbers and magnitudes of all sorts. And when these parts are applied to particular or practical subjects, they constitute the [illegible] or parts called *Mixed Mathematics*.—Mathematics is also distinguis[illegible] *Speculative* and *Practical*; viz. *Speculative*, when it is concerned in [illegible]ing properties and relations; and *Practical*, when applied to practice [illegible] use concerning physical objects.

3. In Mathematics are several general terms or principles; such as, De[illegible]tions, Axioms, Propositions, Theorems, Problems, Lemmas, Corollaries, Scholiums, &c.

4. *A Definition* is the explication of any term or word in a science; showing the sense and meaning in which the term is employed.—Every Definition ought to be clear, and expressed in words that are common and perfectly well understood.

5. *A Proposition* is something proposed to be proved, or something required to be done; and is accordingly either a Theorem or a Problem.

6. *A Theorem* is a demonstrative proposition; in which some property is asserted, and the truth of it required to be proved. Thus, when it is said that, The sum of the three angles of any triangle is equal to two right angles, this is a Theorem, the truth of which is demonstrated by Geometry.—A set or collection of such Theorems constitutes a *Theory*.

7. *A Problem* is a proposition or a question requiring something to be done; either to investigate some truth or property, or to perform some operation. As, to find out the quantity or sum of all the three angles of any triangle, or to draw one line perpendicular to another.—*A Limited Problem* is that which

has but one answer or solution. An *Unlimited Problem* is that which has innumerable answers. And a *Determinate Problem* is that which has a certain number of answers.

8. *Solution* of a problem, is the resolution or answer given to it. *A Numerical* or *Numeral Solution*, is the answer given in numbers. *A Geometrical Solution*, is the answer given by the principles of Geometry. And a *Mechanical Solution*, is one which is gained by trials.

9. *A Lemma* is a preparatory proposition, laid down in order to shorten the demonstration of the main proposition which follows it.

10. *A Corollary*, or *Consectary*, is a consequence drawn immediately from some proposition or other premises.

11. *A Scholium* is a remark or observation made on some foregoing proposition or premises.

12. *An Axiom*, or *Maxim*, is a self-evident proposition; requiring no formal demonstration to prove the truth of it; but is received and assented to as soon as mentioned. Such as, The whole of any thing is greater than a part of it; or, The whole is equal to all its parts taken together; or, Two quantities that are each of them equal to a third quantity, are equal to each other.

13. *A Postulate*, or *Petition*, is something required to be done, which is so easy and evident that no person will hesitate to allow it.

14. *An Hypothesis* is a supposition assumed to be true, in order to argue from, or to found upon it the reasoning and demonstration of some proposition.

15. *Demonstration* is the collecting the several arguments and proofs, and laying them together in proper order, to show the truth of the proposition under consideration.

16. *A Direct, Positive*, or *Affirmative Demonstration*, is that which concludes with the direct and certain proof of the proposition in hand.—This kind of Demonstration is most satisfactory to the mind; for which reason it is called sometimes an *Ostensive Demonstration*.

17. *An Indirect* or *Negative Demonstration*, is that which shows a proposition to be true, by proving that some absurdity would necessarily follow if the proposition advanced were false. This is also sometimes called *Reductio ad Absurdum*; because it shows the absurdity and falsehood of all suppositions contrary to that contained in the proposition.

18. *Method* is the art of disposing a train of arguments in a proper order, to investigate either the truth or falsity of a proposition, or to demonstrate it to others when it has been found out.—This is either Analytical or Synthetical.

19. *Analysis*, or the *Analytic Method*, is the art or mode of finding out the truth of a proposition, by first supposing the thing to be done, and then reasoning back step by step till we arrive at some known truth. This is also called the *Method of Invention*, or *Resolution*; and is that which is commonly used in Algebra.

20. *Synthesis*, or the *Synthetic Method*, is the searching out truth, by first laying down some simple and easy principles, and pursuing the consequences flowing from them till we arrive at the conclusion.—This is also called the *Method of Composition*; and is the reverse of the Analytic method, as this proceeds from known principles to an unknown conclusion; while the other goes in a retrograde order, from the thing sought, considered as if it were true, to some known principle or fact. And therefore, when any truth has been found out by the Analytic method, it may be demonstrated by a process in the

ARITHMETI[illegible]

Arithmetic is the art or science of numbering; b[illegible] matics which treats of the nature and properties o[illegible] of whole numbers, it is called *Vulgar*, or *Commo*[illegible] broken numbers, or parts of numbers, it is called *F*[illegible]

Unity, or an *Unit*, is that by which every thing [illegible] called one; being the beginning of number. As one man, one ball, one g[illegible]

Number is either simply one, or a compound of several units. As one man, three men, ten men.

An *Integer* or *Whole Number*, is some certain precise quantity of units; as one, three, ten.—These are so called as distinguished from Fractions, which are broken numbers, or parts of numbers; as one-half, two-thirds, or three-fourths.

NOTATION AND NUMERA[illegible]

Notation, or Numeration, teaches to denote or express any proposed number, either by words or characters; or to read and write down any sum or number.

The numbers in Arithmetic are expressed by the following ten digits, or Arabic numeral figures, which were introduced into Europe by the Moors about eight or nine hundred years since: viz, 1 one, 2 two, 3 three, 4 four, 5 five, 6 six, 7 seven, 8 eight, 9 nine, 0 cipher or nothing, These characters or fig[illegible] were formerly all called by the general name of *Ciphers*; whence it ca[illegible] pass that the art of Arithmetic was then often called *Ciphering*. Also the [illegible] nine are called *Significant Figures*, as distinguished from the cipher, [illegible] quite insignificant of itself.

Beside this value of those figures, they have also another, which depends upon the place they stand in when joined together; as in the following Table.

&c.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	[illegible]	[illegible]
&c.	9	8	7	6	5	4	3	[illegible]	[illegible]
		9	8	7	6	5	4	[illegible]	[illegible]
			9	8	7	6	5	[illegible]	[illegible]
				9	8	7	6	[illegible]	[illegible]
					9	8	7	[illegible]	[illegible]
						9	8	[illegible]	[illegible]
							9	[illegible]	[illegible]
								[illegible]	[illegible]
									[illegible]

Here any figure in the first place, reckoning from right to left, denotes only its own simple value; but that in the second place, denotes ten times its simple value; and that in the third place, a hundred times its simple value; and so on; the value of any figure, in each successive place, being always ten times its former value.

Thus, in the number 1796, the 6 in the first place denotes only six units, or simply six; 9 in the second place signifies nine tens, or ninety; 7 in the third place, seven hundred; and the 1 in the fourth place, one thousand; so that the whole number is read thus, one thousand seven hundred and ninety-six.

As to the cipher 0, it stands for nothing of itself, but being joined on the right hand side to other figures, it increases their value in the same tenfold proportion: thus, 5 signifies only five; but 50 denotes 5 tens, or fifty; and 500 is five hundred; and so on.

For the more easily reading of large numbers, they are divided into periods and half-periods, each half-period consisting of three figures; the name of the first period being units; of the second, millions; of the third, millions of millions, or bi-millions, contracted to billions; of the fourth, millions of millions of millions, or tri-millions, contracted to trillions; and so on. Also the first part of any period is so many units of it, and the latter part so many thousands.

The following Table contains a summary of the whole doctrine:

Periods.	Quadrill.	Trillions;	Billions;	Millions;	Units.
Half-per.	th. un.	th. un.	th. un.	th. un.	th. un.
Figures.	123,456;	789,098;	765,432;	101,231;	567,890.

NUMERATION is the reading of any number in words that is proposed or set down in figures; which will be easily done by the help of the following rule, deduced from the foregoing tablets and observations, viz.

Divide the figures in the proposed number, as in the summary above, into periods and half-periods; then begin at the left-hand side, and read the figures with the names set to them in the two foregoing tables.

EXAMPLES.

Express in words the following numbers; viz.

34	15080	13405670
96	72003	47050023
[illegible]	[illegible]	309025600
[illegible]	[illegible]	4723507689
[illegible]	[illegible]	271856390000
[illegible]	[illegible]	6578600307024

NOTATION is the writing down in figures any number proposed in words; [illegible] of the words or names belonging [illegible] the vacant places with ciphers

EXAMPLES.

Set down in figures the following numbers:

Fifty-seven.

Two hundred and eighty-six.

Nine thousand, two hundred and ten.

Twenty-seven thousand, five hundred and ninety-four.

Six hundred and forty thousand, four hundred and eighty-one.

Three millions, two hundred and sixty thousand, one hundred and six.

Four hundred and eight millions, two hundred and fifty-five thousand, one hundred and ninety-two.

Twenty-seven thousand and eight millions, ninety-six thousand, two hundred and four.

Two hundred thousand and five hundred and fifty millions, one hundred and ten thousand, and sixteen.

Twenty-one billions, eight hundred and ten millions, sixty-four thousand, one hundred and fifty.

OF THE ROMAN NOTATION.

The Romans, like several other nations, expressed their numbers by certain letters of the alphabet. The Romans used only seven numeral letters, being the seven following capitals: viz. I for *one*; V for *five*; X for *ten*; L for *fifty*; C for a *hundred*; D for *five hundred*; M for a *thousand*. The other numbers they expressed by various repetitions and combinations of these, after the following manner:

1 = I.	
2 = II.	As often as any character is repeated, so many times is its value repeated.
3 = III.	
4 = IIII or IV.	A less character before a greater diminishes its value.
5 = V.	
6 = VI.	A less character after a greater increases its value.
7 = VII.	
8 = VIII.	
9 = IX.	
10 = X.	
50 = L.	
100 = C.	
500 = D or IↃ.	For every Ↄ annexed, this becomes ten times as many.
1000 = M or CIↃ.	For every C and Ↄ, placed one at each end, it becomes ten times as much.
2000 = MM	
5000 = $\overline{\text{V}}$ or IↃↃ.	A bar over any number, increases it 1000 fold.
6000 = $\overline{\text{VI}}$	
10000 = $\overline{\text{X}}$ or CCIↃↃ.	
50000 = $\overline{\text{L}}$ or IↃↃↃ.	
60000 = $\overline{\text{LX}}$	
100000 = $\overline{\text{C}}$ or CCCIↃↃↃ.	
1000000 = $\overline{\text{M}}$ or CCCCIↃↃↃↃ.	
2000000 = $\overline{\text{MM}}$.	
&c. &c.	

ARITHMETIC.

EXPLANATION OF CERTAIN CHARACTERS.

There are various characters or marks used in Arithmetic and Algebra, to denote several of the operations and propositions; the chief of which are as follow:

+ signifies plus, or addition.
− ... ... minus, or subtraction.
× ... ... multiplication.
÷ ... ... division.
: :: : ... ... proportion.
= ... ... equality.
√ ... ... square root.
∛ ... ... cube root, &c.

Thus,
5 + 3, denotes that 3 is to be added to 5.
6 − 2, denotes that 2 is to be taken from 6.
7 × 3, denotes that 7 is to be multiplied by 3.
8 ÷ 4, denotes that 8 is to be divided by 4.
2 : 3 :: 4 : 6, shows that 2 is to 3 as 4 is to 6.
6 + 4 = 10, shows that the sum of 6 and 4 is equal to 10.
$\sqrt{3}$, or $3^{\frac{1}{2}}$, denotes the square root of the number 3.
$\sqrt[3]{5}$, or $5^{\frac{1}{3}}$, denotes the cube root of the number 5.
7^2, denotes that the number 7 is to be squared.
8^3, denotes that the number 8 is to be cubed.

OF ADDITION.

Addition is the collecting or putting of several numbers together, in order to find their sum, or the total amount of the whole. This is done as follows:

Set or place the numbers under each other, so that each figure may stand exactly under the figures of the same value; that is, units under units, tens under tens, hundreds under hundreds, &c; and draw a line under the lowest number, to separate the given numbers from their sum, when it is found.—Then add up the figures in the column or row of units, and find how many tens are contained in their sum.—Set down exactly below, what remains more than those tens, or if nothing remains, a cipher, and carry as many ones to the next row as there are tens.—Next add up the second row, together with the number carried, in the same manner as the first. And thus proceed till the whole is finished, setting down the total amount of the last row.

TO PROVE ADDITION.

First Method.—Begin at the top, and add together all the rows of numbers downwards, in the same manner as they were before added upwards; then if [illegible] work is right.—This method of [illegible] over, a little varied.

[illegible] the uppermost number, and suppose it [illegible] numbers together in the usual way, and

set their sum under the number that is to be proved.—Lastly, add this last found number and the uppermost line together; then if their sum be the same as that found by the first addition, it may be presumed the work is right.—This method of proof is founded on the plain axiom, that "the whole is equal to all its parts taken together."

Third Method.—Add the figures in the uppermost line together, and find how many nines are contained in their sum.—Reject those nines, and set down the remainder towards the right hand directly even with the figures in the line, as in the next example.—Do the same with each of the proposed lines of numbers, setting all these excesses of nines in a column on the right hand, as here 5, 5, 6. Then, if the excess of 9's in this sum, found as before, be equal to the excess of 9's in the total sum 18304, the work is right.—Thus, the sum of the right hand column 5, 5, 6, is 16, the excess of which above 9 is 7. Also the sum of the figures in the sum total 18304 is 16, the excess of which above 9 is also 7, the same as the former.*

EXAMPLE I.

	Excess of nines
3197	5
6512	5
8295	6
18304	7

OTHER EXAMPLES.

2.	3.	
12345	12345	12345
67890	67890	876
98765	9876	9087
43210	543	56
12345	21	234
67890	9	1012
302445	90684	23610
290100	78339	11265
302445	90684	23610

Ex. 5. Add 3426; 9024; 5106; 8390; 1204 together. Ans. 27150.

6. Add 509267; 235809; 72910; 8392; 420; 21; and 9 together. Ans. 826828

7. Add 2; 19; 817; 4298; 50916; 730205; 9120634 together. Ans. 9906891.

8. How many days are in the twelve calendar months? Ans. 365.

* This method of proof depends upon a property of the number 9, which, except the number 3, belongs to no other digit whatever, namely, "that any number divided by 9, will leave the same remainder as the sum of its figures or digits divided by 9;" which may be demonstrated in this manner.

Demonstration.—Let there be any number proposed, as 4658. This, separated into its several parts, becomes $4000 + 600 + 50 + 8$. But $4000 = 4 \times 1000 = 4 \times (999 + 1) = 4 \times 999 + 4$. In like manner $600 = 6 \times 99 + 6$, and $50 = 5 \times 9 + 5$. Therefore the given number $4658 = 4 \times 999 + 4 + 6 \times 99 + 6 + 5 \times 9 + 5 + 8 = 4 \times 999 + 6 \times 99 + 5 \times 9 + 4 + 6 + 5 + 8$; and $4658 \div 9 = (4 \times 999 + 6 \times 99 + 5 \times 9 + 4 + 6 + 5 + 8) \div 9$. But $4 \times 999 + 6 \times 99 + 5 \times 9$ is evidently divisible by 9, without a remainder; therefore if the given number 4658 be divided by 9, it will leave the same remainder as $4 + 6 + 5 + 8$ divided by 9. And the same, it is evident, will hold for any other number whatever.

In like manner, the same property may be shown to belong to the number 3; but the preference is usually given to the number 9, on account of its being more convenient in practice.

Now, from the demonstration above given, the reason of the rule itself is evident; for the excess of 9's in two or more numbers being taken separately, and the excess of 9's taken also out of the sum of the former excesses, it is plain that this last excess must be equal to the excess of 9's contained in the total sum of all these numbers; all the parts taken together being equal to the whole.—This rule was first given by Dr Wallis in his Arithmetic, published in the year [illegible]

[illegible] the 15th day of April to the 24th day of [illegible] Ans. 224.

[illegible] nfantry * or foot, 5110 horse, 6250 dragoons, [illegible] nners, 1410 pioneers, 250 sappers, and 406 [illegible] f men ? Ans. 70995.

[illegible] TRACTION.

[illegible] much one number exceeds another, called [illegible] taking the less from the greater. The [illegible]

[illegible] reater, in the same manner as in Addition, [illegible] tens, and so on; and draw a line below [illegible] take each figure in the lower line or num- [illegible] down the remainder below it.—But if the [illegible] n that above it, first borrow or add 10 to [illegible] er figure from that sum, setting down the [illegible] as borrowed, to the next lower figure, with [illegible] ll the whole is finished.

[illegible] SUBTRACTION.

[illegible] ber, or that which is just above it, and if [illegible] permost number, the work is right.†

[illegible] MPLES.

	2.		3.
[illegible]	5386427	From	1234567
[illegible]	4258792	Take	702973
[illegible]	1127635	Rem.	531594
[illegible]	5386427	Proof.	1234567

[illegible] 918. Ans. 157888.

[illegible] 6809. Ans. 4254105

[illegible] 371. Ans. 7929231.

[illegible] year 1642, and he died in 1727; how old [illegible] Ans. 85 years.

[illegible] nd Christ 1840 years ago; then how long [illegible] Ans. 733 years.

[illegible] the word *Infantry*; and all those that charge on [illegible] conjecture, that the term infantry is derived from [illegible] army commanded by the king her father had been [illegible] people together on foot, with which she eng[illegible] [illegible] s event, and to distinguish the foot soldiers, who [illegible] received the name of Infantry, from her own title [illegible]

[illegible] nt: for if the difference of two numbers be added [illegible] to the greater.

9. Noah's flood happened about the year of the world 1656, and the birth of Christ about the year 4000; then how long was the flood before Christ?

Ans. 2344 years.

10. The Arabian or Indian method of notation was first known in England about the year 1150; then how long is it since, till this present year 1840?

Ans 690 years.

11. Gunpowder was invented in the year 1330; then how long was this before the invention of printing, which was in 1441? Ans. 111 years.

12. The mariner's compass was invented in Europe in the year 1302; then how long was that before the discovery of America by Columbus, which happened in 1492? Ans. 190 years.

OF MULTIPLICATION.

MULTIPLICATION is a compendious method of Addition, teaching how to find the amount of any given number when repeated a certain number of times. As 4 times 6, which is 24.

The number to be multiplied, or repeated, is called the *Multiplicand*.—The number you multiply by, or number of repetitions, is the *Multiplier*.—And the number found, being the total amount, is called the *Product*.—Also, both the multiplier and multiplicand are, in general, named the *Terms* or *Factors*.

Before proceeding to any operations in this rule, it is necessary to learn off very perfectly the following Table of all the products of the first 12 numbers, sometimes called the Multiplication Table, or Pythagoras's Table, from its inventor.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

To multiply any Given Number by a Single Figure, or by any Number not more than 12.

* Set the multiplier under the units figure, or right-hand place, of the multiplicand, and draw a line below it.—Then, beginning at the right-hand, multiply every figure in this by the multiplier.—Count how many tens there are in the product of every single figure, and set down the remainder directly under the figure that is multiplied; and if nothing remains, set down a cipher.—Carry as many units or ones, as there are tens counted, to the product of the next figures, and proceed in the same manner till the whole is finished.

EXAMPLE.

Multiply	9876543210	Multiplicand.
By	2	Multiplier.
	19753086420	Product.

To multiply by a Number consisting of Several Figures.

† Set the multiplier below the multiplicand, placing them as in Addition, namely, units under units, tens under tens, &c. drawing a line below it.—Multiply the whole of the multiplicand by each figure of the multiplier, as in the last article; setting down a line of products for each figure in the multiplier, so as that the first figure of each line may stand straight under the figure multiplying by.—Add all the lines of products together, in the order as they stand, and their sum will be the answer or whole product required.

TO PROVE MULTIPLICATION.

THERE are three different ways of proving Multiplication, which are as below:

First Method.—Make the multiplicand and multiplier change places, and multiply the latter by the former in the same manner as before. Then if the product found in this way be the same as the former, the number is right.

Second Method.—‡ Cast all the 9's out of the sum of the figures in each of the

* The reason of this rule is the same as for the process in Addition, in which 1 is carried for every 10, to the next place, gradually as the several products are produced, one after another, instead of setting them all down below each other, as in the annexed Example.

5678		
4		
32	=	8 × 4
280	=	70 × 4
2400	=	600 × 4
20000	=	5000 × 4
22712	=	5678 × 4

† After having found the product of the multiplicand by the first figure of the multiplier, as in the former case, the multiplier is supposed to be divided into parts, and the product is found for the second figure in the same manner: but as this figure stands in the place of tens, the product must be 10 times its simple value; and therefore the first figure of this product must be set in the place of tens; or, which is the same thing directly under the figure multiplied by. And proceeding in this manner separately with all the figures of the multiplier, it is evident that we shall multiply all the parts of the multiplicand by all the parts of the multiplier, or the whole of the multiplicand by the whole of the multiplier: therefore these several products being added together, will be equal to the whole required product: as in the example annexed.

1234567		the multiplicand.
4567		
8641969	=	7 times the mult.[d]
7407402	=	60 times ditto.
6172835	=	500 times ditto.
4938268	=	4000 times ditto.
5638267489	=	4567 times ditto.

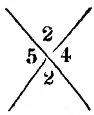
‡ This method of proof is derived from the peculiar property of the number 9, mentioned in the proof of Addition, and the reason for the one may serve for that of the other. Another more ample demon.

two factors, as in Addition, and set down the remainders. Multiply these two remainders together, and cast the 9's out of the product, as also out of the whole product or answer of the question, reserving the remainders of these last two, which remainders must be equal when the work is right.—*Note*, It is common to set the four remainders within the four angular spaces of a cross, as in the example below.

Third Method.—Multiplication is also very naturally proved by Division; for the product divided by either of the factors, will evidently give the other. But this cannot be practised till the rule of Division is learned.

EXAMPLES.

Mult. 3542		or Mult. 6196
by 6196	*Proof.*	by 3542
21252		12392
31878		24784
3542		30980
21252		18588
21946232 Product.		21946232 Proof.



OTHER EXAMPLES.

Multiply 123456789 by 3.	Ans. 370370367.
Multiply 123456789 by 4.	Ans. 493827156.
Multiply 123456789 by 5.	Ans. 617283945.
Multiply 123456789 by 6.	Ans. 740740734.
Multiply 123456789 by 7.	Ans. 864197523.
Multiply 123456789 by 8.	Ans. 987654312.
Multiply 123456789 by 9.	Ans. 1111111101.
Multiply 123456789 by 11.	Ans. 1358024679.
Multiply 123456789 by 12.	Ans. 1481481468.
Multiply 302914603 by 16.	Ans. 4846633648.
Multiply 273580961 by 23.	Ans. 6292362103.
Multiply 402097316 by 195.	Ans. 78408976620.
Multiply 82164973 by 3027.	Ans. 248713373271.
Multiply 7564900 by 579.	Ans. 4380077100.
Multiply 8496427 by 874359.	Ans. 7428927415293.
Multiply 2760325 by 37072.	Ans. 102330768400.

CONTRACTIONS IN MULTIPLICATION.

1. *When there are Ciphers in the Factors.*

IF the ciphers be at the right-hand of the numbers; multiply the other figures only, and annex as many ciphers to the right-hand of the product, as are in both the factors.—And when the ciphers are in the middle parts of the multiplier; neglect them as before, only taking care to place the first figure of every line of products exactly under the figure by which the multiplication is made.

stration of this rule may be as follows:—Let P and Q denote the number of 9's in the factors to be multiplied, and a and b what remain; then $9P + a$ and $9Q + b$ will be the numbers themselves, and their product is $(9P \times 9Q) + (9P \times b) + (9Q \times a) + (a \times b)$; but the first three of these products are each a precise number of 9's, because their factors are so, either one or both: these therefore being cast away, there remains only $a \times b$; and if the 9's be also cast out of this, the excess is the excess of 9's in the total product: but a and b are the excesses in the factors themselves, and $a \times b$ is their product; therefore the rule is true.

EXAMPLES.

1.	2.
Mult. 9001635	Mult. 390720400
by ... 70100	by ... 406000
9001635	23443224
63011445	15628816
631014613500 Prod.	158632482400000

3. Multiply 81503600 by 7030. Ans. 572970308000.
4. Multiply 9030100 by 2100. Ans. 18963210000.
5. Multiply 8057069 by 70050. Ans. 564397683450.

II. *When the Multiplier is the Product of two or more Numbers in the Table; then*

* Multiply by each of those parts separately, instead of the whole number at once.

EXAMPLES.

1. Multiply 51307298 by 56, or by 7 times 8.

```
 51307298
        7
---------
359151086
        8
---------
2873208688
```

2. Multiply 31704592 by 36. Ans. 1141365312.
3. Multiply 29753804 by 72. Ans. 2142273888.
4. Multiply 7128368 by 96. Ans. 684323328.
5. Multiply 160430800 by 108. Ans. 17326526400.
6. Multiply 61835720 by 1320. Ans. 81623150400.
7. There was an army composed of 104 † battalions, each consisting of 500 men; what was the number of men contained in the whole? Ans. 52000.
8. A convoy of ammunition ‡ bread, consisting of 250 waggons, and each waggon containing 320 loaves, having been intercepted and taken by the enemy; what is the number of loaves lost? Ans. 80000.

OF DIVISION.

Division is a kind of compendious method of Subtraction, teaching to find how often one number is contained in another, or may be taken out of it, which is the same thing.

* The reason of this rule is obvious enough; for any number multiplied by the component parts of another, must give the same product as if it were multiplied by that number at once. Thus, in the 1st example, 7 times the product of 8 by the given number, makes 56 times the same number, as plainly as 7 times 8 makes 56.

† A battalion is a body of foot, consisting of 500, or 600, or 700 men, more or less.

‡ The ammunition bread is that which is provided for, and distributed to the soldiers; the usual allowance being a loaf of 6 pounds to every soldier, once in 4 days.

The number to be divided, is called the *Dividend.*—The number to divide by, is the *Divisor.*—And the number of times the dividend contains the divisor, is called the *Quotient.*—Sometimes there is a *Remainder* left, after the division is finished.

The usual manner of placing the terms, is, the dividend in the middle, having the divisor on the left-hand, and the quotient on the right, each separated by a curve line; as to divide 12 by 4, the quotient is 3,

Divisor.	Dividend.	Quotient.
4)	12	(3

showing that the number 4 is 3 times contained in 12, or may be three times subtracted out of it, as in the margin.

* *Rule.*—Having placed the divisor before the dividend, as above directed, find how often the divisor is contained in as many figures of the dividend as are just necessary, and place the number on the right in the quotient. Multiply the divisor by this number, and set the product under the figures of the dividend before-mentioned.—Subtract this product from that part of the dividend under which it stands, and bring down the next figure of the dividend, or more if necessary, to join on the right of the remainder.—Divide this number, so increased, in the same manner as before; and so on till all the figures are brought down and used.

```
12
 4 subtr.
—
 8
 4 subtr.
—
 4
 4 subtr.
—
 0
```

N. B. If it be necessary to bring down more figures than one to any remainder, in order to make it as large as the divisor, or larger, a cipher must be set in the quotient for every figure so brought down more than one.

TO PROVE DIVISION.

† Multiply the quotient by the divisor; to this product add the remainder, if there be any; then the sum will be equal to the dividend when the work is right.

* In this way we resolve the dividend into parts, and find by trial how often the divisor is contained in each of those parts, one after another, and arranging the several figures of the quotient one after another, into one number.

When there is no remainder to a division the quotient is the whole and perfect answer to the question But when there is a remainder, it goes so much towards another time as it approaches to the divisor: so, if the remainder be half the divisor, it will go the half of a time more; if the 4th part o the divisor, it will go one-fourth of a time more; and so on. Therefore, to complete the quotient, set the remainder at the end of it, above a small line, and the divisor below it, thus forming a fractional part of the whole quotient.

† This method of proof is plain enough: for since the quotient is the number of times the dividend contains the divisor, the quotient multiplied by the divisor must evidently be equal to the dividend.

There are also several other methods sometimes used for proving Division, some of the most useful of which are as follow:

Second Method—Subtract the remainder from the dividend, and divide what is left by the quotient, so shall the new quotient from this last division be equal to the former divisor, when the work is right.

Third Method.—Add together the remainder and all the products of the [illegible] quotient figures by the divisor, according to the order in which they stand in the work; and the sum will be equal to the dividend when the work is right.

EXAMPLES.

```
          1.                                          2.
3 ) 1234567 ( 411522 Quot.           37 ) 12345678 ( 333666 Quot.
    12          mult. 3                   111             37
    --         -------                    ---        -------
     3         1234566                     124       2335662
     3           add 1                     111       1000998
    --         -------                     ---       rem. 36
     4         1234567                     135       --------
     3         -------                     111       12345678
    --          Proof.                     ---       --------
     15                                    246        Proof.
     15                                    222
     --                                    ---
      6                                     247
      6                                     222
      --                                    ---
       7                                     258
       6                                     222
       --                                    ---
 Rem.  1                              Rem.  36
```

3. Divide 73146085 by 4. Ans. 18286521$\frac{1}{4}$.
4. Divide 5317986027 by 7. Ans. 759712289$\frac{4}{7}$.
5. Divide 570196382 by 12. Ans. 47516365$\frac{2}{12}$.
6. Divide 74638105 by 37. Ans. 2017246$\frac{3}{37}$.
7. Divide 137896254 by 97. Ans. 1421610$\frac{84}{97}$.
8. Divide 35821649 by 764. Ans. 46886$\frac{745}{764}$.
9. Divide 72091365 by 5201. Ans. 13861$\frac{304}{5201}$.
10. Divide 4637064283 by 57606. Ans. 80496$\frac{11707}{57606}$.

11. Suppose 471 men are formed into ranks of 3 deep, what is the number in each rank? Ans. 157.

12. A party, at the distance of 378 miles from the head quarters, receive orders to join their corps in 18 days; what number of miles must they march each day to obey their orders? Ans. 21.

13. The annual revenue of a gentleman being 37960*l*; how much a day is that equivalent to, there being 365 days in the year? Ans. 104*l*.

CONTRACTIONS IN DIVISION.

THERE are certain contractions in Division, by which the operation in particular cases may be performed in a shorter manner; as follows:

I. *Division by any Small Number*, not greater than 12, may be expeditiously performed, by multiplying and subtracting mentally, omitting to set down the work, except only the quotient immediately below the dividend.

EXAMPLES.

```
     3 ) 56103961          4 ) 52019675          5 ) 1370192
         --------              --------              -------
Quot.    18701320⅓
         --------              --------              -------

     6 ) 38072910          7 ) 81392627          8 ) 23718620
         --------              --------              --------

         --------              --------              --------
     9 ) 43[illegible]62         11 ) 57014230         12 ) 27980318
         --------              --------              --------

         --------              --------              --------
```

II. * *When Ciphers are annexed to the Divisor;* cut off those ciphers from it, and cut off the same number of figures from the right-hand of the dividend; then divide the remaining figures, as usual. And if there be any thing remaining after this division, place the figures cut off from the dividend to the right of it, and the whole will be the true remainder; otherwise, the figures cut off only will be the remainder.

EXAMPLES.

1. Divide 3704196 by 20.

```
2,0 ) 370419,6
      --------
Quot. 185209 16/20
      --------
```

2. Divide 31086901 by 7100.

```
71,00 ) 310869,01 ( 4378 3101/7100.
        284
        ---
         268
         213
         ---
          556
          497
          ---
           599
           568
           ---
            31
           ---
```

3. Divide 7380964 by 23000. Ans. $320\frac{20964}{23000}$.

4. Divide 2304109 by 5800. Ans. $397\frac{1509}{5800}$.

III. *When the Divisor is the exact Product of two or more of the small Numbers not greater than* 12: † Divide by each of those numbers separately, instead of the whole divisor at once.

N. B.—There are commonly several remainders in working by this rule, one to each division; and to find the true or whole remainder, the same as if the division had been performed all at once, proceed as follows: Multiply the last remainder by the preceding divisor, or last but one, and to the product add the preceding remainder; multiply this sum by the next preceding divisor, and to the product add the next preceding remainder; and so on, till you have gone backward through all the divisors and remainders to the first. As in the example following:

EXAMPLES.

1. Divide 31046835 by 56, or 7 times 8.

```
7 ) 31046835                        6 the last rem.
    --------                  mult. 7 preced. divisor.
8 )  4435262—1 first rem.          --
    --------                        42
      554407—6 second rem.      add  1 the 1st rem.
    --------                        --
Ans. 554407 43/56.                  43 whole rem.
    --------                        --
```

* This method is only to avoid a needless repetition of ciphers, which would happen in the common way. And the truth of the principle upon which it is founded, is evident; for, cutting off the same number of ciphers, or figures, from each, is the same as dividing each of them by 10, or 100, or 1000, &c according to the number of ciphers cut off; and it is evident that as often as the whole divisor is contained in the whole dividend, so often must any part of the former be contained in a like part of the latter.

† This follows from the 2d contraction in Multiplication, being only the converse of it; for the half of the third part of any thing, is evidently the same as the sixth part of the whole; and so of any other numbers.—The reason of the method of finding the whole remainder, from the several particular ones, will best appear from the nature of Vulgar Fractions. Thus, in the first example above, the first remainder being 1, when the divisor is 7, makes $\frac{1}{7}$; this must be added to the second remainder 6, making $6\frac{1}{7}$ to the divisor 8, or to be divided by 8. But $6\frac{1}{7} = \frac{6 \times 7 + 1}{7} = \frac{43}{7}$; and this divided by 8 gives $\frac{43}{7 \times 8} = \frac{43}{56}$.

2. Divide 7014596 by 72. Ans. 97424$\frac{68}{72}$.
3. Divide 5130652 by 132. Ans. 38868$\frac{76}{132}$.
4. Divide 83016572 by 240. Ans. 345902$\frac{92}{240}$.

IV. *Common Division may be performed more concisely*, by omitting the several products, and setting down only the remainders; namely, multiply the divisor by the quotient figures as before, and, without setting down the product, subtract each figure of it from the dividend, as it is produced; always remembering to carry as many to the next figure as were borrowed before.

EXAMPLES.

1. Divide 3104679 by 833

```
833 ) 3104679 ( 3727 88/833.
        6056
         2257
          5919
            88
```

2. Divide 79165238 by 238. Ans. 332627$\frac{12}{238}$.
3. Divide 29137062 by 5317. Ans. 5479$\frac{5219}{5317}$.
4. Divide 62015735 by 7803. Ans. 7947$\frac{5294}{7803}$.

OF REDUCTION.

Reduction is the changing of numbers from one name or denomination to another, without altering their value.—This is chiefly concerned in reducing money, weights, and measures.

When the numbers are to be reduced from a higher name to a lower, it is called *Reduction Descending;* but when, contrariwise, from a lower name to a higher, it is *Reduction Ascending.*

Before proceeding to the rules and questions of Reduction, it will be proper to set down the usual Tables of money, weights, and measures, which are as below:

OF MONEY, WEIGHTS, AND MEASURES.

TABLES OF MONEY.*

2 Farthings	=	1 Halfpenny	$\frac{1}{2}$	*qrs.*		*d.*				
4 Farthings	=	1 Penny	*d.*	4	=	1		*s.*		
12 Pence	=	1 Shilling	*s.*	48	=	12	=	1		£.
20 Shillings	=	1 Pound	£.	960	=	240	=	20	=	1.

* £ denotes pounds, *s* shillings and *d* denotes pence.
$\frac{1}{4}$ denotes 1 farthing, or one quarter of any thing.
$\frac{1}{2}$ denotes a half-penny, or the half of any thing.
$\frac{3}{4}$ denotes 3 farthings, or three quarters of any thing.

The full weight and value of the English gold and silver coin, is as here below.

Gold.	Value. £	*s.*	*d.*	Weight. *dwt.*	*gr.*	Silver.	Value. *s.*	*d.*	Weight. *dwt.*	*gr.*
A Guinea	1	1	0	5	$9\frac{1}{2}$	A Crown	5	0	18	$4\frac{4}{11}$
Half-guinea	0	10	6	2	$16\frac{3}{4}$	Half-crown	2	6	9	$2\frac{2}{11}$
Quarter-guinea	0	5	3	1	$8\frac{1}{4}$	Shilling	1	0	3	$15\frac{2}{11}$
Sovereign	1	0	0	5	$3\frac{3}{11}$	Sixpence	0	6	1	$19\frac{7}{11}$
Half Sov.	0	10	0	2	$13\frac{7}{11}$					

PENCE TABLE.

d.		s.	d.
20	are	1	8
30	—	2	6
40	—	3	4
50	—	4	2
60	—	5	0
70	—	5	10
80	—	6	8
90	—	7	6
100	—	8	4
110	—	9	2
120	—	10	0

SHILLINGS TABLE.

s.		d.
1	is	12
2	are	24
3	—	36
4	—	48
5	—	60
6	—	72
7	—	84
8	—	96
9	—	108
10	—	120

TROY WEIGHT.*

Grains .... marked gr.
24 Grains make 1 Pennyweight dwt.
20 Pennyweights 1 Ounce oz.
12 Ounces 1 Pound lb.

gr.		dwt.		oz.		lb.
24	=	1				
480	=	20	=	1		
5760	=	240	=	12	=	1

By this weight are weighed Gold, Silver, and Jewels.

APOTHECARIES' WEIGHT.

Grains .... marked ...... gr.
20 Grains make 1 Scruple —— ...... sc. or ℈
3 Scruples — 1 Dram —— ...... dr. or ʒ
8 Drams — 1 Ounce —— ...... oz. or ℥
12 Ounces — 1 Pound —— ...... lb. or ℔.

gr.		sc.		dr.		oz.		lb.
20	=	1						
60	=	3	=	1				
480	=	24	=	8	=	1		
5760	=	288	=	96	=	12	=	1

The usual value of gold is nearly £4 an ounce, or 2d a grain; and that of silver is nearly 5s an ounce. Also, any quantity of gold is to the same weight of standard silver, nearly as 15 to 1, or more nearly as 15 and 1-14th to 1.

Pure gold, free from mixture with other metals, usually called fine gold, is of such purity of nature, that it will endure the fire without wasting, though it be kept continually melted. But silver, not having the purity of gold, will not endure the fire like it: yet fine silver will waste but a very little by being in the fire any moderate time; whereas copper, tin, lead, &c, will not only waste, but may be calcined or burnt to a powder.

Both gold and silver, in their purity, are so very soft and flexible (like new lead, &c.) that they are not so useful either in coin or otherwise, (except to beat into leaf gold or silver) as when they are alloyed, or mixed and hardened with copper or brass. And though most nations differ more or less in the quantity of such alloy, as well as in the same place at different times, yet in England the standard for gold and silver coin has been for a long time as follows, viz. That 22 parts of fine gold, and 2 parts of copper, being melted together, shall be esteemed the true standard for gold coin: And that 11 ounces and 2 pennyweights of fine silver, and 18 pennyweights of copper, being melted together, is esteemed the true standard for silver coin, called Sterling silver.

* The original of all weights used in England, was a grain or corn of wheat, gathered out of the middle of the ear, and being well dried, 32 of them were to make one pennyweight, 20 pennyweights one ounce, and 12 ounces one pound. But in later times, it was thought sufficient to divide the same pennyweight into 24 equal parts, still called grains, being the least weight now in common use; and from thence the rest are computed, as in the Tables above.

This is the same as Troy weight, only having some different divisions. Apothecaries make use of this weight in compounding their medicines; but they buy and sell their drugs by Avoirdupois weight.

AVOIRDUPOIS WEIGHT.

	Drams			marked	dr.
16	Drams	make	1 Ounce	—	oz.
16	Ounces	—	1 Pound	—	lb.
28	Pounds	—	1 Quarter	—	qr.
4	Quarters	—	1 Hundred Weight	—	cwt.
20	Hundred Weight	—	1 Ton	—	ton

dr.		oz.		lb.		qr.		cwt.		ton.
16	=	1								
256	=	16	=	1						
7168	=	448	=	28	=	1				
28672	=	1792	=	112	=	4	=	1		
573440	=	35840	=	2240	=	80	=	20	=	1

By this Weight, are weighed all things of a coarse or drossy nature, as Corn, Bread, Butter, Cheese, Flesh, Grocery Wares, and some Liquids; also all Metals except Silver and Gold

		oz.	dwt.	gr.	
Note, that 1 lb. Avoirdupois	=	14	11	15½	Troy.
1 oz. ——	=	0	18	5½	—
1 dr. ——	=	0	1	3½	—

LONG MEASURE.

3 Barley-corns	make	1 Inch	marked	In.
12 Inches	—	1 Foot	—	Ft.
3 Feet	—	1 Yard	—	Yd.
6 Feet	—	1 Fathom	—	Fth.
5 Yards and a half	—	1 Pole or Rod	—	Pl.
40 Poles	—	1 Furlong	—	Fur.
8 Furlongs	—	1 Mile	—	Mile
3 Miles	—	1 League	—	Lea.
69½ Miles nearly	—	1 Degree	—	Deg. or °

In.		Ft.		Yd.		Pl.		Fur.		Mile.
12	=	1								
36	=	3	=	1						
198	=	16½	=	5½	=	1				
7920	=	660	=	220	=	40	=	1		
63360	=	5280	=	1760	=	320	=	8	=	1

CLOTH MEASURE.

2 Inches and a quarter	make	1 Nail	marked	Nl.
4 Nails	—	1 Quarter of Yard	—	Qr.
3 Quarters	—	1 Ell Flemish	—	E F
4 Quarters	—	1 Yard	—	Yd.
5 Quarters	—	1 Ell English	—	E E
4 Qrs 1⅕ Inch	—	1 Ell Scotch	—	E S.

SQUARE MEASURE.

144	Square Inches	make	1 Sq. Foot	marked	Ft.
9	Square Feet	—	1 Sq. Yard	—	Yd.
$30\frac{1}{4}$	Square Yards	—	1 Sq. Pole	—	Pole.
40	Square Poles	—	1 Rood	—	Rd.
4	Roods	—	1 Acre	—	Acr.

Sq. Inc.		Sq. Ft.		Sq. Yd.		Sq. Pl.		Rd.		Acr.
144	=	1								
1296	=	9	=	1						
39204	=	$272\frac{1}{4}$	=	$30\frac{1}{4}$	=	1				
1568160	=	10890	=	1210	=	40	=	1		
6272640	=	43560	=	4840	=	160	=	4	=	1

By this measure, Land, Husbandmen and Gardeners work are measured; also Artificers works, such as Board, Glass, Pavements, Plastering, Wainscotting, Tiling, Flooring, and every dimension of length and breadth only.

When three dimensions are concerned, namely, length, breadth, and depth or thickness, it is called cubic or solid measure, which is used to measure Timber, Stone, &c.

The cubic or solid Foot, which is 12 inches in length and breadth and thickness, contains 1728 cubic or solid inches, and 27 solid feet make one solid yard.

DRY, OR CORN MEASURE.

2	Pints	make	1 Quart	marked	Qt.
2	Quarts	—	1 Pottle	—	Pot.
2	Pottles	—	1 Gallon	—	Gal.
2	Gallons	—	1 Peck	—	Pec.
3	Pecks	—	1 Bushel	—	Bu.
8	Bushels	—	1 Quarter	—	Qr.
5	Quarters	—	1 Weigh or Load	—	Wey.
2	Weys	—	1 Last	—	Last.

Pts.		Gal.		Pec.		Bu.		Qr.		Wey.		Last.
8	=	1										
16	=	2	=	1								
64	=	8	=	4	=	1						
512	=	64	=	32	=	8	=	1				
2560	=	320	=	160	=	40	=	5	=	1		
5120	=	640	=	320	=	80	=	10	=	2	=	1

By this are measured all dry wares, as Corn, Seeds, Roots, Fruits, Salt, Coals, Sand, Oysters, &c.

The standard Gallon dry-measure contains $268\frac{4}{5}$ cubic or solid inches, and the Corn or Winchester Bushel $2150\frac{2}{5}$ cubic inches; for the dimensions of the Winchester bushel, by the Statute, are 8 inches deep, and $18\frac{1}{2}$ inches wide or in diameter. But the Coal bushel must be $19\frac{1}{2}$ inches in diameter; and 36 bushels, heaped up, make a London chaldron of coals, the weight of which is 3136 lb Avoirdupois.

ALE AND BEER MEASURE.

2	Pints	make	1 Quart	marked	Qt.
4	Quarts	—	1 Gallon	—	Gal.
36	Gallons	—	1 Barrel	—	Bar.
1	Barrel and a half	—	1 Hogshead	—	Hhd.

2 Barrels	—	1 Puncheon	—	Pun.
2 Hogsheads	—	1 Butt	—	Butt.
2 Butts	—	1 Tun	—	Tun.

Pts.		Qt.								
2	=	1		Gal.						
8	=	4	=	1		Bar.				
288	=	144	=	36	=	1		Hhd.		
432	=	216	=	54	=	$1\frac{1}{2}$	=	1		Butt.
864	=	432	=	108	=	3	=	2	=	1

Note,—The Ale Gallon contains 282 cubic or solid inches,

WINE MEASURE.

2 Pints	make	1 Quart	marked	Qt.
2 Quarts	—	1 Gallon	—	Gal.
42 Gallons	—	1 Tierce	—	Tier
63 Gallons or $1\frac{1}{2}$ Tier. ...	—	1 Hogshead	—	Hhd.
2 Tierces	—	1 Puncheon	—	Pun.
2 Hogsheads	—	1 Pipe or Butt	—	Pi.
2 Pipes	—	1 Tun	—	Tun.

Pts.		Qts.												
2	=	1		Gal.										
8	=	4	=	1		Tier.								
336	=	168	=	42	=	1		Hhd.						
504	=	252	=	63	=	$1\frac{1}{2}$	=	1		Pun.				
672	=	336	=	84	=	2	=	$1\frac{1}{3}$	=	1		Pi.		
1008	=	504	=	126	=	3	=	2	=	$1\frac{1}{2}$	=	1		Tun
2016	=	1008	=	252	=	6	=	4	=	3	=	2	=	1

Note.—By this are measured all Wines, Spirits, Strong waters, Cyder, Mead, Perry, Vinegar, Oil, Honey, &c.

The Wine Gallon contains 231 cubic or solid inches. And it is remarkable that the Wine and Ale Gallons have the same proportion to each other, as the Troy and Avoirdupois Pounds have; that is, as one Pound Troy is to one Pound Avoirdupois, so is one Wine Gallon to one Ale Gallon.

OF TIME.

60 Seconds	make	1 Minute	marked	M or
60 Minutes	—	1 Hour	—	Hr.
24 Hours	—	1 Day	—	Day.
7 Days	—	1 Week	—	Wk.
4 Weeks	—	1 Month	—	Mo.
13 Months, 1 Day, 6 Hours, or 365 Days, 6 Hours	—	1 Julian Year	—	Yr.

Sec.		Min.										
60	=	1		Hr.								
3600	=	60	=	1		Day.						
86400	=	1440	=	24	=	1		Wk.				
604800	=	10080	=	168	=	7	=	1		Mo.		
2419200	=	40320	=	672	=	28	=	4	=	1		
31557600	=	525960	=	8766	=	$365\frac{1}{4}$	=	1 Year				

	Wk.	Da.	Hr.		Mo.	Da.	Hr.		
Or	52	1	6	=	13	1	6	=	1 Julian Year.

	Da.	Hr.	M.	Sec.		
But	365	5	48	48	=	1 Solar Year.

IMPERIAL MEASURES.

By the late Act of Parliament for Uniformity of Weights and Measures, which commenced its operation on the 1st of January, 1826, the chief part of the weights and measures are allowed to remain as they were; the Act simply prescribing scientific modes of determining them, in case they should be lost.

The pound *troy* contains 5760 grains.
The pound *avoirdupois* contains 7000 grains.
The *imperial gallon* contains 277·274 cubic inches.
The *corn bushel* eight times the above, or 2218·192 cubic inches.

Hence with respect to Ale, Wine, and Corn, it will be expedient to possess a

TABLE OF FACTORS,

For converting old measures into new, and the contrary.

	By decimals.			By vulgar fractions nearly.		
	Corn Measure.	Wine Measure.	Ale Measure.	Corn Measure.	Wine Measure.	Ale Measure.
To convert old measures to new.	.96943	.83311	1.01704	$\frac{31}{32}$	$\frac{5}{6}$	$\frac{60}{59}$
To convert new measures to old	1.03153	1.20032	.98324	$\frac{32}{31}$	$\frac{6}{5}$	$\frac{59}{60}$

N. B.—For reducing the *prices*, these numbers must all be reversed.

RULES FOR REDUCTION.

I. *When the Numbers are to be reduced from a Higher Denomination to a Lower.*

Multiply the number in the highest denomination by as many of the next lower as make an integer, or 1, in that higher; to this product add the number, if any, which was in this lower denomination before, and set down the amount.

Reduce this amount in like manner, by multiplying it by as many of the next lower as make an integer of this, taking in the odd parts of this lower, as before. And so proceed through all the denominations to the lowest; so shall the num-

*ber last found be the value of all the numbers which were in the higher denominations, taken together.**

EXAMPLE.

1. In 1234*l.* 15*s.* 7*d*, how many farthings?

	£.	s.	d.
	1234	15	7
	20		
	24695	Shillings.	
	12		
	296347	Pence.	
	4		
Answer	1185388	Farthings.	

II. *When the Numbers are to be reduced from a Lower Denomination to a Higher,*

Divide the given number by as many of that denomination as make 1 of the next higher, and set down what remains, as well as the quotient.

Divide the quotient by as many of this denomination as make 1 of the next higher; setting down the new quotient, and remainder, as before.

Proceed in the same manner through all the denominations, to the highest; and the quotient last found, together with the several remainders, if any, will be of the same value as the first number proposed.

EXAMPLES.

2. Reduce 1185388 farthings, into pounds, shillings, and pence.

4) 1185388

12) 296347 *d*

2,0) 2469,5 *s.*—7*d.*

1234*l.* 15*s.* 7*d.*

3. Reduce 23*l.* to farthings. Ans. 22080.

4. Reduce 337587 farthings to pounds, &c. Ans. 351*l.* 13*s.* $0\frac{3}{4}$*d.*

5. How many farthings are in 35 guineas? Ans. 35280.

6. In 35280 farthings how many guineas? Ans. 35.

7. In 59 lb. 13 dwts. 5 gr. how many grains? Ans. 340157.

8. In 8012131 grains how many pounds, &c? Ans. 1390 lb. 11 oz. 18 dwt. 19 gr.

9. In 35 ton. 17 cwt. 1 qr. 23 lb. 7 oz. 13 dr. how many drams? Ans. 20571005.

10. How many barley-corns will reach round the earth, supposing it, according to the best calculations, to be 25000 miles? Ans. 4752000000.

* The reason of this rule is very evident; for pounds are brought into shillings by multiplying them by 20; shillings into pence, by multiplying them by 12; and pence into farthings, by multiplying by 4; and the reverse of this rule by Division.—And the same, it is evident, will be true in the reduction of numbers consisting of any denominations whatever.

11. *How many seconds are in a solar year, or 365 days 5 hrs. 48 min. 48 sec.?*
Ans. 31556928.

12. In a lunar month, or 29 ds. 12 hrs. 44 min. 3 sec. how many seconds?
Ans. 2551443.

COMPOUND ADDITION.

Compound Addition shows how to add or collect several numbers of different denominations into one sum.

Rule.—Place the numbers so that those of the same denomination may stand directly under each other, and draw a line below them.—Add up the figures in the lowest denomination, and find, by Reduction, how many units, or ones, of the next higher denomination are contained in their sum.—Set down the remainder below its proper column, and carry those units or ones to the next denomination, which, add up in the same manner as before.—Proceed thus through all the denominations, to the highest, whose sum, together with the several remainders, will give the answer sought.

The method of proof is the same as in Simple Addition.

EXAMPLES OF MONEY.

1.			2.			3.			4.		
£.	s.	d.	£.	s.	d.	£.	s.	d.	£.	s.	d.
7	13	3	14	7	5	15	17	10	53	14	8
3	5	10½	8	19	2¼	3	14	6	5	10	2¾
6	18	7	5	3	4¼	23	6	2¾	93	11	6
0	2	5¾	21	2	9	8	3	5	7	5	0
4	0	3	7	16	8½	15	6	4	2	0	9
17	15	4¼	0	4	3	6	12	9¾	0	18	7
39	15	9¾									
32	2	6¾									
39	15	9¾									

5.			6.			7.			8.		
£.	s.	d.	£.	s.	d.	£.	s.	d.	£.	s.	d.
14	0	7¼	37	15	8	61	3	2½	472	15	3
5	13	6	14	12	9¾	7	16	8	9	2	2½
62	4	7	5	6	11	29	13	10¾	27	12	6¼
4	17	8	23	10	9¼	8	14	0	370	16	2½
23	0	4¾	8	6	0	0	7	5¼	25	3	8
6	6	7	14	0	5½	24	13	0	6	10	5¾
91	0	10¼	54	2	7¼	5	0	10¼	30	0	11¾

9. A nobleman, going out of town, is informed by his steward that his butcher's bill comes to 197*l.* 13*s.* 7½*d.*; his baker's to 59*l.* 5*s.* 2¾*d.*; his brewer's to 85*l.*; his wine-merchant's to 103*l.* 13*s.*; to his corn-chandler is due 75*l.* 3*d.*; to his tallow-chandler and cheese-monger, 27*l.* 15*s.* 11¼*d.*; and to his tailor 55*l.* 3*s.* 5¾*d.*; also for rent, servants' wages, and other charges, 127*l.* 3*s.*: now, supposing he would take 100*l.* with him, to defray his charges on the road, for what sum must he send to his banker? Ans. 830*l.* 14*s.* 6¼*d.*

10. The strength of a regiment of foot, of 10 companies, and the amount of their subsistence,* for a month of 30 days, according to the annexed Table, are required?

Number.	Rank.	Subsistence for a month.		
		£.	s.	d.
1	Colonel	27	0	0
1	Lieutenant Colonel	19	10	0
1	Major	17	5	0
7	Captains	78	15	0
11	Lieutenants	57	15	0
9	Ensigns	40	10	0
1	Chaplain	7	10	0
1	Adjutant	4	10	0
1	Quarter-master	5	5	0
1	Surgeon	4	10	0
1	Surgeon's Mate	4	10	0
30	Serjeants	45	0	0
30	Corporals	30	0	0
20	Drummers	20	0	0
2	Fifers	2	0	0
390	Private men	292	10	0
507	Total.	656	10	0

* Subsistence Money, is the money paid to the soldiers weekly, short of their full pay; because their clothes, accoutrements, &c. are to be accounted for. It is likewise the money advanced to officers till their accounts are made up, which is commonly once a year, when they are paid their arrears. The following Table shows the full pay and subsistence of each rank on the English establishment:

COMMISSIONED OFFICERS' REGIMENTAL PAY,—1824.

	Life Guards. Subsistence per Diem nett.	Life Guards. Gross Pay and Allowance per Diem as borne on the Establishment.	Horse Guards. Subsistence per Diem nett.	Horse Guards. Gross Pay and Allowance per Diem as borne on the Establishment.	Foot Guards. Subsistence per Diem nett.	Foot Guards. Gross Pay and Allowance per Diem as borne on the Establishment.	Dr. Gds. & Drs.	R. Wag. Train.	Foot.	R. Staff Corps.	Royal Artillery. Marching and Invalid Battalion.	Royal Artillery. Horse Brigade.	Royal Eng.	Royal Marines	Militia and Fencib.
Colonel Commandant	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	2 14 9½	3 0 0	2 14 9½	2 13 0	. . .
Colonel	1 7 0	1 16 0	1 11 0	2 1 0	1 10 0	1 19 0	1 12 10	. . .	1 2 6	1 16 10	1 6 3	1 12 0	1 6 3	1 2 6	1 2 6
Lieutenant Colonel	1 3 3	1 11 0	1 2 6	1 9 6	1 1 6	1 8 6	1 3 0	1 12 10	0 17 0	1 3 0	0 18 1	1 7 1	{16 1 {18 1	} 17 0	0 15 11
Major	0 19 6	1 6 0	1 1 6	1 7 0	0 18 6	1 4 6	0 19 3	0*18 9	0 16 0	0 19 3	0 16 11	1 2 11	. . .	0 16 0	0 14 1
Captain	0 12 0	0 16 0	0 16 6	1 1 6	0 12 6	0 16 6	0 14 7	0*12 6	0 10 6	0 14 7	0 11 1	0 16 1	0 11 1	0 10 6	0 10 6
Ditto with Brevet of Major, or superior rank	. . .	. . .	. . .	. . .	. . .	. . .	. . .	0*14 6	0 12 6	. . .	0 13 1	0 18 1	0 13 1	0 12 6	. . .
Lieutenant	0 8 3	0 11 0	0 11 6	0 15 0	0 6 0	0 7 10	0 9 0	0 *8 6	0 6 6	0 9 0	0 6 10	0 9 10	0 6 10	0 6 6	0 §6 0
Ditto above 7 years standing	. . .	. . .	. . .	. . .	. . .	. . .	. . .	0 *9 6	0 7 6	. . .	0 7 10	0 10 10	0 7 10	0 7 6	. . .
Cornet, Ensign, and 2d Lieut.	0 7 3	0 8 6	0 11 0	0 14 0	0 4 6	0 5 10	0 8 0	0 *7 3	0 5 3	0 8 0	0 5 7	. . .	0 5 7	0 5 3	0 §5 3
Paymaster	. . .	. . .	. . .	. . .	. . .	. . .	0 15 0	0 15 0	0 15 0	0 15 0	. . .	. . .	. . .	. . .	0‡15 0
Adjutant	0 13 0	0 13 0	0 10 0	0 10 0	0 10 0	0 10 0	0 10 0	0 10 0	0 8 6	0 10 0	0 8 †6	0 10 6	0 6 0	0 8 6	0 8 6
Quarter Master	0 4 9	0 6 0	0 8 6	0 8 6	0 6 6	0 6 6	0 *8 0	. . .	0 6 6	0 6 6	0 7 10	0 10 10	0 6 0	0 6 6	0 6 6
Surgeon Major	. . .	. . .	. . .	. . .	0 16 8	1 0 0	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
Battalion Surgeon	. . .	. . .	. . .	. . .	0 10 0	0 12 0	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
Surgeon	0 9 0	0 12 0	0 9 0	0 12 0	. . .	. . .	0 11 4	0 11 4	0 11 4	0 11 4	0 11 4	. . .	. . .	. . .	0 11 4
Assistant Surgeon	0 8 6	0 8 6	0 8 6	0 8 6	0 7 6	0 7 6	0 8 6	. . .	0 7 6	0 7 6	0 7 6	. . .	. . .	. . .	. . .
Surgeon's Mate	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	0 §5 0
Veterinary Surgeon	0 8 0	0 8 0	0 8 0	0 8 0	. . .	. . .	0 8 0	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

The difference between the Subsistence and Gross Pay of the Officers of these Regiments, after deducting Poundage, Hospital, and Agency, is paid as "*Arrears.*"

* These rates include 2*s.* a day for a horse. † If second Captain 12*s.* 8*d.* Including pay as a Subaltern.

§ If holding another Commission, Lieutenants of Militia receive only 4*s.* 8*d.*, Ensigns 3*s.* 8*d.*, and Surgeons' Mates 3*s.* 6*d.*

MEM.—Regimental Surgeons of the Line, those of the Royal Artillery, and Veterinary Surgeons, after certain periods of Service, receive the following Rates of Pay, viz.

Surgeons of the Line and Royal Artillery. *s.* *d.*
After 7 years service .... 14 1 per diem.
— 20 ditto .... 18 10 ditto.

Veterinary Surgeons. *s.*
After 3 years service .... 10 per diem.
— 10 ditto .... 12 ditto.
— 20 ditto .... 15 ditto.

EXAMPLES OF WEIGHTS, MEASURES, &c.

TROY WEIGHT

1.

lb.	oz.	dwt.
17	3	15
4	6	3
0	10	7
9	5	0
176	2	17
23	11	12

2.

oz.	dwt.	gr.
37	9	3
9	5	3
3	16	21
17	7	8
5	9	0
3	0	19

APOTHECARIES WEIGHT.

3.

lb.	oz.	dr.	sc.
3	5	7	2
13	7	3	0
9	11	0	1
0	9	1	2
36	3	5	0
5	8	6	1

4.

oz.	dr.	sc.	gr.
3	5	1	17
7	3	2	5
16	7	0	12
9	5	1	5
4	1	2	18
36	4	1	14

AVOIRDUPOIS WEIGHT.

5.

lb.	oz.	dr.
17	10	13
5	14	8
8	6	15
27	1	6
0	4	0
6	14	10

6.

cwt.	qr.	lb.
15	2	15
6	3	24
7	0	10
9	1	17
10	2	6
3	0	8

LONG MEASURE.

7.

mls.	fur.	pl.
29	3	14
19	6	29
5	4	20
9	1	37
7	0	3
4	5	9

8.

yds.	ft.	in
127	1	5
12	2	9
0	2	6
54	1	11
5	2	7
23	0	5

CLOTH MEASURE.

9.

yds.	qr.	nls.
26	3	1
13	1	2
6	2	0
217	0	3
9	1	0
55	3	1

10.

el. En.	qrs.	nls.
270	1	0
57	4	3
8	2	1
0	3	2
10	1	0
4	4	1

LAND MEASURE.

11.

ac.	ro.	p.
25	3	37
16	1	25
9	0	13
4	2	9
42	1	19
7	0	6

12.

ac.	ro.	p.
19	0	16
270	3	29
9	1	3
23	0	34
7	2	16
75	0	23

WINE MEASURE.

13.

tr.	hds.	gal.
13	3	15
8	1	35
4	2	26
25	0	12
3	1	9
72	3	21

14.

hds.	gal.	pts.
15	61	5
7	16	3
29	23	7
3	15	1
16	8	0
4	36	6

ALE AND BEER MEASURE.

15.

hds.	gal.	pts.
17	37	3
4	13	5
3	6	2
5	14	0
12	9	6
8	42	4

16.

hds.	gal.	pts.
29	43	5
7	9	2
14	16	6
6	8	1
57	13	4
5	6	0

COMPOUND SUBTRACTION.

Compound Subtraction shows how to find the difference between any two numbers of different denominations.—To perform which, observe the following Rule:

Rule.*—Place the less number below the greater, so that those parts which are of the same denomination may stand directly under each other; and draw a line below them. Begin at the right hand and subtract each number or part in the lower line from the one just above it, and set the remainder straight below it. —But if any number in the lower line be greater than that above it, add so many to the upper number as make 1 of the next higher denomination; then take the lower number from the upper one thus increased, and set down the remainder.—Carry the unit borrowed to the next number in the lower line; after which subtract this number from the one above it, as before; and so proceed till the whole is finished. Then the several remainders, taken together, will be the whole difference sought.

The method of proof is the same as in Simple Subtraction.

EXAMPLES OF MONEY.

	1.			2.			3.			4.		
	£.	s.	d.	£.	s.	d.	£.	s.	d.	£.	s.	d.
From	79	17	$8\frac{3}{4}$	103	3	$2\frac{1}{4}$	57	0	10	251	13	0
Take	35	12	$4\frac{1}{4}$	71	12	$5\frac{1}{2}$	29	13	$3\frac{1}{4}$	35	4	$7\frac{3}{4}$
Rem.	44	5	$4\frac{1}{2}$	31	10	$8\frac{3}{4}$						
Proof.	79	17	$8\frac{3}{4}$	103	3	$2\frac{1}{4}$						

5. What is the difference between 73*l.* $5\frac{1}{4}$*d.* and 19*l.* 13*s.* 10*d*?
Ans. 53*l.* 6*s.* $7\frac{1}{4}$*d.*

6. A lends to B 100*l.*, how much is B in his debt, after A has taken goods of him to the amount of 73*l.* 12*s.* $4\frac{3}{4}$*d*? Ans. 26*l.* 7*s.* $7\frac{1}{4}$*d.*

7. Suppose that my rent for half a year is 10*l.* 12*s.*, and that I have laid out for the land-tax 14*s.* 6*d.*, and for several repairs 1*l.* 3*s.* $3\frac{1}{4}$*d.*, what have I to pay of my half year's rent? Ans. 8*l.* 14*s.* $2\frac{3}{4}$*d.*

8. A trader, failing, owes to A 35*l.* 7*s.* 6*d.*, to B 91*l.* 13*s.* $0\frac{1}{2}$*d.*, to C 53*l.* $7\frac{1}{4}$*d.*, to D 87*l.* 5*s.*, and to E 111*l.* 3*s.* $5\frac{3}{4}$*d.* When this happened, he had by him in cash 23*l.* 7*s.* 5*d.*, in wares 53*l.* 11*s.* $10\frac{1}{4}$*d.*, in household furniture 63*l.* 17*s.* $7\frac{3}{4}$*d.*, and in recoverable book-debts 25*l.* 7*s.* 5*d.* What will his creditors lose by him, suppose these things delivered to them? Ans. 212*l.* 5*s.* $3\frac{1}{2}$*d.*

* The reason of this Rule will easily appear from what has been said in Simple Subtraction, for the borrowing depends upon the same principle, and is only different as the numbers to be subtracted are of different denominations.

EXAMPLES OF WEIGHTS, MEASURES, &C.

TROY WEIGHT.

	1.				2.			
	lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.
From	7	3	14	11	4	9	1	13
Take	3	7	5	19	3	7	16	12
Rem.								
Proof.								

APOTHECARIES WEIGHT.

	3.				
	lb.	oz.	dr.	scr	gr.
From	73	4	7	0	14
Take	26	7	2	1	16
Rem.					
Proof.					

AVOIRDUPOIS WEIGHT.

	4.			5.		
	cwt.	qr.	lb.	lb.	oz.	dr.
From	5	0	17	71	5	9
Take	3	2	11	14	6	14
Rem.						
Proof.						

LONG MEASURE.

	6.			7.		
	m.	fu.	pl.	yd.	ft.	in.
From	14	3	17	96	1	4
Take	3	7	9	41	2	7
Rem.						
Proof.						

CLOTH MEASURE.

	8.			9		
	yd.	qr.	nl.	yd.	qr.	nl
From	17	2	1	9	0	2
Take	5	2	1	6	1	2
Rem.						
Proof.						

LAND MEASURE.

	10.			11.		
	ac.	ro.	p.	ac.	ro.	p.
From	17	1	14	57	1	16
Take	9	3	6	24	2	25
Rem.						
Proof.						

WINE MEASURE.

	12.			13.		
	tr.	hd.	gal.	hd.	gal.	pt.
From	17	2	23	5	0	4
Take	4	3	39	3	2	7
Rem.						
Proof.						

ALE AND BEER MEASURE.

	14.			15.		
	hd.	gal.	pt.	hd.	gal.	pt.
From	14	29	3	71	16	5
Take	7	34	5	17	3	2
Rem.						
Proof.						

DRY MEASURE.

	16.			17.		
	[illegible]	qr.	bu.	bu.	gal.	pt.
From	9	4	7	13	7	1
Take	3	7	2	7	3	4
Rem.						
Proof.						

TIME.

	18.			19.		
	mo.	wk.	da.	ds.	hrs.	min.
From	71	2	5	114	17	26
Take	14	3	0	75	12	35
Rem.						
Proof.						

20. The line of defence in a certain polygon being 236 yards, and that part of it which is terminated by the curtain and shoulder being 146 yards 1 foot 4 inches; what then was the length of the face of the bastion?

Ans. 89 yds. 1 f. 8 in.

COMPOUND MULTIPLICATION.

Compound Multiplication shows how to find the amount of any given number of different denominations repeated a certain proposed number of times.

Rule.—Set the multiplier under the lowest denomination of the multiplicand, and draw a line below it.—Multiply the number in the lowest denomination by the multiplier, and find how many units of the next higher denomination are contained in the product, setting down what remains.—In like manner, multiply the number in the next denomination, and to the product carry or add the units, before found, and find how many units of the next higher denomination are in this amount, which carry in like manner to the next product, setting down the overplus.—Proceed thus to the highest denomination proposed; so shall the last product, with the several remainders, taken as one compound number, be the whole amount required.

The method of Proof, and the reason of the Rule, are the same as in Simple Multiplication.

EXAMPLES OF MONEY.

1. To find the amount of 8lb. of Tea, at *5s.* 8½*d.* per lb.

	s.	*d.*	
	5	8½	
		8	
£. 2	5	8	Answer.

		£.	*s.*	*d.*
2. 4 lb. of Tea, at 7*s.* 8*d.* per. lb.	Ans.	1	10	8
3. 6 lb. of Butter, at 9½*d.* per lb.	Ans.	0	4	9
4. 7 lb. of Tobacco, at 1*s.* 8½*d.* per lb.	Ans.	0	11	11½
5. 9 cwt. of Cheese, at 1*l.* 11*s.* 5*d.* per cwt.	Ans.	14	2	9
6. 10 cwt. of Cheese, at 2*l.* 17*s.* 10*d.* per cwt.	Ans.	28	18	4
7. 12 cwt. of Sugar, at 3*l.* 7*s.* 4*d.* per cwt.	Ans.	40	8	0

CONTRACTIONS.

I. If the multiplier exceed 12, multiply successively by its component parts, instead of the whole number at once.

EXAMPLES.

1. 15 cwt. of Cheese, at 17*s.* 6*d.* per cwt.

£.	s.	d.
0	17	6
		3
2	12	6
13	2	6 Answer.

		£.	s.	d.
2. 20 cwt. of Hops, at 4*l.* 7*s.* 2*d.* per cwt.	Ans.	87	3	4
3. 24 tons of Hay, at 3*l.* 7*s.* 6*d.* per ton.	Ans.	81	0	0
4. 45 ells of Cloth, at 1*s.* 6*d.* per ell.	Ans.	3	7	6
5. 63 gallons of Oil, at 2*s.* 3*d.* per gallon.	Ans.	7	1	9
6 70 barrels of Ale, at 1*l.* 4*s.* per barrel.	Ans.	84	0	0
7. 84 quarters of Oats, at 1*l.* 12*s.* 8*d.* per qr.	Ans.	137	4	0
8. 96 quarters of Barley, at 1*l.* 3*s.* 4*d.* per qr	Ans.	112	0	0
9. 120 days' Wages, at 5*s.* 9*d.* per day.	Ans.	34	10	0
10. 144 reams of Paper, at 13*s.* 4*d.* per ream.	Ans.	96	0	0

II. If the multiplier cannot be exactly produced by the multiplication of simple numbers, take the nearest number to it either greater or less, which can be so produced, and multiply by its parts as before.—Then multiply the given multiplicand by the difference between this assumed number and the multiplier, and add the product to that before found when the assumed number is less than the multiplier, but subtract the same when it is greater.

EXAMPLES.

1. 26 yards of Cloth, at 3*s.* 0¾*d.* per yard.

	£.	s.	d.
	0	3	0¾
			5
	0	15	3¾
			5
	3	16	6¾
		3	0¾
£	3	19	7½ Answer.

		£.	s.	d.
2. 29 quarters of Corn, at 2*l.* 5*s.* 3¼*d.* per quarter.	Ans.	65	12	10¼
3. 53 loads of Hay, at 3*l.* 15*s.* 2*d.* per load.	Ans.	199	3	10
4. 79 bushels of Wheat, at 11*s.* 5¾*d.* per bushel.	Ans.	45	6	10¼
5. 94 casks of Beer, at 12*s.* 2*d.* per cask.	Ans.	57	3	8
6. 114 stone of Meat, at 15*s.* 3¼*d.* per stone.	Ans.	87	5	7½

EXAMPLES OF WEIGHTS AND MEASURES.

1.

lb.	oz.	dwt.	gr.
21	1	7	13
			4

2.

lb.	oz.	dr.	sc.	gr.
2	4	2	1	0
				7

3.

cwt.	qr.	lb.	oz.
27	1	13	12
			12

4.

mls.	fu.	pls.	yds.
24	3	20	2
			6

5.

yds.	qrs.	na.
127	2	2
		8

6.

ac.	ro.	po.
27	2	1
		9

7.

tuns.	hhd.	gal.	pts.
29	1	20	3
			5

8.

wa.	qr.	bu.	pe.
27	1	7	2
			7

9.

mo.	we.	da.	ho.	min.
175	3	6	20	59
				11

COMPOUND DIVISION.

Compound Division teaches how to divide a number of several denominations by any given number, or into any number of equal parts.

Rule.—Place the divisor on the left of the dividend, as in Simple Division.—Begin at the left hand, and divide the number of the highest denomination by the divisor, setting down the quotient in its proper place.—If there be any remainder after this division, reduce it to the next lower denomination, which add to the number, if any, belonging to that denomination, and divide the sum by the divisor.—Set down again this quotient, reduce its remainder to the next lower denomination again, and so on through all the denominations to the last.

EXAMPLES OF MONEY.

1. Divide 225*l.* 2*s.* 4*d.* by 2.

	£.	s.	d.	
2)	225	2	4	
£	112	11	2	the Quotient.

	£.	s.	d.			£	s.	d.
2. Divide	751	14	7½	by 3.	Ans.	250	11	6½
3. Divide	821	17	9¾	by 4.	Ans.	205	9	5¼ ¾
4. Divide	2382	13	5½	by 5.	Ans.	476	10	8¼ ⅕
5. Divide	28	2	1½	by 6.	Ans.	4	13	8¼
6. Divide	55	14	0¾	by 7.	Ans.	7	19	1¾ $\frac{2}{7}$
7. Divide	6	5	4	by 8.	Ans.	0	15	8
8. Divide	135	10	7	by 9.	Ans.	15	1	2 $\frac{4}{9}$
9. Divide	21	18	4	by 10.	Ans.	2	3	10
10. Divide	227	10	5	by 11.	Ans.	20	13	8$\frac{0}{4}$ $\frac{4}{11}$
11. Divide	1332	11	8½	by 12.	Ans.	111	0	11½ $\frac{10}{12}$

CONTRACTIONS.

I. If the divisor exceed 12, find what simple numbers, multiplied together, will produce it, and divide by them separately, as in Simple Division.

EXAMPLES.

1. What is Cheese per cwt. if 16 cwt. cost 30*l.* 18*s.* 8*d.*?

```
      £.   s.  d.
  4 ) 30   18  8
  ---------------
  4 )  7   14  8
  ---------------
     £ 1   18  8 the Answer.
```

		£	s.	d.
2. If 20 cwt. of Tobacco come to 120*l.* 10*s.*, what is that per cwt.?	Ans.	6	0	6
3. Divide 57*l.* 3*s.* 7*d.* by 35.	Ans.	1	12	8$\frac{0}{4}$ $\frac{12}{35}$
4. Divide 85*l.* 6*s.* by 72.	Ans.	1	3	8¼ $\frac{2}{6}$
5. Divide 31*l.* 2*s.* 10¼*d.* by 99.	Ans.	0	6	3½
8. At 18*l.* 18*s.* per cwt., how much per lb.?	Ans.	0	3	4½

II. If the divisor cannot be produced by the multiplication of small numbers, divide by the whole divisor at once, after the manner of Long Division.

EXAMPLES.

1. Divide 74*l.* 13*s.* 6*d.* by 17.

```
      £.  s.  d.   £.  s.  d.
17 ) 74   13  6  ( 4   7   10 Answer.
     68
     --
      6
     20
    ---
    133
    119
    ---
     14
     12
    ---
    174
    170
    ---
      4
    ---
```

	£.	s.	d.			£.	s.	d.
2. Divide	23	15	$7\frac{1}{4}$	by 37.	Ans.	0	12	10
3. Divide	199	3	10	by 53.	Ans.	3	15	2
4. Divide	675	12	6	by 138.	Ans.	4	17	11
5. Divide	315	3	$10\frac{1}{4}$	by 365.	Ans.	0	17	$3\frac{1}{4}$

EXAMPLES OF WEIGHTS AND MEASURES.

1. Divide 23 lb. 7 oz. 6 dwts. 12 gr. by 7. Ans. 3 lb. 4 oz. 9 dwts. 12 gr.
2. Divide 13 lb. 1 oz. 2 dr. 0 scr. 10 gr. by 12. Ans. 1 lb. 1 oz. 0 dr. 2 scr. $10\frac{10}{12}$ gr.
3. Divide 1061 cwt. 2 qrs. by 28. Ans. 37 cwt. 3 qrs. 18 lb.
4. Divide 375 mi. 2 fur. 7 po. 2 yds. 1 ft. 2 in. by 39. Ans. 9 mi. 4 fur. 39 po. 0 yds. 2 ft. $8\frac{26}{39}$ in.
5. Divide 571 yds. 2 qrs. 1 nl. by 47. Ans. 12 yds. 0 qrs. $2\frac{27}{47}$ nls.
6. Divide 51 ac. 1 ro. 11 po. by 51. Ans. 1 ac. 0 ro. 1 pl.
7. Divide 10 tun 2 hhds. 17 gals. 2 pi. by 67. Ans. 39 gals. 6 pi.
8. Divide 120 lasts, 0 qr. 1 bu. 2 pk. by 74. Ans. 1 last, 6 qrs. 1 bu. 3 pk.
9. Divide 120 mo. 2 wk. 3 da. 4 hr. 12 min. by 111. Ans. 1 mo. 0 wk. 2 da. 10 ho. 12 min.

GOLDEN RULE, OR RULE OF THREE.

The rule of three teaches how to find a fourth proportional to three numbers given. Whence it is also sometimes called the Rule of Proportion. It is called the Rule of Three, because three terms or numbers are given, to find a fourth. And because of its great and extensive usefulness, it is often called the Golden Rule.

This Rule is usually considered as of two kinds, namely, Direct, and Inverse.

The Rule of Three Direct, is that in which more requires more, or less requires less. As in this; if 3 men dig 21 yards of trench in a certain time, how much will 6 men dig in the same time? Here more requires more, that is, 6 men, which are more than 3 men, will also perform more work in the same time. Or when it is thus; if 6 men dig 42 yards, how much will 3 men dig in the same time? Here then less requires less, or 3 men will perform proportionally less work than 6 men in the same time. In both these cases, then, the Rule, or the Proportion, is Direct; and the stating must be

thus, As 3 : 21 :: 6 : 42,
or thus, As 6 : 42 :: 3 : 21.

But the Rule of Three Inverse, is when more requires less, or less requires more. As in this; if 3 men dig a certain quantity of trench in 14 hours, in how many hours will 6 men dig the like quantity? Here it is evident that 6 men, being more than 3, will perform an equal quantity of work in less time, or

fewer hours. Or thus; if 6 men perform a certain quantity of work in 7 hours, in how many hours will 3 men perform the same? Here less requires more, *for 3 men will take more hours than 6 to perform the same work. In both these cases, then, the Rule, or the Proportion, is Inverse; and the stating must be*

thus, As 6 : 14 :: 3 : 7,
or thus, As 3 : 7 :: 6 : 14.

And in all these statings, the fourth term is found, by multiplying the 2d and 3d terms together, and dividing the product by the 1st term.

Of the three given numbers, two of them contain the supposition, and the third a demand. And for stating and working questions of these kinds, observe the following general Rule:

Rule.—State the question, by setting down in a straight line the three given numbers, in the following manner, viz, so that the 2d term be that number of supposition which is of the same kind that the answer or 4th term is to be; making the other number of supposition the 1st term, and the demanding number the 3d term, when the question is in direct proportion; but contrariwise, the other number of supposition the third term, and the demanding number the 1st term, when the question has inverse proportion.

Then, in both cases, multiply the 2d and 3d terms together, and divide the product by the 1st, which will give the answer, or 4th term sought, of the same denomination as the second term.

Note, If the first and third terms consist of different denominations, reduce them both to the same: and if the second term be a compound number, it is mostly convenient to reduce it to the lowest denomination mentioned.—If, after division, there be any remainder, reduce it to the next lower denomination, and divide by the same divisor as before, and the quotient will be of this last denomination. Proceed in the same manner with all the remainders, till they be reduced to the lowest denomination which the second term admits of, and the several quotients taken together will be the answer required.

Note also, The reason for the foregoing Rules will appear when we come to treat of the nature of Proportions.—Sometimes also two or more statings are necessary, which may always be known from the nature of the question.

EXAMPLES.

1. If 8 yards of cloth cost 1*l.* 4*s.*, what will 96 yards cost?

```
   yds.  £.  s.    yds.   £.  s.
As  8  :  1   4  ::  96  :  14   8 the answer.
             20
            ---
             24
             96
            ---
            144
           216
           ----
      8 ) 2304
          -----
    2,0 )  28,8s.
          -----
     £   14 8 Answer.
```

2. An engineer having raised 100 yards of a certain work in 24 days with 5 men; how many men must he employ to finish a like quantity of work in 15 days?

	da.		men		da.		men	
As	15	:	5	: :	24	:	8	Ans.
					5			
				15)	120	(8	Answer	
					120			

3. What will 72 yards of cloth cost, at the rate of 9 yards for *5l.* 12*s.* Ans. 44*l.* 16*s.*

4. A person's annual income being 146*l.*; how much is that per day? Ans. 8*s.*

5. If 3 paces or common steps of a certain person be equal to 2 yards; how many yards will 160 of his paces make? Ans. 106 yds. 2 f

6. What length must be cut off a board, that is 9 inches broad, to make a square foot, or as much as 12 inches in length and 12 in breadth contains? Ans. 16 inches.

7. If 750 men require 22,500 rations of bread for a month; how many rations will a garrison of 1200 men require? Ans. 36,000.

8. If 7 cwt. 1 qr. of sugar cost 26*l.* 10*s.* 4*d.*, what will be the price of 43 cwt. 2 qrs.? Ans. 159*l.* 2*s.*

9. The clothing of a regiment of foot of 750 men amounting to 2831*l.* 5*s.*; what will the clothing of a body of 3500 amount to? Ans. 13,212*l.* 10*s.*

10. How many yards of matting, that is 2 ft. 6 in. broad, will cover a floor that is 27 feet long and 20 feet broad? Ans. 72 yds.

11. What is the value of 6 bushels of coals, at the rate of 1*l.* 14*s.* 6*d.* the chaldron? Ans. 5*s.* 9*d.*

12. If 6352 stones of 3 feet long complete a certain quantity of walling; how many stones of 2 feet long will raise a like quantity? Ans. 9528.

13. What must be given for a piece of silver weighing 73 lb. 5 oz. 15 dwts. at the rate of 5*s.* 9*d.* per ounce? Ans. 253*l.* 10*s.* $0\frac{3}{4}$*d.*

14. A garrison of 536 men has provisions for 12 months, how long will those provisions last if the garrison be increased to 1124 men? Ans. 174 days and $\frac{64}{1124}$.

15. What will the tax upon 763*l.* 15*s.* be, at the rate of 3*s.* 6*d.* per pound sterling? Ans. 133*l.* 13*s.* $1\frac{1}{2}$*d.*

16. A certain work being raised in 12 days, by working 4 hours each day; how long would it have been in raising by working 6 hours per day? Ans. 8 days.

17. What quantity of corn can I buy for 90 guineas, at the rate of 6*s.* the bushel? Ans. 39 qrs. 3 bu.

18. A person failing in trade, owes in all 977*l.*; at which time he has in money, goods, and recoverable debts, 420*l.* 6*s.* $3\frac{1}{4}$*d*; now supposing these things delivered to his creditors, how much will they get per pound? Ans. 8*s.* $7\frac{1}{4}$*d.*

19. A plain of a certain extent having supplied a body of 3000 horse with forage for 18 days; then how many days would the same plain have supplied a body of 2000 horse? Ans. 27 days.

20. Suppose a gentleman's income is 500 guineas a year, and that he spends

19*s.* 7*d.* per day, one day with another; how much will he have saved at the year's end? Ans. 167*l.* 12*s.* 1*d.*

21. What cost 30 pieces of lead, each weighing 1 cwt. 12 lb., at the rate of 16*s.* 4*d.* per cwt.? Ans. 27*l.* 2*s.* 6*d.*

22. The governor of a besieged place having provisions for 54 days, at the rate of $1\frac{1}{2}$lb. of bread; but being desirous to prolong the siege to 80 days, in expectation of succour, in that case what must the ration of bread be?

Ans. $1\frac{1}{80}$ lb.

23. At half a guinea per week, how long can I be boarded for 20 pounds?

Ans. $38\frac{12}{126}$ weeks.

24. How much will 75 chaldrons 7 bushels of coals come to, at the rate of 1*l.* 13*s.* 6*d.* per chaldron? Ans. 125*l.* 19*s.* $0\frac{1}{6}$*d.*

25. If the penny loaf weigh 9 ounces when the bushel of wheat cost 6*s.* 3*d.*; what ought the penny loaf to weigh when the wheat is at 8*s.* $2\frac{1}{2}$*d.*?

Ans. 6 oz. $13\frac{127}{197}$ dr.

26. How much a year will 173 acres, 2 roods, 14 poles of land give, at the rate of 1*l.* 7*s.* 8*d.* per acre? Ans. 240*l.* 2*s.* $7\frac{1}{20}$*d.*

27. To how much amounts 172 pieces of lead, each weighing 3 cwt. 2 qrs. $17\frac{1}{2}$ lb., at 8*l.* 17*s.* 6*d.* per fother of $19\frac{1}{2}$ cwt.? Ans. 286*l.* 4*s.* $4\frac{1}{2}$*d.*

28. How many yards of stuff, of 3 qrs. wide, will line a cloak that is $5\frac{1}{2}$ yards in length and $1\frac{1}{4}$ yard wide? Ans. 9 yds. 0 qrs. $2\frac{2}{3}$ nl.

29. If 5 yards of cloth cost 14*s.* 2*d.*, what must be given for 9 pieces, containing each 21 yards 1 quarter? Ans. 27*l.* 1*s.* $10\frac{1}{2}$*d.*

30. If a gentleman's estate be worth 2107*l.* 12*s.* a year; what may he spend per day, to save 500*l.* in the year? Ans. 4*l.* 8*s.* $1\frac{19}{365}$*d.*

31. Wanting just an acre of land cut off from a piece which is $13\frac{1}{2}$ poles in breadth, what length must the piece be? Ans. 11 po. 4 yds. 2 ft. $0\frac{18}{27}$ in.

32. At 13*s.* $2\frac{1}{2}$*d.* per yard, what is the value of a piece of cloth containing $52\frac{3}{4}$ ells English? Ans 43*l.* 10*s.* $11\frac{3}{32}$*d.*

33. If the carriage of 5 cwt. 14 lb. for 96 miles be 1*l.* 12*s.* 6*d.*; how far may I have 3 cwt. 1 qr. carried for the same money? Ans. 151 m. 3 fur. $3\frac{1}{13}$. pol.

34. Bought a silver tankard, weighing 1 lb. 7 oz. 14 dwts.; what did it cost me at 6*s.* 4*d.* per ounce? Ans. 6*l.* 4*s.* $9\frac{1}{3}$*d.*

35. What is the half year's rent of 547 acres of land, at 15*s.* 6*d.* the acre?

Ans. 211*l.* 19*s.* 3*d.*

36. A wall that is to be built to the height of 27 feet, was raised 9 feet high by 12 men in 6 days; then how many men must be employed to finish the wall in 4 days, at the same rate of working? Ans. 36 men.

37. What will be the charge of keeping 11 horses for a year, at the rate of $11\frac{1}{2}$*d.* per day for each horse? Ans. 192*l.* 7*s.* $8\frac{1}{2}$*d.*

38. If 15 ells of stuff that is $\frac{3}{4}$ yard wide cost 37*s.* 6*d.*; what will 40 ells of the same goodness cost, being yard wide? Ans. 6*l.* 13*s.* 4*d.*

39. How many yards of paper that is 30 inches wide, will hang a room that is 20 yards in circuit, and 9 feet high? Ans. 72 yds.

40. If a gentleman's estate be worth 384*l.* 16*s.* a year, and the land-tax be assessed at 2*s.* $9\frac{1}{2}$*d.* per pound, what is his nett annual income?

Ans. 331*l.* 1*s.* $9\frac{1}{5}$*d.*

41. The circumference of the earth is about 25,000 miles; at what rate per hour is a person at the middle of its surface carried round, one whole rotation being made in 23 hours 56 minutes? Ans. $1044\frac{816}{1436}$ miles.

42. If a person drink 20 bottles of wine per month, when it costs 8*s.* a gallon; how many bottles per month may he drink, without increasing the expense, when wine costs 10*s.* the gallon? Ans. 16 bottles.

43. What cost 43 qrs. 5 bushels of corn, at 1*l.* 8*s.* 6*d.* the quarter?
Ans. 62*l.* 3*s.* $3\frac{3}{4}$*d.*

44. How many yards of canvass that is ell wide, will line 20 yards of say that is 3 quarters wide? Ans. 12 yds.

45. If an ounce of gold cost 4 guineas, what is the value of a grain?
Ans. $2\frac{1}{10}$*d.*

46. If 3 cwt. of tea cost 40*l.* 12*s.*; at how much a pound must it be retailed, to gain 10*l.* by the whole? Ans. $3\frac{4}{3\tau}$*s.*

COMPOUND PROPORTION.

Compound Proportion teaches how to resolve such questions as require two or more statings by Simple Proportion: and that, whether they be Direct or Inverse.

In these questions, there is always given an odd number of terms, either five, or seven, or nine, &c. These are distinguished into terms of supposition, and terms of demand, there being always one term more of the former than of the latter, which is of the same kind with the answer sought.

Rule.—Set down in the middle place that term of supposition which is of the same kind with the answer sought.—Take one of the other terms of supposition, and one of the demanding terms which is of the same kind with it; then place one of them for a first term, and the other for a third, according to the directions given in the Rule of Three.—Do the same with another term of supposition, and its corresponding demanding term; and so on if there be more terms of each kind; setting the numbers under each other which fall all on the left hand side of the middle term, and the same for the others on the right hand side.—Then, to work.

By several Operations.—Take the two upper terms and the middle term, in the same order as they stand, for the first Rule of Three question to be worked, whence will be found a fourth term. Then take this fourth number, so found, for the middle term of a second Rule of Three question, and the next two under terms in the general stating, in the same order as they stand, finding a fourth term from them. And so on, as far as there are any numbers in the general stating, making always the fourth number resulting from each simple stating to be the second term of the next following one. So shall the last resulting number be the answer to the question.

By One Operation.—Multiply together all the terms standing under each other, on the left hand side of the middle term; and in like manner, multiply together all those on the right hand side of it. Then multiply the middle term by the latter product, and divide the result by the former product, so shall the quotient be the answer sought.

EXAMPLES.

1. How many men can complete a trench of 135 yards long in 8 days when 16 men can dig 54 yards in 6 days?

General Stating.

yds.	54	: 16 men ::	135	yds.
days	8		6	days
	432		810	
			16	
			4860	
			810	

432) 12960 (30 Ans. by one operation.
      1296
      0

The same by two operations.

1st.

As 54 : 16 :: 135 : 40
              16
              810
              135
54) 2160 (40
     216
     0

2d.

As 8 : 40 :: 6 : 30
              6
8) 240 (30 Ans.
    24
    0

2. If 100*l.* in one year gain 5*l.* interest, what will be the interest of 750*l.* for 7 years? Ans. 262*l.* 10*s.*

3. If a family of 9 persons expend 120*l.* in 8 months; how much will serve a family of 24 people 16 months? Ans. 640*l.*

4. If 27*s.* be the wages of 4 men for 7 days; what will be the wages of 14 men for 10 days? Ans. 6*l.* 15*s.*

5. If a footman travel 130 miles in 3 days, when the days are 12 hours long; in how many days, of 10 hours each, may he travel 360 miles? Ans. $9\frac{63}{65}$ days

6. If 120 bushels of corn can serve 14 horses 56 days; how many days will 94 bushels serve 6 horses? Ans $102\frac{16}{45}$ days.

7. If 3000 lb. of beef serve 340 men 15 days; how many lbs. will serve 120 men for 25 days? Ans. 1764 lb. $11\frac{5}{17}$ oz.

8. If a barrel of beer be sufficient to last a family of 7 persons 12 days; how many barrels will be drunk by 14 persons in the space of a year? Ans. $60\frac{5}{6}$ barrels.

9. If 240 men, in 5 days, of 11 hours each, can dig a trench 230 yards long, 3 wide, and 2 deep; in how many days, of 9 hours long, will 24 men dig a trench of 420 yards long, 5 wide, and 3 deep? Ans. $278\frac{68}{69}$ days.

OF VULGAR FRACTIONS.

A Fraction, or broken number, is an expression of a part, or some parts, of something considered as a whole.

It is denoted by two numbers, placed one below the other, with a line between them;

$$\text{thus, } \frac{3}{4}\begin{matrix}\text{numerator}\\ \text{denominator}\end{matrix}\Bigg\}\text{ which is named three-fourths.}$$

The Denominator, or number placed below the line, shows how many equal parts the whole quantity is divided into; and represents the Divisor in Division. And the Numerator, or number set above the line, shows how many of those parts are expressed by the Fraction; being the remainder after division.—Also, both these numbers are, in general, named the Terms of the Fraction.

Fractions are either Proper, Improper, Simple, Compound, or Mixed.

A Proper Fraction, is when the numerator is less than the denominator; as $\frac{1}{2}$, or $\frac{2}{3}$, or $\frac{3}{4}$, &c.

An Improper Fraction, is when the numerator is equal to, or exceeds, the denominator; as $\frac{3}{3}$, or $\frac{5}{4}$, or $\frac{7}{5}$; &c.

A Simple Fraction, is a single expression denoting any number of parts of the integer; as $\frac{2}{3}$, or $\frac{3}{5}$.

A Compound Fraction, is the fraction of a fraction, or several fractions connected with the word *of* between them; as $\frac{1}{2}$ of $\frac{2}{3}$, or $\frac{3}{5}$ of $\frac{5}{6}$ of 3, &c.

A Mixed Number, is composed of a whole number and a fraction together; as $3\frac{1}{4}$, or $12\frac{4}{5}$, &c.

A whole or integer number may be expressed like a fraction, by writing 1 below it, as a denominator; so 3 is $\frac{3}{1}$, or 4 is $\frac{4}{1}$, &c.

A fraction denotes division; and its value is equal to the quotient obtained by dividing the numerator by the denominator; so $\frac{12}{4}$ is equal to 3, and $\frac{20}{5}$ is equal to 4.

Hence then, if the numerator be less than the denominator, the value of the fraction is less than 1. If the numerator be the same as the denominator, the fraction is just equal to 1. And if the numerator be greater than the denominator, the fraction is greater than 1.

REDUCTION OF VULGAR FRACTIONS.

Reduction of Vulgar Fractions, is the bringing them out of one form or denomination into another; commonly to prepare them for the operations of Addition, Subtraction, &c. of which there are several cases.

To find the greatest common measure of two or more numbers.

The Common Measure of two or more numbers, is that number which will divide them both without a remainder: so 3 is a common measure of 18 and 24; the quotient of the former being 6, and of the latter 8. And the greatest number

that will do this, is the greatest common measure: so 6 is the greatest common measure of 18 and 24; the quotient of the former being 3, and of the latter 4, which will not both divide farther.

RULE.—If there be two numbers only; divide the greater by the less; then divide the divisor by the remainder; and so on, dividing always the last divisor by the last remainder, till nothing remains; then shall the last divisor of all be the greatest common measure sought.

When there are more than two numbers; find the greatest common measure of two of them, as before; then do the same for that common measure and another of the numbers; and so on, through all the numbers; then will the greatest common measure last found be the answer.

If it happen that the common measure thus found is 1; then the numbers are said to be incommensurable, or to have no common measure.

EXAMPLES.

1. To find the greatest common measure of 1998, 918, and 522.

```
918 ) 1998 ( 2
      1836
      ----
       162 ) 918 ( 5
             810
             ---
             108 ) 162 ( 1
                   108
                   ---
                    54 ) 108 ( 2
                         108
                         ---
```

So 54 is the greatest common measure of 1998 and 918.

```
Hence 54 ) 522 ( 9
           486
           ---
            36 ) 54 ( 1
                 36
                 --
                 18 ) 36 ( 2
                      36
                      --
```

so that 18 is the answer required.

2. What is the greatest common measure of 246 and 372? Ans. 6.

3. What is the greatest common measure of 336, 720, and 1736? Ans. 8.

CASE I.

To abbreviate or reduce fractions to their lowest terms.

RULE.*—Divide the terms of the given fraction by any number that will divide them without a remainder; then divide these quotients again in the same man-

* That dividing both the terms of the fraction by the same number, whatever it be, will give another fraction equal to the former, is evident. And when those divisions are performed as often as can be done, or when the common divisor is the greatest possible, the terms of the resulting fraction must be the least possible.

Note 1. Any number ending with an even number, or a cipher, is divisible, or can be divided by 2.

2. Any number ending with 5, or 0, is divisible by 5

3. If the right hand place of any number be 0, the whole is divisible by 10; if there be two ciphers, it is divisible by 100; if 3 ciphers, by 1000; and so on; which is only cutting off those ciphers.

4. If the two right hand figures of any number be divisible by 4, the whole is divisible by 4. And if the three right hand figures be divisible by 8, the whole is divisible by 8. And so on.

5. If the sum of the digits in any number be divisible by 3, or by 9, the whole is divisible by 3, or by 9.

6. If the right hand digit be even, and the sum of all the digits be divisible by 6, then the whole will be divisible by 6.

7 A number is divisible by 11, when the sum of the 1st, 3d, 5th, &c., or of all the odd places, is equal to the sum of the 2d, 4th, 6th, &c., or of all the even places of digits.

8. If a number cannot be divided by some quantity less than the square of the same, that number is a prime, or cannot be divided by any number whatever.

ner; and so on; till it appears that there is no number greater than 1 which will divide them; then the fraction will be in its lowest terms.

Or, Divide both the terms of the fraction by their greatest common measure, and the quotients will be the terms of the fraction required, of the same value as at first.

EXAMPLES.

1. Reduce $\frac{144}{240}$ to its least terms.

$$\frac{144}{240} = \frac{72}{120} = \frac{36}{60} = \frac{18}{30} = \frac{9}{15} = \frac{3}{5}, \text{ the answer.}$$

Or thus:

```
144 ) 240 ( 1
      144
      ---
       96 ) 144 ( 1
             96
             --
             48 ) 96 ( 2
                  96
                  --
```

Therefore 48 is the greatest common measure, and 48) $\frac{144}{240} = \frac{3}{5}$ the answer, the same as before.

2. Reduce $\frac{192}{576}$ to its lowest terms. Ans. $\frac{1}{3}$.
3. Reduce $\frac{252}{364}$ to its lowest terms. Ans. $\frac{9}{13}$.
4. Reduce $\frac{1344}{1536}$ to its lowest terms. Ans. $\frac{7}{8}$.

CASE II.

To reduce a mixed number to its equivalent improper fraction.

RULE.*—Multiply the whole number by the denominator of the fraction, and add the numerator to the product; then set that sum above the denominator for the fraction required.

EXAMPLES.

1. Reduce $23\frac{2}{5}$ to a fraction.

```
 23
  5
---
115
  2
---
117
---
  5
```

Or,

$$\frac{(23 \times 5) + 2}{5} = \frac{117}{5}, \text{ the answer.}$$

9. All prime numbers, except 2 and 5, have either 1, 3, 7, or 9, in the place of units; and all other numbers are composite, or can be divided.

10. When numbers, with the sign of addition or subtraction between them, are to be divided by any number, then each of those numbers must be divided by it. Thus, $\frac{10 + 8 - 4}{2} = 5 + 4 - 2 = 7$

11. But if the numbers have the sign of multiplication between them, only one of them must be divided. Thus, $\frac{10 \times 8 \times 3}{6 \times 2} = \frac{10 \times 4 \times 3}{6 \times 1} = \frac{10 \times 4 \times 1}{2 \times 1} = \frac{10 \times 2 \times 1}{1 \times 1} = \frac{20}{1} = 20.$

* This is no more than first multiplying a quantity by some number, and then dividing the result back again by the same, which it is evident does not alter the value: for any fraction represents a division of the numerator by the denominator

2. Reduce $12\frac{7}{9}$ to a fraction. Ans. $\frac{115}{9}$
3. Reduce $14\frac{7}{10}$ to a fraction. Ans. $\frac{147}{10}$
4. Reduce $183\frac{5}{21}$ to a fraction. Ans. $\frac{3848}{21}$

CASE III.

To reduce an improper fraction to its equivalent whole or mixed number.

RULE.*—Divide the numerato by the denominator, and the quotient will be the whole or mixed number sought.

EXAMPLES.

1. Reduce $\frac{12}{3}$ to its equivalent number.
   Here $\frac{12}{3}$ or $12 \div 3 = 4$, the answer.
2. Reduce $\frac{15}{7}$ to its equivalent number.
   Here $\frac{15}{7}$ or $15 \div 7 = 2\frac{1}{7}$, the answer.
3. Reduce $\frac{749}{17}$ to its equivalent number.

```
Thus, 17 ) 749 ( 44 1/17
           68
           --
            69
            68
            --
             1
            --
```

So that $\frac{749}{17} = 44\frac{1}{17}$, the answer.

4. Reduce $\frac{56}{7}$ to its equivalent number. Ans. 8.
5. Reduce $\frac{1902}{35}$ to its equivalent number. Ans. $54\frac{12}{35}$.
6. Reduce $\frac{2918}{17}$ to its equivalent number. Ans. $171\frac{11}{17}$.

CASE IV.

To reduce a whole number to an equivalent fraction, having a giver denominator.

RULE.†—Multiply the whole number by the given denominator, then set the product over the said denominator, and it will form the fraction required.

EXAMPLES.

1. Reduce 9 to a fraction whose denominator shall be 7.
   Here $9 \times 7 = 63$, then $\frac{63}{7}$ is the answer.
   For $\frac{63}{7} = 63 \div 7 = 9$, the proof.
2. Reduce 13 to a fraction whose denominator shall be 12. Ans. $\frac{156}{12}$
3. Reduce 27 to a fraction whose denominator shall be 11. Ans. $\frac{297}{11}$.

* This rule is evidently the reverse of the former; and the reason of it is manifest from the nature of Common Division.

† Multiplication and Division being here equally used, the result must be the same as the quantity first proposed.

CASE V.

To reduce a compound fraction to an equivalent simple one.

RULE.*—Multiply all the numerators together for a numerator, and all the denominators together for the denominator, and they will form the simple fraction sought.

When part of the compound fraction is a whole or mixed number, it must first be reduced to a fraction by one of the former cases.

And, when it can be done, any two terms of the fraction may be divided by the same number, and the quotients used instead of them. Or when there are terms that are common, they may be omitted.

EXAMPLES.

1. Reduce $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ to a simple fraction.

Here $\frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{6}{24} = \frac{1}{4}$, the answer.

Or, $\frac{1 \times \cancel{2} \times \cancel{3}}{\cancel{2} \times \cancel{3} \times 4} = \frac{1}{4}$, by omitting the twos and threes.

2. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{10}{11}$ to a simple fraction.

Here $\frac{2 \times 3 \times 10}{3 \times 5 \times 11} = \frac{60}{165} = \frac{12}{33} = \frac{4}{11}$, the answer

Or, $\frac{2 \times 3 \times 10}{3 \times 5 \times 11} = \frac{4}{11}$, the same as before.

3. Reduce $\frac{3}{7}$ of $\frac{4}{5}$ to a simple fraction. Ans. $\frac{12}{35}$.
4. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{8}$ to a simple fraction. Ans. $\frac{1}{4}$.
5. Reduce $\frac{2}{5}$ of $\frac{5}{8}$ of $3\frac{1}{2}$ to a simple fraction. Ans. $\frac{7}{8}$.
6. Reduce $\frac{2}{7}$ of $\frac{5}{8}$ of $\frac{7}{2}$ of 4 to a simple fraction. Ans. $\frac{5}{2}$.

CASE VI.

To reduce fractions of different denominators to equivalent fractions, having a common denominator.

RULE.†—Multiply each numerator into all the denominators except its own for the new numerators; and multiply all the denominators together for a common denominator.

Note. It is evident, that in this and several other operations, when any of the proposed quantities are integers, or mixed numbers, or compound fractions, they must be reduced, by their proper rules, to the form of simple fractions.

* The truth of this rule may be shown as follows: Let the compound fraction be $\frac{2}{3}$ of $\frac{5}{7}$. Now $\frac{1}{3}$ of $\frac{5}{7}$ is $\frac{5}{7} \div 3$, which is $\frac{5}{21}$; consequently $\frac{2}{3}$ of $\frac{5}{7}$ will be $\frac{5}{21} \times 2$ or $\frac{10}{21}$; that is the numerators are multiplied together, and also the denominators, as in the rule.—When the compound fraction consists of more than two single ones; having first reduced two of them as above, then the resulting fraction and a third will be the same as a compound fraction of two parts; and so on to the last of all.

† This is evidently no more than multiplying each numerator and its denominator by the same quantity, and consequently the value of the fraction is not altered.

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ to a common denominator.

$1 \times 3 \times 4 = 12$ the new numerator for $\frac{1}{2}$.
$2 \times 2 \times 4 = 16$ .... ditto for $\frac{2}{3}$.
$3 \times 2 \times 3 = 18$ .... ditto for $\frac{3}{4}$.
$2 \times 3 \times 4 = 24$ the common denominator.

Therefore the equivalent fractions are $\frac{12}{24}$. $\frac{16}{24}$, and $\frac{18}{24}$.

Or the whole operation of multiplying may be very well performed mentally, and only set down the results and given fractions thus: $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $= \frac{12}{24}$, $\frac{16}{24}$, $\frac{18}{24} = \frac{6}{12}$, $\frac{8}{12}$, $\frac{9}{12}$, by abbreviation.

2. Reduce $\frac{2}{7}$ and $\frac{5}{9}$ to fractions of a common denominator. Ans. $\frac{18}{63}$, $\frac{35}{63}$.

3. Reduce $\frac{2}{3}$, $\frac{3}{5}$, $\frac{3}{4}$, to a common denominator. Ans. $\frac{40}{60}$, $\frac{36}{60}$, $\frac{45}{60}$.

4. Reduce $\frac{5}{6}$, $2\frac{3}{5}$, and 4, to a common denominator. Ans. $\frac{25}{30}$, $\frac{78}{30}$, $\frac{120}{30}$.

Note 1. When the denominators of two given fractions have a common measure, let them be divided by it; then multiply the terms of each given fraction by the quotient arising from the other's denominator.

2. When the less denominator of two fractions exactly divides the greater, multiply the terms of that which hath the less denominator by the quotient.

3. When more than two fractions are proposed; it is sometimes convenient, first to reduce two of them to a common denominator; then these and a third; and so on till they be all reduced to their least common denominator.

CASE VII.

To find the value of a fraction in parts of the integer.

Rule.—Multiply the integer by the numerator, and divide the product by the denominator, by Compound Multiplication and Division, if the integer be a compound quantity.

Or, if it be a single integer, multiply the numerator by the parts in the next inferior denomination, and divide the product by the denominator. Then, if any thing remains, multiply it by the parts in the next inferior denomination, and divide by the denominator as before; and so on as far as necessary; so shall the quotients, placed in order, be the value of the fraction required.*

EXAMPLES.

1. What is the $\frac{4}{5}$ of 2*l.* 6*s.*?
By the former part of the rule,

	2*l.*	6*s.*
		4
5)	9	4
Ans.	1*l.* 16*s.* 9*d.* $2\frac{2}{5}$*q.*	

2. What is the value of $\frac{2}{3}$ of 1*l.*?
By the 2d part of the rule,

2
20
3) 40 (13*s.* 4*d.* Ans.
1
12
3) 12 (4*d.*

* The numerator of a fraction being considered as a remainder, in Division, and the denominator as the divisor, this rule is of the same nature as Compound Division, or the Valuation of remainders in the Rule of Three, before explained.

3. What is the value of $\frac{3}{8}$ of a pound sterling? Ans. 7*s.* 6*d.*
4. What is the value of $\frac{2}{9}$ of a guinea? Ans. 4*s.* 8*d.*
5. What is the value of $\frac{3}{4}$ of half-a-crown? Ans. 1*s.* 10½*d.*
6. What is the value of $\frac{2}{5}$ of 4*s.* 10*d.*? Ans. 1*s.* $11\frac{1}{5}$*d.*
7. What is the value of $\frac{3}{5}$ of a pound troy? Ans. 7 oz. 4 dwts.
8. What is the value of $\frac{5}{16}$ of a cwt.? Ans. 1 qr. 7 lb.
9. What is the value of $\frac{5}{8}$ of an acre? Ans. 2 ro. 20 po.
10. What is the value of $\frac{3}{10}$ of a day? Ans. 7 hrs. 12 min.

CASE VIII.

To reduce a fraction from one denomination to another.

RULE.*—Consider how many of the less denomination make one of the greater; then multiply the numerator by that number, if the reduction be to a less name, or the denominator, if to a greater.

EXAMPLES.

1. Reduce $\frac{2}{9}$ of a pound to the fraction of a penny

$\frac{2}{9} \times \frac{20}{1} \times \frac{12}{1} = \frac{480}{9} = \frac{160}{3}$, the answer.

2. Reduce $\frac{5}{6}$ of a penny to the fraction of a pound.

$\frac{5}{6} \times \frac{1}{12} \times \frac{1}{20} = \frac{5}{1440} = \frac{1}{288}$, the answer.

3. Reduce $\frac{2}{15}$*l.* to the fraction of a penny. Ans. $\frac{32}{1}$*d.*
4. Reduce $\frac{2}{3}$ of a farthing to the fraction of a pound. Ans. $\frac{1}{1440}$.
5. Reduce $\frac{2}{7}$ cwt. to the fraction of a lb. Ans. $\frac{32}{1}$.
6. Reduce $\frac{4}{5}$ dwt. to the fraction of a lb. troy. Ans. $\frac{1}{300}$.
7. Reduce $\frac{5}{8}$ of a crown to the fraction of a guinea. Ans. $\frac{25}{168}$.
8. Reduce $\frac{5}{8}$ of a half-crown to the fraction of a shilling. Ans. $\frac{25}{16}$.

ADDITION OF VULGAR FRACTIONS.

To add fractions together that have a common denominator.

RULE.—Add all the numerators together, and place the sum over the common denominator, and that will be the sum of the fractions required.

If the fractions proposed have not a common denominator, they must be reduced to one. Also compound fractions must be reduced to simple ones; and mixed numbers to improper fractions; also fractions of different denominations to those of the same denomination.†

* This is the same as the rule of Reduction in whole numbers, from one denomination to another.

† Before fractions are reduced to a common denominator, they are quite dissimilar, as much as shillings and pence are, and therefore cannot be incorporated with one another, any more than these can. But when they are reduced to a common denominator, and made parts of the same thing, their sum, or difference, may then be as properly expressed by the sum or difference of the numerators, as the sum or difference of any two quantities whatever, by the sum or difference of their individuals. Whence the reason of the rule is manifest both for Addition and Subtraction.

When several fractions are to be collected, it is commonly best first to add two of them together that most easily reduce to a common denominator; then add their sum and a third, and so on.

EXAMPLES.

1. To add $\frac{3}{5}$ and $\frac{4}{5}$ together.
   Here $\frac{3}{5} + \frac{4}{5} = \frac{7}{5} = 1\frac{2}{5}$, the answer.
2. To add $\frac{3}{5}$ and $\frac{5}{6}$ together.
   $\frac{3}{5} + \frac{5}{6} = \frac{18}{30} + \frac{25}{30} = \frac{43}{30} = 1\frac{13}{30}$ the answer.
3. To add $\frac{5}{8}$ and $7\frac{1}{2}$ and $\frac{1}{3}$ of $\frac{3}{4}$ together.
   $\frac{5}{8} + 7\frac{1}{2} + \frac{1}{3}$ of $\frac{3}{4} = \frac{5}{8} + \frac{15}{2} + \frac{1}{4} = \frac{5}{8} + \frac{60}{8} + \frac{2}{8} = \frac{67}{8} = 8\frac{3}{8}$, the answer.
4. To add $\frac{3}{7}$ and $\frac{6}{7}$ together. Ans. $1\frac{2}{7}$.
5. To add $\frac{3}{4}$ and $\frac{5}{9}$ together. Ans. $1\frac{11}{36}$.
6. Add $\frac{2}{7}$ and $\frac{5}{14}$ together Ans. $\frac{9}{14}$.
7. What is the sum of $\frac{2}{3}$ and $\frac{3}{5}$ and $\frac{5}{7}$? Ans. $1\frac{103}{105}$.
8. What is the sum of $\frac{5}{9}$ and $\frac{3}{5}$ and $2\frac{1}{6}$? Ans. $3\frac{29}{90}$.
9. What is the sum of $\frac{3}{5}$ and $\frac{4}{5}$ of $\frac{1}{3}$, and $9\frac{3}{20}$? Ans. $10\frac{1}{60}$.
10. What is the sum of $\frac{2}{3}$ of a pound and $\frac{5}{9}$ of a shilling?
    Ans. $\frac{125}{9}s.$ or $13s.$ $10d.$ $2\frac{2}{3}q.$
11. What is the sum of $\frac{3}{5}$ of a shilling and $\frac{4}{15}$ of a penny? Ans. $\frac{112}{15}d.$ or $7d.$ $1\frac{13}{15}q.$
12. What is the sum of $\frac{1}{7}$ of a pound, and $\frac{2}{9}$ of a shilling, and $\frac{5}{12}$ of a penny?
    Ans. $\frac{3139}{1008}s.$ or $3s.$ $1d.$ $1\frac{10}{21}q.$

SUBTRACTION OF VULGAR FRACTIONS.

Rule.—Prepare the fractions the same as for Addition; then subtract the one numerator from the other, and set the remainder over the common denominator, for the difference of the fractions sought.

EXAMPLES.

1. To find the difference between $\frac{5}{6}$ and $\frac{1}{6}$.
   Here $\frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$, the answer.
2. To find the difference between $\frac{3}{4}$ and $\frac{5}{7}$.
   $\frac{3}{4} - \frac{5}{7} = \frac{21}{28} - \frac{20}{28} = \frac{1}{28}$, the answer.
3. What is the difference between $\frac{5}{12}$ and $\frac{7}{12}$? Ans. $\frac{1}{6}$.
4. What is the difference between $\frac{3}{13}$ and $\frac{4}{39}$? Ans. $\frac{5}{39}$.
5. What is the difference between $\frac{5}{12}$ and $\frac{7}{15}$? Ans. $\frac{19}{150}$.
6 What is the difference between $5\frac{3}{8}$ and $\frac{2}{7}$ of $4\frac{1}{6}$? Ans. $4\frac{31}{168}$.
7. What is the difference between $\frac{5}{9}$ of a pound, and $\frac{2}{3}$ of $\frac{3}{4}$ of a shilling?
   Ans. $\frac{191}{18}s.$ or $10s.$ $7d.$ $1\frac{1}{3}q.$
8. What is the difference between $\frac{2}{7}$ of $5\frac{1}{6}$ of a pound, and $\frac{3}{5}$ of a shilling?
   Ans. $\frac{3037}{2100}l.$ or $1l.$ $8s.$ $11\frac{8}{35}q.$

MULTIPLICATION OF VULGAR FRACTIONS.

Rule.*—Reduce mixed numbers, if there be any, to equivalent fractions;

* Multiplication of any thing by a fraction implies the taking some part or parts of the thing; it may therefore be truly expressed by a compound fraction; which is resolved by multiplying together the numerators and the denominators.

then multiply all the numerators together for a numerator, and all the denominators together for a denominator, which will give the product required.

EXAMPLES.

1. Required the product of $\frac{3}{4}$ and $\frac{2}{9}$.
   Here, $\frac{3}{4} \times \frac{2}{9} = \frac{6}{36} = \frac{1}{6}$, the answer.
   Or, $\frac{3}{4} \times \frac{2}{9} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.
2. Required the continued product of $\frac{2}{3}$, $3\frac{1}{4}$, 5, and $\frac{3}{4}$ of $\frac{3}{5}$.
   Here $\frac{2}{3} \times \frac{13}{4} \times \frac{5}{1} \times \frac{3}{4} \times \frac{3}{5} = \frac{13 \times 3}{4 \times 2} = \frac{39}{8} = 4\frac{7}{8}$, answer.
3. Required the product of $\frac{2}{7}$ and $\frac{5}{8}$. Ans. $\frac{5}{28}$.
4. Required the product of $\frac{4}{15}$ and $\frac{5}{24}$. Ans. $\frac{1}{18}$.
5. Required the product of $\frac{3}{7}$, $\frac{4}{9}$, and $\frac{14}{15}$. Ans. $\frac{8}{45}$.
6. Required the product of $\frac{1}{2}$, $\frac{2}{3}$, and 3. Ans. 1.
7. Required the product of $\frac{7}{9}$, $\frac{3}{5}$, and $4\frac{5}{14}$. Ans. $2\frac{1}{30}$.
8. Required the product of $\frac{5}{6}$, and $\frac{2}{3}$ of $\frac{6}{7}$. Ans. $\frac{10}{21}$.
9. Required the product of 6, and $\frac{2}{3}$ of 5. Ans. 20.
10. Required the product of $\frac{3}{9}$ of $\frac{2}{5}$, and $\frac{5}{8}$ of $3\frac{3}{7}$. Ans. $\frac{23}{84}$.
11. Required the product of $3\frac{2}{7}$, and $4\frac{14}{33}$. Ans. $14\frac{124}{231}$.
12. Required the product of 5, $\frac{2}{3}$, $\frac{2}{7}$ of $\frac{3}{5}$, and $4\frac{1}{6}$, Ans. $2\frac{8}{21}$.

DIVISION OF VULGAR FRACTIONS.

RULE.*—Prepare the fractions as before in Multiplication; then divide the numerator by the numerator, and the denominator by the denominator, if they will exactly divide; but if not, then invert the terms of the divisor, and multiply the dividend by it, as in Multiplication.

EXAMPLES.

1. Divide $\frac{25}{9}$ by $\frac{5}{3}$.
   Here $\frac{25}{9} \div \frac{5}{3} = \frac{5}{3} = 1\frac{2}{3}$, by the first method.
2. Divide $\frac{5}{9}$ by $\frac{2}{15}$.
   Here $\frac{5}{9} \div \frac{2}{15} = \frac{5}{9} \times \frac{15}{2} = \frac{5}{3} \times \frac{5}{2} = \frac{25}{6} = 4\frac{1}{6}$, by the latter way.
3. Divide $\frac{16}{25}$ by $\frac{4}{5}$. Ans. $\frac{4}{5}$.
4. Divide $\frac{7}{16}$ by $\frac{3}{4}$. Ans. $\frac{7}{12}$.
5. Divide $\frac{14}{9}$ by $\frac{7}{6}$. Ans. $1\frac{1}{3}$.
6. Divide $\frac{5}{6}$ by $\frac{15}{7}$. Ans. $\frac{7}{18}$.
7. Divide $\frac{12}{35}$ by $\frac{3}{5}$. Ans. $\frac{4}{7}$.
8. Divide $\frac{2}{7}$ by $\frac{3}{5}$. Ans. $\frac{10}{21}$.
9. Divide $\frac{9}{16}$ by 3. Ans. $\frac{3}{16}$.
10. Divide $\frac{3}{5}$ by 2. Ans. $\frac{3}{10}$.
11. Divide $7\frac{1}{3}$ by $9\frac{5}{9}$. Ans. $\frac{33}{43}$.
12. Divide $\frac{2}{3}$ of $\frac{1}{3}$ by $\frac{5}{7}$ of $7\frac{3}{5}$. Ans. $\frac{7}{171}$.

Note. A fraction is best multiplied by an integer, by dividing the denominator by it; but if it will not exactly divide, then multiply the numerator by it.

* Division being the reverse of Multiplication, the reason of the Rule is evident.

Note. A fraction is best divided by an integer, by dividing the numerator by it; but if it will not exactly divide, then multiply the denominator by it.

RULE OF THREE IN VULGAR FRACTIONS.

RULE.*—Make the necessary preparations as before directed; then multiply continually together, the second and third terms, and the first with its terms inverted as in Division, for the answer.

EXAMPLES.

1. If $\frac{3}{8}$ of a yard of velvet cost $\frac{2}{5}$ of a pound sterling; what will $\frac{5}{16}$ of a yard cost?

Here, $\frac{3}{8} : \frac{2}{5} :: \frac{5}{16} : \frac{8}{3} \times \frac{2}{5} \times \frac{5}{16} = \frac{1}{3}l. = 6s.\ 8d.$ the answer.

2. What will $3\frac{3}{8}$ oz. of silver cost, at 6*s.* 4*d.* an ounce? Ans. 1*l.* 1*s.* $4\frac{1}{2}d.$

3. If $\frac{3}{16}$ of a ship be worth 273*l.* 2*s.* 6*d.*, what is $\frac{5}{32}$ of her worth? Ans. 227*l.* 12*s.* 1*d.*

4. What is the purchase of 1230*l.* bank-stock, at $108\frac{5}{8}$ per cent? Ans. 1336*l.* 1*s.* 9*d.*

5. What is the interest of 273*l.* 15*s.* for a year, at $3\frac{1}{4}$ per cent? Ans. 8*l.* 17*s.* $11\frac{1}{4}d.$

6. If $\frac{1}{8}$ of a ship be worth 73*l.* 1*s.* 3*d.*, what part of her is worth 250*l.* 10*s.*? Ans. $\frac{3}{7}$.

7. What length must be cut off a board, that is $7\frac{3}{4}$ inches broad, to contain a square foot, or as much as another piece of 12 inches long and 12 broad? Ans. $18\frac{18}{31}$ inches.

8. What quantity of shalloon, that is $\frac{3}{4}$ of a yard wide, will line $9\frac{1}{2}$ yards of cloth, that is $2\frac{1}{2}$ yards wide? Ans. $31\frac{2}{3}$ yds.

9. If the penny-loaf weigh $6\frac{9}{10}$ oz. when the price of wheat is 5*s.* the bushel; what ought it to weigh when the wheat is at 8*s.* 6*d.* the bushel? Ans. $4\frac{1}{17}$ oz.

10. How much in length, of a piece of land that is $11\frac{11}{12}$ poles broad, will make an acre of land, or as much as 40 poles in length and 4 in breadth? Ans. $13\frac{61}{143}$ poles.

11. If a courier perform a certain journey in $35\frac{1}{2}$ days, travelling $13\frac{5}{8}$ hours a-day; how long would he be in performing the same, travelling only $11\frac{9}{10}$ hours a-day? Ans. $40\frac{615}{952}$ days.

12. A regiment of soldiers, consisting of 976 men, are to be new clothed, each coat to contain $2\frac{1}{2}$ yards of cloth that is $1\frac{5}{8}$ yard wide, and lined with shalloon $\frac{7}{8}$ yard wide; how many yards of shalloon will line them? Ans. 4531 yds. 1 qr. $2\frac{6}{7}$ nails.

DECIMAL FRACTIONS.

A DECIMAL FRACTION is that which has for its denominator, a unit (1) with as many ciphers annexed as the numerator has places; and it is usually expressed by setting down the numerator only, with a point before it on the left hand.

* This is only multiplying the second and third terms together, and dividing the product by the first, as in the Rule of Three in whole numbers.

Thus, $\frac{5}{10}$ is ·5, and $\frac{25}{100}$ is ·25, and $\frac{75}{1000}$ is ·075, and $\frac{124}{100000}$ is ·00124; where ciphers are prefixed to make up as many places as are in the numerator, when there is a deficiency of figures.

A mixed number is made up of a whole number with some decimal fraction, the one being separated from the other by a point. Thus 3·25 is the same as $3\frac{25}{100}$, or $\frac{325}{100}$.

Ciphers on the right hand of decimals make no alteration in their value; for ·5 or ·50 or ·500, are decimals having all the same value, being each $= \frac{5}{10}$ or $\frac{1}{2}$. But if they are placed on the left hand, they decrease the value in a tenfold proportion. Thus ·5 is $\frac{5}{10}$ or 5 tenths, but ·05 is only $\frac{5}{100}$ or 5 hundredths, and ·005 is but $\frac{5}{1000}$ or 5 thousandths.

The first place of decimals, counted from the left hand towards the right, is called the place of primes, or 10ths; the second is the place of seconds, or 100ths; the third is the place of thirds, 1000ths; and so on. For, in decimals, as well as in whole numbers, the values of the places increase towards the left hand, and decrease towards the right, both in the same tenfold proportion; as in the following Scale or Table of Notation.

millions.	hundred thousands.	ten thousands.	thousands.	hundreds.	tens.	units.		tenth parts.	hundredth parts.	thousandth parts.	ten thousandth parts.	hundred thousandth parts.	millionth parts.
3	3	3	3	3	3	3	·	3	3	3	3	3	3

ADDITION OF DECIMALS.

RULE.—Set the numbers under each other according to the value of their places, like as in whole numbers; in which state the decimal separating points will stand all exactly under each other. Then, beginning at the right hand, add up all the columns of numbers as in integers; and point off as many places, for decimals, as are in the greatest number of decimal places in any of the lines that are added; or place the point directly below all the other points.

EXAMPLES.

1. To add together 29·0146, and 3146·5, and 2109, and ·62417 and 14·16.

```
  29·0146
3146·5
2109·
    ·62417
  14·16

5299·29877, the sum.
```

2. To find the sum of 376·25 + 86·125 + 637·4725 + 6·5 + 41·02 + 358·865. Ans. 1506·2325.

3. Required the sum of 3·5 + 47·25 + 2·0073 + 927·01 + 1·5. Ans. 981·2673.

4. Required the sum of 276 + 54·321 + 112 + 0·65 + 12·5 + ·0463. Ans. 455·5173.

SUBTRACTION OF DECIMALS.

RULE.—Place the numbers under each other according to the value of their places, as in the last rule. Then, beginning at the right hand, subtract as in whole numbers, and point off the decimals as in Addition.

EXAMPLES.

1. To find the difference between 91·73 and 2·138.

```
     91·73
      2·138
Ans. 89·592 the difference.
```

2. Find the difference between 1·9185 and 2·73. Ans. 0·8115.
3. Find the difference between 214·81 and 4·90142. Ans. 209·90858.
4. Find the difference between 2714 and ·916. Ans. 2713·084.

MULTIPLICATION OF DECIMALS.

RULE.*—Place the factors, and multiply them together the same as if they were whole numbers.—Then point off in the product just as many places of decimals as there are decimals in both the factors. But if there be not so many figures in the product, then supply the defect by prefixing ciphers.

EXAMPLES.

```
1. Multiply ·321096
         by ·2465
           1605480
          1926576
         1284384
         642192
Ans. ·0791501640 the product.
```

* The rule will be evident from this example: Let it be required to multiply ·12 by ·361, these numbers are equivalent to $\frac{12}{100}$ and $\frac{361}{1000}$ the product of which is $\frac{4332}{100000}$ = ·04332, by the nature of Notation, which consists of as many places as there are ciphers, that is, of as many places as there are in both numbers. And in like manner for any other numbers.

2. Multiply 79·347 by 23·15. Ans. 1836·88305
3. Multiply ·63478 by ·8204. Ans. ·520773512.
4. Multiply ·385746 by ·00464. Ans. ·00178986144.

CONTRACTION I.

To multiply decimals by 1 *with any number of ciphers, as* 10, *or* 100, *or* 1000, *&c.*

This is done by only removing the decimal point so many places farther to the right hand as there are ciphers in the multiplier; and subjoining ciphers if need be.

EXAMPLES.

1. The product of 51·3 and 1000 is 51300
2. The product of 2·714 and 100 is
3. The product of ·916 and 1000 is
4. The product of 21·31 and 10000 is

CONTRACTION II.

To contract the operation, so as to retain only as many decimals in the product as may be thought necessary, when the product would naturally contain several more places.

Set the units' place of the multiplier under that figure of the multiplicand whose place is the same as is to be retained for the last in the product; and dispose of the rest of the figures in the inverted or contrary order to what they are usually placed in.—Then, in multiplying, reject all the figures that are more to the right than each multiplying figure; and set down the products, so that their right hand figures may fall in a column straight below each other; but observing to increase the first figure of every line with what would arise from the figures omitted, in this manner, namely 1 from 5 to 14, 2 from 15 to 24, 3 from 25 to 34, &c.; and the sum of all the lines will be the product as required, commonly to the nearest unit in the last figure.

EXAMPLES.

1. To multiply 27·14986 by 92·41035, so as to retain only four places of decimals in the product.

Contracted way.	Common way.	
27·14986	27·14986	
53014·29	92·41035	
24434874	13	574930
542997	81	44958
108599	2714	986
2715	108599	44
81	542997	2
14	24434874	
2508·9280	2508·9280	650510

2. Multiply 480·14936 by 2·72416, retaining only four decimals in the product.
3. Multiply 2490·3048 by ·573286, retaining only five decimals in the product.
4. Multiply 325·701428 by ·7218393, retaining only three decimals in the product.

DIVISION OF DECIMALS.

Rule.—Divide as in whole numbers; and point off in the quotient as many places for decimals, as the decimal places in the dividend exceed those in the divisor.*

When the places of the quotient are not so many as the rule requires, let the defect be supplied by prefixing ciphers.

When there happens to be a remainder after the division; or when the decimal places in the divisor are more than those in the dividend; then ciphers may be annexed to the dividend, and the quotient carried on as far as required.

EXAMPLES.

1.

179) ·48624097 (·00271643

1282

294

1150

769

537

000

2.

·2685) 27·00000 (100·55865

15000

15750

23250

17700

15900

24750

3. Divide 234·70525 by 64·25. Ans. 3·653.
4. Divide 14 by ·7854. Ans. 17·825.
5. Divide 2175·68 by 100. Ans. 21·7568.
6. Divide ·8727587 by ·162. Ans. 5·38739.

CONTRACTION I.

When the divisor is an integer, with any number of ciphers annexed; cut off those ciphers, and remove the decimal point in the dividend as many places farther to the left as there are ciphers cut off, prefixing ciphers if need be; then proceed as before.†

EXAMPLES.

1. Divide 45·5 by 2100.

21·00) ·455 (·0216, &c.

35

140

14

2. Divide 41020 by 32000.
3. Divide 953 by 21600.
4. Divide 61 by 79000.

* The reason of this rule is evident; for, since the divisor multiplied by the quotient gives the dividend, therefore the number of decimal places in the dividend is equal to those in the divisor and quotient taken together, by the nature of Multiplication; and consequently the quotient itself must contain as many as the dividend exceeds the divisor.

† This is no more than dividing both divisor and dividend by the same number, either 10, or 100, or 1000, &c., according to the number of ciphers cut off; which, leaving them in the same proportion, does not affect the quotient. And, in the same way, the decimal point may be moved the same number of places in both the divisor and dividend, either to the right or left, whether they have ciphers or not.

CONTRACTION II.

HENCE, if the divisor be 1 with ciphers, as 10, or 100, or 1000, &c.; then the quotient will be found by merely moving the decimal point in the dividend so many places farther to the left as the divisor has ciphers; prefixing ciphers if need be.

So, 217·3 ÷ 100 = 2·173. and 419 ÷ 10 =
And 5·16 ÷ 100 = . and ·21 ÷ 1000 =

CONTRACTION III.

WHEN there are many figures in the divisor; or only a certain number of decimals are necessary to be retained in the quotient, then take only as many figures of the divisor as will be equal to the number of figures, both integers and decimals, to be in the quotient, and find how many times they may be contained in the first figures of the dividend, as usual.

Let each remainder be a new dividend; and for every such dividend, leave out one figure more on the right hand side of the divisor; remembering to carry for the increase of the figures cut off, as in the 2d contraction in Multiplication.

Note. When there are not so many figures in the divisor as are required to be in the quotient, begin the operation with all the figures, and continue it as usual till the number of figures in the divisor be equal to those remaining to be found in the quotient, after which begin the contraction.

EXAMPLES.

1. Divide 2508·92806 by 92·41035, so as to have only four decimals in the quotient, in which case the quotient will contain six figures.

Contracted.	Common way.
92·4103,5) 2508·928,06 (27·1498	92·4103,5) 2508·928,06 (27·1498
660721	66072106
13849	13848610
4608	46075750
912	91116100
80	79467850
6	5539570

2. Divide 4109·2351 by 230·409, so that the quotient may contain only four decimals.

3. Divide 37·10438 by 5713·96, that the quotient may contain only five decimals.

4. Divide 913·08 by 2137·2, that the quotient may contain only three decimals.

REDUCTION OF DECIMALS.

CASE I.

To reduce a vulgar fraction to its equivalent decimal.

RULE.—Divide the numerator by the denominator as in Division of Decimals, annexing ciphers to the numerator as far as necessary; so shall the quotient be the decimal required.

EXAMPLES.

1. **Reduce** $\frac{7}{24}$ to a decimal.

$24 = 4 \times 6$. Then $4\ )\ 7\cdot$

$6\ )\ 1\cdot750000$

$\cdot291666$, &c.

2. Reduce $\frac{1}{4}$, and $\frac{1}{2}$, and $\frac{3}{4}$, to decimals. Ans. ·25, and ·5, and ·75.
3. Reduce $\frac{3}{8}$ to a decimal. Ans ·375.
4. Reduce $\frac{1}{25}$ to a decimal. Ans. ·04.
5. Reduce $\frac{3}{192}$ to a decimal. Ans. ·015625.
6. Reduce $\frac{275}{3842}$ to a decimal. Ans. ·071577, &c.

CASE II.

To find the value of a decimal in terms of the inferior denominations.

RULE.—Multiply the decimal by the number of parts in the next lower denomination; and cut off as many places for a remainder, to the right hand, as there are places in the given decimal.

Multiply that remainder by the parts in the next lower denomination again, cutting off for another remainder as before.

Proceed in the same manner through all the parts of the integer; then the several denominations separated on the left hand, will make up the answer.

Note. This operation is the same as Reduction Descending in whole numbers.

EXAMPLES.

1. Required to find the value of ·775 pounds sterling.

$$\begin{array}{rr} & \cdot775 \\ & 20 \\ \hline s. & 15\cdot500 \\ & 12 \\ \hline d. & 6\cdot000 \end{array}$$

Ans. 15*s.* 6*d.*

2. What is the value of ·625*s.*? Ans. $7\frac{1}{2}d.$
3. What is the value of ·8635*l.*? Ans. 17*s.* 3·24*d.*
4. What is the value of ·0125 lb. troy? Ans. 3 dwts.
5. What is the value of ·4694 lb. troy? Ans. 5 oz. 12 dwt. 15·744 gr.
6. What is the value of ·625 cwt.? Ans. 2 qr. 14 lb.
7. What is the value of ·009943 miles? Ans. 17 yd. 1 ft. 5·98848 in.
8. What is the value of ·6875 yd.? Ans. 2 qr. 3 nls.
9. What is the value of ·3375 ac.? Ans. 1 rd. 14 poles.
10. What is the value of ·2083 hhd. of wine? Ans. 13·1229 gal.

CASE III.

To reduce integers or decimals to equivalent decimals of higher denominations.

RULE.—Divide by the number of parts in the next higher denomination; continuing the operation to as many higher denominations as may be necessary, the same as in Reduction Ascending of whole numbers.

EXAMPLES.

1. Reduce 1 dwt. to the decimal of a pound troy

20	1 dwt.
12	0·05 oz.
	0·004166, &c. lb. answer.

2. Reduce 9*d.* to the decimal of a pound. Ans. ·0375*l.*
3. Reduce 7 dr. to the decimal of a pound avoird. Ans. ·02734375 lb.
4. Reduce ·26*d.* to the decimal of a £. Ans. ·0010833, &c. £.
5. Reduce 2·15 lb. to the decimal of a cwt. Ans. ·019196 + cwt.
6. Reduce 24 yards to the decimal of a mile. Ans. ·013636, &c. miles
7. Reduce ·056 poles to the decimal of an acre. Ans. ·00035 ac.
8. Reduce 1·2 pints of wine to the decimal of a hhd. Ans. ·00238 + hhd.
9. Reduce 14 minutes to the decimal of a day. Ans. ·009722, &c. da.
10. Reduce ·21 pints to the decimal of a peck. Ans. ·013125 pec.

NOTE.—*When there are several numbers, to be reduced all to the decimal of the highest.*

Set the given numbers directly under each other, for dividends, proceeding orderly from the lowest denomination to the highest.

Opposite to each dividend, on the left hand, set such a number for a divisor as will bring it to the next higher name; drawing a perpendicular line between all the divisors and dividends.

Begin at the uppermost, and perform all the divisions; only observing to set the quotient of each division, as decimal parts, on the right hand of the dividend next below it; so shall the last quotient be the decimal required.

EXAMPLES.

1. Reduce 15*s.* 9$\frac{3}{4}$*d.* to the decimal of a pound.

4	3·
12	9·75
20	15·8125
	£ 0·790625, answer.

2. Reduce 19*l.* 17*s.* 3$\frac{1}{4}$*d.* to £. Ans. 19·86354166, &c. £.
3. Reduce 15*s.* 6*d.* to the decimal of a £. Ans. ·775£.
4. Reduce 7$\frac{1}{2}$*d.* to the decimal of shil. Ans. ·625*s.*
5. Reduce 5 oz. 12 dwts. 16 gr. to lbs. Ans. ·46944, &c. lb.

RULE OF THREE IN DECIMALS.

RULE.—Prepare the terms by reducing the vulgar fractions to decimals, any compound numbers either to decimals of the higher denominations, or to integers of the lower, also the first and third terms to the same name: then multiply and divide as in whole numbers.

Note. Any of the convenient examples in the Rule of Three or Rule of Five in Integers, or Vulgar Fractions, may be taken as proper examples to the same rules in Decimals.—The following example, which is the first in Vulgar Fractions, is wrought here to show the method.

If $\frac{3}{8}$ of a yard of velvet cost $\frac{2}{5}l$, what will $\frac{5}{16}$ yd. cost?

	yd.		l.		yd.		l.	s.	d.
$\frac{3}{8}$ = ·375	·375	:	·4	::	·3125	:	·333, &c. or	6	8
					·4				
$\frac{2}{5}$ = ·4			·375)		·12500	(	·333333, &c.		
					1250		20		
					125				
$\frac{5}{16}$ = ·3125							*s.* 6·66666, &c.		
							12		
	Answer, 6*s.* 8*d.*						*d.* 7·99999, &c. = 8*d.*		

DUODECIMALS.

Duodecimals, or Cross Multiplication, is a rule made use of by workmen and artificers, in computing the contents of their works.

Dimensions are usually taken in feet, inches, and quarters; any parts smaller than these being neglected as of no consequence. And the same in multiplying them together, or casting up the contents.

Rule.—Set down the two dimensions, to be multiplied together, one under the other, so that feet stand under feet, inches under inches, &c.

Multiply each term in the multiplicand, beginning at the lowest, by the feet in the multiplier, and set the result of each straight under its corresponding term, observing to carry 1 for every 12, from the inches to the feet.

In like manner, multiply all the multiplicand by the inches and parts of the multiplier, and set the result of each term one place removed to the right hand of those in the multiplicand; omitting however what is below parts of inches, only carrying to these the proper number of units from the lowest denomination.

Or, instead of multiplying by the inches, take such parts of the multiplicand as these are of a foot.

Then add the two lines together, after the manner of Compound Addition, carrying 1 to the feet for 12 inches, when these come to so many.

EXAMPLES.

1. Multiply	4 f.	7 inc.
by	6	4
	27	6
	1	6$\frac{1}{3}$
Answer,	29	0$\frac{1}{3}$

2. Multiply	14 f.	9 inc.
by	4	6
	59	0
	7	4$\frac{1}{2}$
Answer,	66	4$\frac{1}{2}$

3. Multiply 4 f. 7 inc. by 9 f. 6 inc. Ans. 43 f. $6\frac{1}{2}$ inc.
4. Multiply 12 f. 5 inc. by 6 f. 8 inc. Ans. 82 $9\frac{1}{3}$
5. Multiply 35 f. $4\frac{1}{2}$ inc. by 12 f. 3 inc. Ans. 433 $4\frac{1}{8}$
6. Multiply 64 f. 6 inc. by 8 f. $9\frac{1}{4}$ inc. Ans. 565 $8\frac{5}{8}$

INVOLUTION

INVOLUTION is the raising of Powers from any given number, as a root.

A Power is a quantity produced by multiplying any given number, called the Root, a certain number of times continually by itself. Thus,

$$2 = 2 \text{ is the root, or first power of 2.}$$
$$2 \times 2 = 4 \text{ is the 2d power, or square of 2.}$$
$$2 \times 2 \times 2 = 8 \text{ is the 3d power, or cube of 2.}$$
$$2 \times 2 \times 2 \times 2 = 16 \text{ is the 4th power of 2, \&c.}$$

And in this manner may be calculated the following Table of the first nine powers of the first nine numbers.

TABLE OF THE FIRST NINE POWERS OF NUMBERS.

1st	2d	3d	4th	5th	6th	7th	8th	9th
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

The Index or Exponent of a Power, is the number denoting the height or degree of that power; and it is 1 more than the number of multiplications used in producing the same. So 1 is the index or exponent of the 1st power or root, 2 of the 2d power or square, 3 of the 3d power or cube, 4 of the 4th power, and so on.

Powers, that are to be raised, are usually denoted by placing the index above the root or first power.

$$\text{So } 2^2 = 4 \text{ is the 2d power of 2.}$$
$$2^3 = 8 \text{ is the 3d power of 2.}$$
$$2^4 = 16 \text{ is the 4th power of 2.}$$
$$540^4 \text{ is the 4th power of 540, \&c.}$$

When two or more powers are multiplied together, their product will be that power whose index is the sum of the exponents of the factors or powers multi

plied. Or the multiplication of the powers, answers to the addition of the indices. Thus, in the following powers of 2.

	1st	2d	3d	4th	5th	6th	7th	8th	9th	10th
	2	4	8	16	32	64	128	256	512	1024
or,	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}

Here, $4 \times 4 = 16$, and $2 + 2 = 4$ its index;
and $8 \times 16 = 128$, and $3 + 4 = 7$ its index;
also $16 \times 64 = 1024$, and $4 + 6 = 10$ its index.

OTHER EXAMPLES.

1. What is the 2d power of 45? Ans. 2025.
2. What is the square of 4·16? Ans. 17·3056.
3. What is the 3d power of 3·5? Ans. 42·875.
4. What is the 5th power of ·029? Ans. ·000000020511149.
5. What is the square of $\frac{2}{3}$? Ans. $\frac{4}{9}$.
6. What is the 3d power of $\frac{5}{9}$? Ans. $\frac{125}{729}$.
7. What is the 4th power of $\frac{3}{4}$? Ans. $\frac{81}{256}$.

EVOLUTION.

EVOLUTION, or the reverse of Involution, is the extracting or finding the roots of any given powers.

The root of any number, or power, is such a number, as being multiplied into itself a certain number of times, will produce that power. Thus, 2 is the square root or 2d root of 4, because $2^2 = 2 \times 2 = 4$; and 3 is the cube root or 3d root of 27, because $3^3 = 3 \times 3 \times 3 = 27$.

Any power of a given number or root may be found exactly, namely, by multiplying the number continually into itself. But there are many numbers of which a proposed root can never be exactly found. Yet, by means of decimals we may approximate or approach towards the root, to any degree of exactness.

Those roots which only approximate, are called Surd roots; but those which can be found quite exact, are called Rational roots. Thus, the square root of 3 is a surd root; but the square root of 4 is a rational root, being equal to 2: also the cube root of 8 is rational, being equal to 2; but the cube root of 9 is surd or irrational.

Roots are sometimes denoted by writing the character $\surd$ before the power, with the index of the root against it. Thus, the third root of 20 is expressed by $\sqrt[3]{20}$; and the square root or 2d root of it is $\sqrt{20}$, the index 2 being always omitted, when the square root is designed.

When the power is expressed by several numbers, with the sign + or — between them, a line is drawn from the top of the sign over all the parts of it: thus, the third root of 45 — 12 is $\sqrt[3]{45 - 12}$, or thus, $\sqrt[3]{}(45 - 12)$, inclosing the numbers in parentheses.

But all roots are now often designed like powers, with fractional indices: thus, the square root of 8 is $8^{\frac{1}{2}}$, the cube root of 25 is $25^{\frac{1}{3}}$, and the 4th root of 45 — 18 is $\overline{45 - 18}|^{\frac{1}{4}}$, or, $(45 - 18)^{\frac{1}{4}}$.

TO EXTRACT THE SQUARE ROOT.

RULE.*—Divide the given number into periods of two figures each, by setting a point over the place of units, another over the place of hundreds, and so on, over every second figure, both to the left hand in integers, and to the right in decimals.

Find the greatest square in the first period on the left hand, and set its root on the right hand of the given number, after the manner of a quotient figure in Division.

Subtract the square thus found from the said period, and to the remainder annex the two figures of the next following period, for a dividend.

Double the root above mentioned for a divisor; and find how often it is contained in the said dividend, exclusive of its right hand figure; and set that quotient figure both in the quotient and divisor.

Multiply the whole augmented divisor by this last quotient figure, and subtract the product from the said dividend, bringing down to the next period of the given number, for a new dividend.

Repeat the same process over again, viz. find another new divisor, by doubling all the figures now found in the root; from which, and the last dividend, find the next figure of the root as before; and so on through all the periods, to the last.

Note. The best way of doubling the root, to form the new divisors, is by adding the last figure always to the last divisor, as appears in the following examples.—Also, after the figures belonging to the given number are all exhausted, the operation may be continued into decimals at pleasure, by adding any number of periods of ciphers, two in each period.

* The reason for separating the figures of the dividend into periods or portions of two places each, is, that the square of any single figure never consists of more than two places; the square of a number of two figures, of not more than four places, and so on. So that there will be as many figures in the root as the given number contains periods so divided or parted off.

And the reason of the several steps in the operation, appears from the algebraic form of the square of any number of terms, whether two, or three, or more. Thus, $\overline{a+b}|^2 = a^2 + 2ab + b^2 = a^2 + \overline{2a+b} \cdot b$, the square of two terms; where it appears, that a is the first term of the root, and b the second term; also a the first divisor, and the new divisor is $2a + b$, or double the first term increased by the second. And hence the manner of extraction is thus:

$$
\begin{array}{rl}
\text{1st division } a\,) & a^2 + 2ab + b^2 \;(\, a + b \text{ the root.} \\
 & \underline{a^2} \\
\text{2d divisor } 2a + b & \big|\; 2ab + b^2 \\
b & \big|\; \underline{2ab + b^2}
\end{array}
$$

Again, for a root of three parts a, b, c, thus:

$$\overline{a+b+c}^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2 =$$
$$a^2 + \overline{2a+b} \cdot b + \overline{2a+2b+c} \cdot c,$$ the square of three terms; where a is the first term of the root, b the second, and c the third term; also a the first divisor, $2a + b$ the second, and $2a + 2b + c$ the third, each consisting of the double of the root increased by the next term of the same. And the mode of extraction is thus:

$$
\begin{array}{rl}
\text{1st divisor } a\,) & a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \;(\, a + b + c \text{ the root.} \\
 & \\
\text{2d divisor } 2a + b & 2ab + b^2 \\
 & 2ab + b^2 \\
\text{3d divisor } 2a + 2b + c & \big|\; 2ac + 2bc + c^2 \\
c & \big|\; 2ac + 2bc + c^2
\end{array}
$$

EXAMPLES.

1. To find the square root of 29506624.

	29506624 (5432 the root.
	25
104	450
4	416
1083	3466
3	3249
10862	21724
2	21724

NOTE.—*When the root is to be extracted to many places of figures, the work may be considerably shortened, thus:*

Having proceeded in the extraction after the common method till there be found half the required number of figures in the root, or one figure more; then, for the rest, divide the last remainder by its corresponding divisor, after the manner of the third contraction in Division of Decimals; thus,

2. To find the root of 2 to nine places of figures.

	2 (1·4142
	1
24	100
4	96
281	400
1	281
2824	11900
4	11296
28282	60400
2	56564
28284)	3836 (1356
. . .	1008
	160
	19
	2

Answer, 1·41421356 the root required.

3. What is the square root of 2025? Ans. 45.
4. What is the square root of 17·3056? Ans. 4·16.
5. What is the square root of ·000729? Ans. ·027.
6. What is the square root of 3? Ans. 1·732050.
7. What is the square root of 5? Ans. 2·236068.
8. What is the square root of 6? Ans. 2·449489.
9. What is the square root of 7? Ans. 2·645751.
10. What is the square root of 10? Ans. 3·162277
11. What is the square root of 11? Ans. 3·316624.
12. What is the square root of 12? Ans. 3·464101.

RULES FOR THE SQUARE ROOTS OF VULGAR FRACTIONS AND MIXED NUMBERS.

FIRST, prepare all vulgar fractions, by reducing them to their least terms, both for this and all other roots. Then,

1. Take the root of the numerator and of the denominator for the respective terms of the root required. And this is the best way if the denominator be a complete power: but if it be not, then,

2. Multiply the numerator and denominator together; take the root of the product: this root being made the numerator to the denominator of the given fraction, or made the denominator to the numerator of it, will form the fractional root required.

$$\text{That is, } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{ab}}{b} = \frac{a}{\sqrt{ab}}$$

And this rule will serve whether the root be finite or infinite.

3. Or reduce the vulgar fraction to a decimal, and extract its root.

4. Mixed numbers may be either reduced to improper fractions, and extracted by the first or second rule; or the vulgar fraction may be reduced to a decimal, then joined to the integer, and the root of the whole extracted.

EXAMPLES.

1. What is the root of $\frac{25}{36}$? Ans. $\frac{5}{6}$.
2. What is the root of $\frac{27}{147}$? Ans. $\frac{3}{7}$.
3. What is the root of $\frac{9}{12}$? Ans. 0·866025.
4. What is the root of $\frac{5}{12}$? Ans. 0·645497.
5. What is the root of $17\frac{3}{8}$? Ans. 4·168333.

By means of the square root also may readily be found the 4th root, or the 8th root, or the 16th root, &c.; that is, the root of any power whose index is some power of the number 2; namely, by extracting so often the square root as is denoted by that power of 2; that is, two extractions for the fourth root, three for the 8th root, and so on.

So, to find the 4th root of the number 21035·8, extract the square root two times as follows;

```
         .  .  .  .  .              .  .  .  .  .
        21035·8000              ( 145·037237  (12·0431407, the 4th root.
        1                         1
      ---------                 ---------
   24 | 110                  22 | 45
    4 |  96                   2 | 44
   -----------              -------------
  285 | 1435               2404 | 10372
    5 | 1425                  4 |  9616
  ---------------           ---------------
29003 | 108000            24083 | 75637
    6 |  87009                6 | 72249
        ---------                 ---------
           20991 ( 7237               3388 ( 1407
             687                       980
             107                        17
              20
```

TO EXTRACT THE CUBE ROOT.

1. DIVIDE the page into three columns (I), (II), (III), in order, from left to right, so that the breadth of the columns may increase in the same order. In column (III) write the given number, and divide it into periods of three figures* each, by putting a point over the place of units, and also over every third figure, from thence to the left, in whole numbers, and to the right in decimals.

2. Find the nearest less cube number to the first or left-hand period; set its root in column (III), separating it from the right of the given number by a curve line, and also in column (I); then multiply the number in (I) by the root figure, thus giving the square of the first root figure, and write the result in (II); multiply the number in (II) by the root figure, thus giving the cube of the first root figure, and write the result below the first or left-hand period in (III); subtract it therefrom, and annex the next period to the remainder for a dividend.

3. In (I) write the root figure below the former, and multiply the sum of these by the root figure; place the product in (II), and add the two numbers together for a trial divisor. Again, write the root figure in (I), and add it to the former sum.

4. With the number in (II) as a trial divisor of the dividend, omitting the two figures to the right of it, find the next figure of the root, and annex it to the former, and also to the number in (I). Multiply the number now in (I) by the new figure of the root, and write the product as it arises in (II), but extended two places of figures more to the right, and the sum of these two numbers will be the corrected divisor; then multiply the corrected divisor by the last root figure, placing the product as it arises below the dividend; subtract it therefrom, annex another period, and proceed precisely as described in (3), for correcting the columns (I) and (II). Then with the new trial divisor in (II), and the new dividend in (III), proceed as before.†

Note I. When the trial divisor is not contained in the dividend, after two figures are omitted on the right, the next root figure is 0, and therefore one cipher must be annexed to the number in (I); two ciphers to the number in (II); and another period to the dividend in (III).

* The number is divided into periods of three figures each, because the cube of one figure never amounts to more than three figures; the cube of two figures to more than six, but always more than three; and so on. For a similar reason, a number is divided into periods of n figures, when the nth root is to be extracted.

† The truth of this rule will be obvious from the composition of the algebraic expression for the cube of a binomial. Thus $(a+b)^3=a^3+3a^2b+3ab^2+b^3$; then by the rule

(I)	(II)	(III)
a	a^2	$a^3+3a^2b+3ab^2+b^3$ ($a+b$=root.
a	$2a^2$	a^3
$2a$	$3a^2$	$3a^2b+3ab^2+b^3$
a	$3ab+b^2$	$3a^2b+3ab^2+b^3$
$3a$	$3a^2+3ab+b^2$	. . . .
$+b$		
$3a+b$		

Note II. When the root is interminable, we may contract the work very considerably, after obtaining a few figures in the decimal part of the root, if we omit to annex another period to the remainder in (III); cut off one figure from the right of (II), and two figures from (I), which will evidently have the effect of cutting off *three* figures from each column; and then work with the numbers on the left, as in contracted multiplication and division of decimals.

EXAMPLE.

Find the cube root of 21035·8 to ten places of decimals.

(I)	(II)	(III)
2	4	21035·8 (27·60491055944
2	8	8
4	12	13035
2	469	11683
67	1669	1352800
7	518	1341576
74	2187..	11224......
7	4896	9142444864
816	223596	2081555136
6	4932	2057415281
822	228528....	24139855
6	331216	22860923
82804	2285611216	1278932
4	331232	1143046
82808	228594244|8	135886
4	7453|1	114305
1·8|28|12	228601697|9	21581
	7453|1	20575
	2286091511	1006
	8|3	914
	228609234	92
	83	91
	2|2|8|6|0|9|3|2	1

EXAMPLES FOR PRACTICE.

Required the cube roots of the following numbers:—

(1) 48228544, 46656, and 15069223. Ans. 364, 36, and 247.

(2) 64481·201, and 28991029248. Ans. 40·1, and 3072.

(3) 12821119155125, and ·000076765625. Ans. 23405, and ·0425.

(4) $\frac{13824}{421875}$, and 16. Ans. $\frac{24}{75}$, and 2·519842.

(5) $91\frac{1}{8}$, and $7\frac{7}{8}$. Ans. 4·5, and 1·98802366.

TO EXTRACT ANY ROOT WHATEVER.*

Let N be the given power or number, n the index of the power, A the assumed power, r its root, R the required root of N.

Then, as the sum of $n+1$ times A and $n-1$ times N, is to the sum of $n+1$ times N and $n-1$ times A, so is the assumed root r, to the required root R.

Or, as half the said sum of $n+1$ times A and $n-1$ times N, is to the difference between the given and assumed powers, so is the assumed root r, to the difference between the true and assumed roots; which difference, added or subtracted, as the case requires, gives the true root nearly.

That is, $\overline{n+1}\cdot A + \overline{n-1}\cdot N : \overline{n+1}\cdot N + \overline{n-1}\cdot A :: r : R$.

Or, $\overline{n+1}\cdot \tfrac{1}{2}A + \overline{n-1}\cdot \tfrac{1}{2}N : A \backsim N :: r : R \backsim r$.

And the operation may be repeated as often as we please, by using always the last found root for the assumed root, and its nth power for the assumed power A.

EXAMPLE.

To extract the 5th root of 21035·8.

Here it appears that the 5th root is between 7·3 and 7·4. Taking 7·3, its 5th power is 20730·71593. Hence then we have,

$N = 21035{\cdot}8;\ r = 7{\cdot}3;\ n = 5;\ \tfrac{1}{2}\cdot\overline{n+1} = 3;\ \tfrac{1}{2}\cdot\overline{n-1} = 2$

$A = 20730{\cdot}716$

$N - A = 305{\cdot}084$

```
A = 20730·716     N = 21035·8
            3                2
  ------------     ----------
3A = 62192·148       42071·6
2N = 42071·6
  ------------
As 104263·7 : 305·084 :: 7·3 : ·0213605
                  7·3
              -------
               915252
              2135588
              -------
104263·7 ) 2227·1132 (  ·0213605, the difference.
               14184     7·3 = r add
                3758     -------------
                 630     7·321360 = R, the root, true to the
                   5                 last figure.
```

OTHER EXAMPLES.

1. What is the 3d root of 2? Ans. 1·259921.
2. What is the 4th root of 2? Ans. 1·189207.
3. What is the 4th root of 97·41? Ans. 3·1415999.
4. What is the 5th root of 2? Ans. 1·148699.
5. What is the 6th root of 21035·8? Ans. 5·254037.
6. What is the 6th root of 2? Ans. 1·122462.
7. What is the 7th root of 21035·8? Ans. 4·145392.
8. What is the 7th root of 2? Ans. 1·104089.
9. What is the 8th root of 21035·8? Ans. 3·470323.
10. What is the 8th root of 2? Ans. 1·090508.
11. What is the 9th root of 21035·8? Ans. 3·022239.
12. What is the 9th root of 2? Ans. 1·080059.

* This is a very general approximating rule for the extraction of any root of a given number, and is the best adapted for practice, and for memory, of any that I have yet seen. It was first discovered by myself, and the investigation and use of it were given at large in my Tracts, vol. 1, p. 45, &c.

OF RATIOS, PROPORTIONS, AND PROGRESSIONS.

NUMBERS are compared to each other in two different ways: the one comparison considers the difference of the two numbers, and is named Arithmetical Relation; and the difference sometimes the Arithmetical Ratio: the other considers their quotient, and is called Geometrical Relation, and the quotient the Geometrical Ratio. So, of these two numbers 6 and 3, the difference, or arithmetical ratio, is $6 - 3$ or 3; but the geometrical ratio is $\frac{6}{3}$ or 2.

There must be two numbers to form a comparison: the number which is compared, being placed first, is called the Antecedent; and that to which it is compared, the Consequent. So, in the two numbers above, 6 is the antecedent, and 3 the consequent.

If two or more couplets of numbers have equal ratios, or equal differences, the equality is named Proportion, and the terms of the ratios Proportionals. So, the two couplets, 4,2 and 8,6, are arithmetical proportionals, because $4 - 2 = 8 - 6 = 2$; and the two couplets 4,2 and 6,3, are geometrical proportionals, because $\frac{4}{2} = \frac{6}{3} = 2$, the same ratio.

To denote numbers as being geometrically proportional, a colon is set between the terms of each couplet, to denote their ratio; and a double colon, or else a mark of equality, between the couplets or ratios. So, the four proportionals, 4, 2, 6, 3, are set thus, $4 : 2 :: 6 : 3$, which means, that 4 is to 2 as 6 is to 3; or thus, $4 : 2 = 6 : 3$; or thus, $\frac{4}{2} = \frac{6}{3}$, both which mean, that the ratio of 4 to 2, is equal to the ratio of 6 to 3.

Proportion is distinguished into Continued and Discontinued When the difference or ratio of the consequent of one couplet and the antecedent of the next couplet. is not the same as the common difference or ratio of the couplets, the proportion is discontinued. So, 4, 2, 8, 6, are in discontinued arithmetical proportion, because $4 - 2 = 8 - 6 = 2$, whereas, $2 - 8 = -6$; and 4, 2, 6, 3, are in discontinued geometrical proportion, because $\frac{4}{2} = \frac{6}{3} = 2$, but $\frac{2}{6} = \frac{1}{3}$, which is not the same.

But when the difference or ratio of every two succeeding terms is the same quantity, the proportion is said to be continued, and the numbers themselves a series of continued proportionals, or a progression. So, 2, 4, 6, 8, form an arithmetical progression, because $4 - 2 = 6 - 4 = 8 - 6 = 2$, all the same common difference; and 2, 4, 8, 16, a geometrical progression, because $\frac{4}{2} = \frac{8}{4} = \frac{16}{8} = 2$, all the same ratio.

When the following terms of a Progression exceed each other, it is called an Ascending Progression or Series; but if the terms decrease, it is a Descending one.

So, 0, 1, 2, 3, 4, &c., is an ascending arithmetical progression,
but 9, 7, 5, 3, 1, &c., is a descending arithmetical progression:
Also, 1, 2, 4, 8, 16, &c., is an ascending geometrical progression,
and 16, 8, 4, 2, 1, &c., is a descending geometrical progression.

ARITHMETICAL PROPORTION AND PROGRESSION.

THE first and last terms of a Progression, are called the Extremes; and the other terms, lying between them, the Means.

The most useful part of arithmetical proportions, is contained in the following theorems:

THEOREM 1.—If four quantities be in arithmetical proportion, the sum of the two extremes will be equal to the sum of the two means.

Thus, of the four 2, 4, 6, 8, here $2 + 8 = 4 + 6 = 10$.

THEOREM 2.—In any continued arithmetical progression, the sum of the two extremes, is equal to the sum of any two means that are equally distant from them, or equal to double the middle term when there is an uneven number of terms.

Thus, in the terms 1, 3, 5, it is $1 + 5 = 3 + 3 = 6$.

And in the series 2, 4, 6, 8, 10, 12, 14, it is $2 + 14 = 4 + 12 = 6 + 10 = 8 + 8 = 16$.

THEOREM 3.—The difference between the extreme terms of an arithmetical progression, is equal to the common difference of the series multiplied by one less than the number of the terms.

So, of the ten terms, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, the common difference is 2, and one less than the number of terms 9; then the difference of the extremes is $20 - 2 = 18$, and $2 \times 9 = 18$ also.

Consequently, the greatest term is equal to the least term added to the product of the common difference multiplied by 1 less than the number of terms.

THEOREM 4.—The sum of all the terms, of any arithmetical progression, is equal to the sum of the two extremes multiplied by the number of terms, and divided by 2; or the sum of the two extremes multiplied by the number of the terms gives double the sum of all the terms in the series.

This is made evident by setting the terms of the series in an inverted order under the same series in a direct order, and adding the corresponding terms together in that order. Thus,

in the series	1,	3,	5,	7,	9,	11,	13,	15;
ditto inverted,	15,	13,	11,	9,	7,	5,	3,	1;
the sums are	16 +	16 +	16 +	16 +	16 +	16 +	16 +	16,

which must be double the sum of the single series, and is equal to the sum of the extremes repeated so often as are the number of the terms.

From these theorems may readily be found any one of these five parts; the two extremes, the number of terms, the common difference, and the sum of all the terms, when any three of them are given; as in the following Problems:

PROB. I.

Given the extremes, and the number of terms; to find the sum of all the terms.

RULE.—ADD the extremes together, multiply the sum by the number of terms and divide by 2.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the sum of the terms?

$$\begin{array}{r} 19 \\ 3 \\ \hline 22 \\ 9 \\ \hline 2\,)\,198 \\ \hline \text{Answer, } 99 \end{array}$$

Or, $\frac{19 + 3}{2} \times 9 = \frac{22}{2} \times 9 = 11 \times 9 = 99$.

2. It is required to find the number of all the strokes a clock strikes in one whole revolution of the index, or in 12 hours? Ans. 78.

3. How many strokes do the clocks of Venice strike in the compass of the day, which go right on from 1 to 24 o'clock? Ans. 300.

4. What debt can be discharged in a year, by weekly payments in arithmetical progression, the first payment being 1*s*., and the last or 52d payment 5*l*. 3*s*.

Ans. 135*l*. 4*s*.

PROB. II.

Given the extremes, and the number of terms; to find the common difference.

RULE.—Subtract the less extreme from the greater, and divide the remainder by 1 less than the number of terms, for the common difference.

EXAMPLES.

1. The extremes being 3 and 19, and the number of terms 9; required the common difference?

$$\begin{array}{r} 19 \\ 3 \\ \hline 8\,)\,16 \\ \hline \text{Ans. } 2 \end{array} \qquad \text{Or, } \frac{19-3}{9-1} = \frac{16}{8} = 2.$$

2. If the extremes be 10 and 70, and the number of terms 21; what is the common difference, and the sum of the series?

Ans. the com. diff. is 3, and the sum is 840.

3. A certain debt can be discharged in one year, by weekly payments in arithmetical progression, the first payment being 1*s*., and the last 5*l*. 3*s*.; what is the common difference of the terms? Ans. 2.

PROB. III.

Given one of the extremes, the common difference, and the number of terms; to find the other extreme, and the sum of the series.

RULE.—Multiply the common difference by 1 less than the number of terms, and the product will be the difference of the extremes: therefore add the product to the less extreme, to give the greater; or subtract it from the greater to give the less.

EXAMPLES.

1. Given the least term 3, the common difference 2, of an arithmetical series of 9 terms; to find the greatest term, and the sum of the series?

$$\begin{array}{rl} 2 & \\ 8 & \\ \hline 16 & \\ 3 & \\ \hline 19 & \text{the greatest term.} \\ 3 & \text{the least.} \\ \hline 22 & \text{sum.} \\ 9 & \text{number of terms.} \\ \hline 2\,)\,198 & \\ \hline 99 & \text{the sum of the series.} \end{array}$$

2. *If the greatest term be 70, the common difference 3, and the number of* **terms 21; what is the least term and the sum of the series?**

Ans. the least term is 10, and the sum is 840.

3. A debt can be discharged in a year, by paying 1*s*. the first week, 3*s*. the second, and so on, always 2*s*. more every week; what is the debt, and what will the last payment be? Ans. the last payment will be 5*l*. 3*s*., and the debt is 135*l*. 4*s*

PROB. IV.

To find an arithmetical mean proportional between two given terms.

RULE.—Add the two given extremes or terms together, and take half their sum for the arithmetical mean required. Or, subtract the less extreme from the greater, and half the remainder will be the common difference; which being added to the less extreme, or subtracted from the greater, will give the mean required.

EXAMPLE.

To find an arithmetical mean between the two numbers 4 and 14.

Here, 14
4
2) 18
Ans. 9

Or, 14
4
2) 10
5 the com. dif.
4 the less extreme.
9

Or, 14
5
9

So that 9 is the mean required by both methods.

PROB. V.

To find two arithmetical means between two given extremes.

RULE.—Subtract the less extreme from the greater, and divide the difference by 3, so will the quotient be the common difference; which being continually added to the less extreme, or taken from the greater, gives the means.

EXAMPLE.

To find two arithmetical means between 2 and 8.

Here 8
2
3) 6
com. dif. 2

Then $2 + 2 = 4$ the one mean,
and $4 + 2 = 6$ the other mean.

PROB. VI.

To find any number of arithmetical means between two given terms or extremes.

RULE.—Subtract the less extreme from the greater, and divide the difference by 1 more than the number of means required to be found, which will give the common difference; then this being added continually to the least term, or subtracted from the greatest, will give the mean terms required.

EXAMPLE.

To find five arithmetical means between 2 and 14.

Here 14
2
6) 12
com. dif. 2

Then by adding this com. dif. continually, the means are found 4, 6, 8, 10, 12.

Note. More of Arithmetical Progression is given in the Algebra.

GEOMETRICAL PROPORTION AND PROGRESSION.

The most useful part of Geometrical Proportion, is contained in the following theorems:

THEOREM 1.—If four quantities be in geometrical proportion, the product of the two extremes will be equal to the product of the two means.

Thus, in the four 2, 4, 3, 6, it is $2 \times 6 = 3 \times 4 = 12$.

And hence, if the product of the two means be divided by one of the extremes, the quotient will give the other extreme. So, of the above numbers, the product of the means $12 \div 2 = 6$ the one extreme, and $12 \div 6 = 2$ the other extreme; and this is the foundation and reason of the practice in the Rule of Three.

THEOREM 2.—In any continued geometrical progression, the product of the two extremes is equal to the product of any two means that are equally distant from them, or equal to the square of the middle term when there is an uneven number of terms.

Thus, in the terms 2, 4, 8, it is $2 \times 8 = 4 \times 4 = 16$.
And in the series 2, 4, 8, 16, 32, 64, 128,
it is $2 \times 128 = 4 \times 64 = 8 \times 32 = 16 \times 16 = 256$.

THEOREM 3.—The quotient of the extreme terms of a geometrical progression, is equal to the common ratio of the series raised to the power denoted by 1 less than the number of the terms.

So, of the ten terms 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, the common ratio is 2, one less than the number of terms 9; then the quotient of the extremes is $\frac{1024}{2} = 512$, and $2^9 = 512$ also.

Consequently the greatest term is equal to the least term multiplied by the said power of the ratio whose index is 1 less than the number of terms.

THEOREM 4.—The sum of all the terms, of any geometrical progression, is found by adding the greatest term to the difference of the extremes divided by 1 less than the ratio.

So, the sum of 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, (whose ratio is 2,) is $1024 + \frac{1024 - 2}{2 - 1} = 1024 + 1022 = 2046$.

The foregoing, and several other properties of geometrical proportion, are demonstrated more at large in the Algebraic part of this work. A few examples may here be added of the theorems, just delivered, with some problems concerning mean proportionals.

EXAMPLES.

1. The least of ten terms, in geometrical progression, being 1, and the ratio 2: what is the greatest term, and the sum of all the terms?

Ans. the greatest term is 512, and the sum 1023.

2. What debt may be discharged in a year, or 12 months, by paying 1*l.* the first month, 2*l.* the second, 4*l.* the third, and so on, each succeeding payment being double the last; and what will the last payment be?

Ans. the debt 4095*l.*, and the last payment 2048*l.*

PROB. I.

To find one geometrical mean proportional between any two numbers.

Rule.—Multiply the two numbers together, and extract the square root of the product, which will give the mean proportional sought.

Or, divide the greater term by the less, and extract the square root of the quotient, which will give the common ratio of the three terms: then multiply the less term by the ratio, or divide the greater term by it, either of these will give the middle term required.

EXAMPLE.

To find a geometrical mean between the two numbers 3 and 12.

First way.	*Second way.*
12	3) 12 (4, its root is 2 the ratio.
3	
36 (6 the mean.	Then, $3 \times 2 = 6$ the mean.
36	Or, $12 \div 2 = 6$ ditto.

PROB. II.

To find two geometrical mean proportionals between any two numbers.

Rule.—Divide the greater number by the less, and extract the cube root of the quotient, which will give the common ratio of the terms. Then multiply the least given term by the ratio for the first mean, and this mean again by the ratio for the second mean: or, divide the greater of the two given terms by the ratio for the greater mean, and divide this again by the ratio for the less mean.

EXAMPLE.

To find two geometrical mean proportionals between 3 and 24.

Here, 3) 24 (8, its cube root, 2 is the ratio.
Then, $3 \times 2 = 6$, and $6 \times 2 = 12$, the two means.
Or, $24 \div 2 = 12$, and $12 \div 2 = 6$, the same.
That is, the two means between 3 and 24, are 6 and 12.

PROB. III.

To find any number of geometrical mean proportionals between two numbers.

Rule.—Divide the greater number by the less, and extract such root of the quotient whose index is 1 more than the number of means required, that is, the 2d root for 1 mean, the 3d root for 2 means, the 4th root for 3 means, and so on; and that root will be the common ratio of all the terms. Then with the ratio multiply continually from the first term, or divide continually from the last or greatest term.

EXAMPLE.

To find four geometrical mean proportionals between 3 and 96.

Here, 3) 96 (32, the 5th root of which is 2, the ratio.
Then, $3 \times 2 = 6$, and $6 \times 2 = 12$, and $12 \times 2 = 24$, and $24 \times 2 =$
Or, $96 \div 2 = 48$, and $48 \div 2 = 24$, and $24 \div 2 = 12$, and $12 \div 2 =$
That is, 6, 12, 24, 48, are the four means between 3 and 96.

OF MUSICAL PROPORTION.

THERE is also a third kind of proportion, called Musical, which being but of little or no common use, a very short account of it may here suffice.

Musical Proportion is when, of three numbers, the first has the same proportion to the third, as the difference between the first and second, has to the difference between the second and third.

As in these three, 6, 8, 12;
Where, $6 : 12 :: 8 - 6 : 12 - 8$,
that is, $6 : 12 :: 2 : 4$.

When four numbers are in Musical Proportion; then the first has the same Proportion to the fourth, as the difference between the first and second has to the difference between the third and fourth.

As in these, 6, 8, 12, 18;
where, $6 : 18 :: 8 - 6 : 18 - 12$,
that is, $6 : 18 :: 2 : 6$.

When numbers are in Musical Progression, their reciprocals are in Arithmetical Progression; and the converse, that is, when numbers are in Arithmetical Progression, their reciprocals are in Musical Progression.

So, in these Musicals 6, 8, 12, their reciprocals $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$, are in arithmetical progression; for $\frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$; and $\frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$; that is, the sum of the extremes is equal to double the mean, which is the property of arithmeticals.

The method of finding out numbers in Musical Proportion, is best expressed by letters in Algebra.

FELLOWSHIP OR PARTNERSHIP.

FELLOWSHIP is a rule, by which any sum or quantity may be divided into any number of parts, which shall be in any given proportion to one another.

By this rule are adjusted the gains, or losses, or charges of partners in company; or the effects of bankrupts, or legacies in case of a deficiency of assets or effects; or the shares of prizes, or the numbers of men to form certain detachments; or the division of waste lands among a number of proprietors.

Fellowship is either Single or Double. It is Single, when the shares or portions are to be proportional each to one single given number only; as when the stocks of partners are all employed for the same time: and Double, when each portion is to be proportional to two or more numbers; as when the Stocks of partners are employed for different times.

SINGLE FELLOWSHIP

General Rule.—Add together the numbers that denote the proportion of the shares. Then,

As the sum of the said proportional numbers,
Is to the whole sum to be parted or divided,
So is each several proportional number,
To the corresponding share or part.

Or, As the whole stock, is to the whole gain or loss,
So is each man's particular stock, to his particular share of the gain or loss.

To prove the work.—Add all the shares or parts together, and the sum will be equal to the whole number to be shared, when the work is right.

EXAMPLES.

1. To divide the number 240 into three such parts, as shall be in proportion to each other as the three numbers 1, 2, and 3.

Here $1 + 2 + 3 = 6$ the sum of the proportional numbers.

Then, as 6 : 240 :: 1 : 40 the 1st part,
and, as 6 : 240 :: 2 : 80 the 2d part,
also as 6 : 240 :: 3 : 120 the 3d part.

Sum of all 240, the proof.

2. Three persons, A, B, C, freighted a ship with 340 tuns of wine; of which, A loaded 110 tuns, B 97, and C the rest: in a storm the seamen were obliged to throw overboard 85 tuns; how much must each person sustain of the loss?

Here, $110 + 97 = 207$ tuns, loaded by A and B;
theref., $340 - 207 = 133$ tuns, loaded by C.
hence, as 340 : 85 :: 110
or, as 4 : 1 :: 110 : $27\frac{1}{2}$ tuns = A's loss;
and, as 4 : 1 :: 97 : $24\frac{1}{4}$ tuns = B's loss;
also, as 4 : 1 :: 133 : $33\frac{1}{4}$ tuns = C's loss.

Sum 85 tuns, the proof

3. Two merchants, C and D, made a stock of 120*l.*; of which C contributed 75*l.*, and D the rest; by trading they gained 30*l.*; what must each have of it?

Ans. C 18*l.* 15*s.*, and D 11*l.* 5*s.*

4. Three merchants, E, F, G, made a stock of 700*l.*; of which E contributed 123*l.*, F 358*l.*, and G the rest; by trading they gain 125*l.* 10*s.*; what must each have of it?

Ans. E must have 22*l.* 1*s.* 0*d.* $2\frac{2}{35}q.$
F .... 64 3 8 $0\frac{32}{35}$.
G .... 39 5 3 $1\frac{1}{35}$.

5. A general imposing a contribution* of 700*l.*, on four villages, to be paid in proportion to the number of inhabitants contained in each; the 1st containing 250, the 2d 350, the 3d 400, and the 4th 500 persons: what part must each village pay?

Ans. the 1st to pay 116*l.* 13*s.* 4*d.*
the 2d 163 6 8
the 3d 186 13 4
the 4th 233 6 8

* Contribution is a tax paid by provinces, towns, villages, &c., to excuse them from being plundered, and is paid in provisions or in money, and sometimes in both.

6. A piece of ground, consisting of 37 ac. 2 ro. 14 ps. is to be divided among three persons, L, M, and N, in proportion to their estates: now if L's estate be worth 500*l.* a year, M's 320*l.*, and N's 75*l.*; what quantity of land must each one have?

Ans.	L must have	20 ac.	3 ro.	$39\frac{139}{179}$ ps.
	M	13	1	$30\frac{46}{179}$
	N	3	0	$23\frac{173}{179}$

7. A person is indebted to O 57*l.* 15*s.*, to P 108*l.* 3*s.* 8*d.*, to Q 22*l.* 10*d.*, and to R 73*l.*; but at his decease, his effects are found to be worth no more than 170*l.* 14*s.*: how must it be divided among his creditors?

Ans.	O must have	37*l.*	15*s.*	5*d.*	$2\frac{8302}{10439}$*q.*
	P	70	15	2	$2\frac{7498}{10439}$
	Q	14	8	4	$0\frac{4729}{10439}$
	R	47	14	11	$2\frac{3368}{10439}$

8. A ship worth 900*l.*, being entirely lost, of which $\frac{1}{8}$ belonged to S, $\frac{1}{4}$ to T, and the rest to V; what loss will each sustain, supposing 540*l.* of her were insured? Ans. S will lose 45*l.*, T 90*l.*, and V 225*l.*

9. Four persons, W, X, Y, and Z, spent among them 25*s*, and agree that W shall pay $\frac{1}{2}$ of it, X $\frac{1}{3}$, Y $\frac{1}{4}$, and Z $\frac{1}{5}$; that is, their shares are to be in proportion as, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$; what are their shares?

Ans.	W must pay	9*s.*	8*d.*	$3\frac{41}{77}$*q.*
	X	6	5	$3\frac{53}{77}$
	Y	4	10	$1\frac{59}{77}$
	Z	3	10	$3\frac{1}{77}$

10. A detachment, consisting of five companies, being sent into a garrison, in which the duty required 76 men a day; what number of men must be furnished by each company, in proportion to their strength; the 1st consisting of 54 men, the 2d of 51 men, the 3d of 48 men, the 4th of 39 men, and the 5th of 36 men? Ans. The 1st must furnish 18, the 2d 17, the 3d 16, the 4th 13, and the 5th 12 men.*

DOUBLE FELLOWSHIP.

Double Fellowship, as has been said, is concerned in cases in which the stocks of partners are employed or continued for different times.

Rule.†—Multiply each person's stock by the time of its continuance; then divide the quantity, as in Single Fellowship, into shares in proportion to these products, by saying,

> As the total sum of all the said products,
> Is to the whole gain or loss, or quantity to be parted,
> So is each particular product,
> To the corresponding share of the gain or loss.

* Questions of this nature frequently occurring in military service, general Haviland, an officer of great merit, contrived an ingenious instrument, for more expeditiously resolving them; which is distinguished by the name of the inventor, being called a Haviland.

† The proof of this rule is as follows: when the times are equal, the shares of the gain or loss are evidently as the stocks, as in Single Fellowship; and when the stocks are equal, the shares are as the times: therefore, when neither are equal, the shares must be as their products.

EXAMPLES.

1. A had in company 50*l.* for 4 months, and B had 60*l.* for 5 months; at the end of which they find 24*l.* gained: how must it be divided between them?

$$\text{Here}\quad \begin{array}{r} 50 \\ 4 \\ \hline 200 \end{array} + \begin{array}{r} 60 \\ 5 \\ \hline 300 \end{array} = 500.$$

Then, as 500 : 24 :: 200 : $9\frac{3}{5}$ = 9*l.* 12*s.* = A's share,
and, as 500 : 24 :: 300 : $14\frac{2}{5}$ = 14 8 = B's share.

2. C and D hold a piece of ground in common, for which they are to pay 36*l.* C put in 23 horses for 27 days, and D 21 horses for 39 days; how much ought each man to pay of the rent? Ans. C must pay 15*l.* 10*s.* 6*d.*
D .... 20 9 6.

3. Three persons, E, F, G, hold a pasture in common, for which they are to pay 30*l.* per annum; into which E put 7 oxen for 3 months, F put 9 oxen for 5 months, and G put in 4 oxen for 12 months: how much must each person pay of the rent? Ans. E must pay 5*l.* 10*s.* 6*d.* $1\frac{5}{19}$*q.*
F .... 11 16 10 $0\frac{8}{19}$
G .... 12 12 7 $2\frac{6}{19}$.

4. A ship's company take a prize of 1000*l.*, which they agree to divide among them according to their pay and the time they have been on board: now the officers and midshipmen have been on board 6 months, and the sailors 3 months; the officers have 40*s.* a month, the midshipmen 30*s.*, and the sailors 22*s.* a month; moreover, there are 4 officers, 12 midshipmen, and 110 sailors: what will each man's share be? Ans. each officer, must have 23*l.* 2*s.* 5*d.* $0\frac{92}{173}$*q*
each midship., ... 17 6 9 $3\frac{69}{173}$.
each seaman, ... 6 7 2 $0\frac{8}{173}$.

5. H, with a capital of 1000*l.*, began trade the first of January, and, meeting with success in business, took in I as a partner, with a capital of 1500*l.*, on the first of March following. Three months after that they admit K as a third partner, who brought into stock 2800*l.* After trading together till the end of the year, they find there has been gained 1776*l.* 10*s.*: how must this be divided among the partners? Ans. H must have 457*l.* 9*s.* $4\frac{1}{4}$*d.*
I .... 571 16 $8\frac{1}{4}$
K .... 747 3 $11\frac{1}{4}$.

6. X, Y, and Z, made a joint-stock for 12 months; X at the first put in 20*l.*, and 4 months after 20*l.* more; Y put in at the first 30*l.*, at the end of 3 months he put in 20*l.* more, and 2 months after he put in 40*l.* more; Z put in at first 60*l.*, and 5 months after he put in 10*l.* more, 1 month after which he took out 30*l.*; during the 12 months they gained 50*l*; how much of it must each have?
Ans. X must have 10*l.* 18*s.* 6*d.* $3\frac{40}{61}$*q.*
Y .... 22 8 1 $0\frac{12}{61}$
Z 16 13 4 0

SIMPLE INTEREST.

Interest is the premium or sum allowed for the loan, or forbearance of money.

The money lent, or forborn, is called the Principal.

The sum of the principal and its interest, added together, is called the Amount

Interest is allowed at so much per cent. per annum; which premium per cent. per annum, or interest of a 100*l.* for a year, is called the Rate of Interest:—So.

When interest is at 3 per cent. the rate is 3;
.... 4 per cent. 4;
.... 5 per cent. 5;
.... 6 per cent. .... 6.

But, by law, interest ought not to be taken higher than at the rate of 5 per cent.

Interest is of two sorts; Simple and Compound.

Simple Interest is that which is allowed for the principal lent or forborn only, for the whole time of forbearance.

As the interest of any sum, for any time, is directly proportional to the principal sum, and also to the time of continuance; hence arises the following general rule of calculation.

General Rule.—As 100*l.* is to the rate of interest, so is any given principal to its interest for one year. And again,

As 1 year is to any given time, so is the interest for a year, just found, to the interest of the given sum for that time.

Otherwise.—Take the interest of 1 pound for a year, which, multiply by the given principal, and this product again by the time of loan or forbearance, in years and parts, for the interest of the proposed sum for that time.

Note. When there are certain parts or years in the time, as quarters, or months, or days; they may be worked for either by taking the aliquot or like parts of the interest of a year, or by the Rule of Three, in the usual way. Also, to divide by 100, is done by only pointing off two figures for decimals.

EXAMPLES.

1. To find the interest of 230*l.* 10*s.*, for 1 year, at 4 per cent. per annum.

Here, as 100 : 4 :: 230*l.* 10*s.* : 9*l.* 4*s.* $4\frac{3}{4}d.$

```
              4
        ---------
100 ) 9|22    0
        20
      ----
      4|40
        12
      ----
      4|80
        4
      ----
      3|20     Ans. 9l. 4s. 4¾d.
```

2. To find the interest of 547*l.* 15*s.*, for 3 years, at 5 per cent. per annum.

```
As, 100 : 5 :: 547 15 :
Or,  20 : 1 :: 547·75 : 27.3875 interest for 1 year.
                              3
                      ---------
                   l. 82.1625 ditto for 3 years.
                             20
                      ---------
                   s.  3|2500
                         12
                      ---------
                   d.  3|00     Ans. 82l. 3s. 3d.
```

3. To find the interest of 210 guineas, for 4 years, 7 months, and 25 days at 4½ per cent. per annum.

```
                                            ds.      l.      ds.
        210                              as, 365 : 9·45 :: 25 :    l.
          4½                             or,  73 : 9·45 ::  5 : ·6472
        ----                                                5
        840                                            -----
        105                                 73 ) 47·25 ( ·6472
        ----                                      345
        9.45  interest for 1 year.                 530
           4                                        19
        ----
       37·80    ditto  4 years.
6 mo. = ½  4·725   ditto  6 months.
1 mo. = ⅙  ·7875   ditto  1 month.
           ·6472   ditto  25 days.
       ----------
    l. 43|9597
           20
    s. 19|1940
           12
       ---------
    d.  2|3280
            4
       ---------
    q.  1|3120       Ans. 43l. 19s. 2¼d.
```

4. To find the interest of 450*l.*, for a year, at 5 per cent. per annum.
Ans. 22*l.* 10*s.*

5. To find the interest of 230*l.* 10*s.*, for a year, at 4 per cent. per annum.
Ans. 9*l.* 4*s.* 4¾*d.*

6. To find the interest of 715*l.* 12*s.* 6*d.*, for a year, at 4½ per cent. per annum.
Ans. 32*l.* 4*s.* 0¾*d.*

7. To find the interest of 720*l.*, for 3 years, at 5 per cent. per annum.
Ans. 108*l.*

8. To find the interest of 355*l.* 15*s.*, for 4 years, at 4 per cent. per annum.
Ans. 56*l.* 18*s.* 4¾*d.*

9. To find the interest of 32*l.* 5*s.* 8*d.*, for 7 years, at 4½ per cent. per annum.
Ans. 9*l.* 12*s.* 1*d.*

10. To find the interest of 170*l.*, for 1½ year, at 5 per cent. per annum.
Ans. 12*l.* 15*s.*

11. To find the insurance of 205*l.* 15*s.*, for ¼ of a year at 4 per cent. per annum.
Ans. 2*l.* 1*s.* 1¾*d.*

12. To find the interest of 319*l.* 6*d.*, for 5¾ years, at 3¾ per cent. per annum.
Ans. 68*l.* 15*s.* 9½*d.*

13. To find the insurance on 107*l.*, for 117 days, at 4¾ per cent. per annum.
Ans. 1*l.* 12*s.* 7*d.*

14. To find the interest of 17*l.* 5*s.*, for 117 days, at 4¾ per cent. per annum.
Ans. 5*s.* 3*d.*

15. To find the insurance on 712*l.* 6*s.*, for 8 months, at 7½ per cent. per annum.
Ans. 35*l.* 12*s.* 3½*d.*

Note. The rules for Simple Interest, serve also to calculate Insurances, or the Purchase of Stocks, or any thing else that is rated at so much per cent.

See also more on the subject of Interest, with the algebraical expression and investigation of the rules, at the end of the Algebra, next following.

COMPOUND INTEREST

Compound Interest, called also Interest upon Interest, is that which arises from the principal and interest, taken together, as it becomes due, at the end of each stated time of payment.

Although it be not lawful to lend money at Compound Interest, yet in purchasing annuities, pensions, or leases in reversion, it is usual to allow Compound Interest to the purchaser for his ready money.

Rules.—1. Find the amount of the given principal, for the time of the first payment, by Simple Interest. Then consider this amount as a new principal for the second payment, whose amount calculate as before And so on, through all the payments to the last, always accounting the last amount as a new principal for the next payment. The reason of which is evident from the definition of Compound Interest. Or else,

2. Find the amount of 1 pound for the time of the first payment, and raise or involve it to the power whose index is denoted by the number of payments. Then that power multiplied by the given principal, will produce the whole amount. From which the said principal being subtracted, leaves the Compound Interest of the same. As is evident from the first rule.

EXAMPLES.

1. To find the amount of 720*l.*, for 4 years, at 5 per cent. per annum.

Here, 5 is the 20th part of 100, and the interest of 1*l.* for a year, is $\frac{1}{20}$ or ·05, and its amount 1·05. Therefore,

1. By the 1st rule.

	l.	*s.*	*d.*	
20)	720	0	0	1st year's principal.
	36	0	0	1st year's interest.
20)	756	0	0	2d year's principal.
	37	16	0	2d year's interest.
20)	793	16	0	3d year's principal.
	39	13	$9\frac{1}{2}$	3d year's interest.
20)	833	9	$9\frac{1}{2}$	4th year's principal.
	41	13	$5\frac{3}{4}$	4th year's interest.
£	875	3	$3\frac{1}{4}$	the whole amount, or answer required.

2. By the 2d rule.

1·05	amount of 1*l.*
1·05	
1·1025	2d power of it.
1·1025	ditto.
1·21550625	4th power of it.
720	
l. 875|1645	
20	
s. 3|2900	
12	
d. 3|4800	

2. To find the amount of 50*l.*, in 5 years, at 5 per cent. per annum, compound interest. Ans. 63*l.* 16*s.* $3\frac{1}{4}$*d.*

3. To find the amount of 50*l.*, in 5 years, or 10 half years, at 5 per cent. per annum, compound interest, the interest payable half yearly. Ans. 64*l.* 0*s.* 1*d.*

4. To find the amount of 50*l.*, in 5 years, or 20 quarters, at 5 per cent. per annum, compound interest, the interest payable quarterly. Ans. 64*l.* 2*s.* $0\frac{1}{4}$*d.*

5. To find the compound interest of 370*l.*, forborn for 6 years, at 4 per cent. per annum. Ans. 98*l.* 3*s.* $4\frac{1}{4}$*d.*

6. To find the compound interest of 410*l.*, forborn for $2\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent. per annum, the interest payable half yearly. Ans. 48*l.* 4*s.* $11\frac{1}{4}$*d.*

7. To find the amount, at compound interest, of 217*l.*, forborn for $2\frac{1}{4}$ years, at 5 per cent. per annum, the interest payable quarterly. Ans. 242*l.* 13*s.* $4\frac{1}{4}$*d.*

POSITION.

Position is a method of performing certain questions, which cannot be resolved by the common direct rules. It is sometimes called False Position, or False Supposition, because it makes a supposition of False numbers, to work with, the same as if they were the true ones, and by their means discovers the true numbers sought. It is sometimes also called Trial and Error, because it proceeds by *trials* of false numbers, and thence finds out the true ones by a comparison of the *errors*.

Position is either Single or Double.

SINGLE POSITION.

Single Position is that by which a question is resolved by means of one supposition only.

Questions which have their results proportional to their suppositions, belong to Single Position; such as those which require the multiplication or division of the number sought by any proposed number; or when it is to be increased or diminished by itself, or any parts of itself, a certain proposed number of times.

Rule.—Take or assume any number for that required, and perform the same operations with it, as are described or performed in the question.

Then say, as the result of the said operation, is to the position, or number assumed; so is the result in the question, to the number sought.*

EXAMPLES.

1. A person, after spending $\frac{1}{3}$ and $\frac{1}{4}$ of his money, has yet remaining 60*l.*; what had he at first?

Suppose he had at first 120*l.*		Proof.	
Now $\frac{1}{3}$ of 120 is	40	$\frac{1}{3}$ of 144 is	48
$\frac{1}{4}$ of it is	30	$\frac{1}{4}$ of 144 is	36
their sum is	70	their sum	84
which taken from	120	taken from	144
leaves	50	leaves	60 as per question.

Then, 50 : 120 :: 60 : 144, the answer.

2. What number is that, which multiplied by 7, and the product divided by 6, the quotient may be 14? Ans. 12.

* The reason of the rule is evident, because it is supposed that the results are proportional to the suppositions.

Thus, $nx : x :: na : a$,

or, $\frac{x}{n} : x :: \frac{a}{n} : a$,

or, $\frac{x}{n} \pm \frac{x}{m}$, &c. $: x :: \frac{a}{n} \pm \frac{a}{m}$, &c. $: a$,

and so on.

3. What number is that, which being increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the sum shall be 125? Ans. 60.

4. A general, after sending out a foraging $\frac{1}{2}$ and $\frac{1}{3}$ of his men, had yet remaining 700; what number had he in command? Ans. 420.

5. A gentleman distributed 78 pence among a number of poor people, consisting of men, women, and children; to each man he gave 6*d.*, to each woman 4*d.*, and to each child 2*d.*: moreover there were twice as many women as men, and thrice as many children as women. How many were there of each?

Ans. 3 men, 6 women, and 18 children.

6. One being asked his age, said, if $\frac{3}{5}$ of the years I have lived, be multiplied by 7, and $\frac{2}{3}$ of them be added to the product, the sum will be 292. What was his age? Ans. 60 years.

DOUBLE POSITION.

Double Position is the method of resolving certain questions by means of two suppositions of false numbers.

To the Double Rule of Position belong such questions as have their results not proportional to their positions: such are those, in which the numbers sought, or their parts, or their multiples, are increased or diminished by some given absolute number, which is no known part of the number sought.

Rule 1.*—Take or assume any two convenient numbers, and proceed with each of them separately, according to the conditions of the question, as in Single Position; and find how much each result is different from the result mentioned in the question, noting also whether the results are too great or too little.

Then multiply each of the said errors by the contrary supposition, namely, the first position by the second error, and the second position by the first error.

If the errors are alike, divide the difference of the products by the difference of the errors, and the quotient will be the answer.

But if the errors are unlike, divide the sum of the products by the sum of the errors, for the answer.

Note. The errors are said to be alike when they are either both too great, or both too little; and unlike, when one is too great and the other too little.

* *Demonstration.*—The rule is founded on this supposition, namely, that the first error is to the second, as the difference between the true and first supposed number, is to the difference between the true and second supposed number; when that is not the case, the exact answer to the question cannot be found by this rule.—That the rule i rue, according to that supposition, may be thus proved.

Let a and b be the two suppositions, and A and B their results, produced by similar operations; also r and s their errors, or the differences between the results A and B from the true result N; and let x denote the number sought, answering to the true result N of the question.

Then, is $N - A = r$, and $N - B =$ And, according to the supposition on which the rule is founded, $r : s :: x - a : x - b$; hence, by multiplying extremes and means, $rx - ro = sx - sa$; then, by transposition, $rx - sx = rb - sa$; and, by division, $x = \frac{rb - sa}{r - s} =$ the number sought, which is the rule when the results are both too little.

If the results be both too great, so that A and B are both greater than N; then $N - A = -r$, and $N - B = -s$, or r and s are both negative; hence $-r : -s :: x - a : x - b$, but $-r : -s : +r : +s$, therefore $r : s :: x - a : x - b$, and the rest will be exactly as in the former case.

But if one result A only be too little, and the other B be too great, or one error r positive, and the other s negative, then the theorem becomes $x = \frac{rb + sa}{r + s}$, which is the rule in this case, or when the errors are unlike

EXAMPLE.

1. What number is that, which being multiplied by 6, the product increased by 18, and the sum divided by 9, the quotient shall be 20.

Suppose the two numbers, 18 and 30. Then,

	First position,		Second position.		Proof.
	18		30		27
	6	mult.	6		6
	108		180		162
	18	add.	18		18
	9) 126		9) 198		9) 180
	14	results.	22		20
	20	true res.	20		
	+ 6	errors unlike.	— 2		
2d pos.	30	mult.	18	1st pos.	
errors. {	2 180		36		
	6 36				
sum	8) 216	sum of products.			
	27	answer sought.			

Rule II.*—Find, by trial, two numbers, as near the true number as possible, and operate with them as in the question; marking the errors which arise from each of them.

Multiply the difference of the two numbers, found by trial, by the least error, and divide the product by the difference of the errors, when they are alike, but by their sum when they are unlike.

Add the quotient, last found, to the number belonging to the least error, when that number is too little, but subtract it when too great, and the result will give the true quantity sought.

EXAMPLES.

1. A son asking his father how old he was, received this answer: Your age is now one fourth of mine; but 5 years ago, your age was only one fifth of mine. What then are their two ages? Ans. 20 and 80.

2. A workman was hired for 30 days, at 2*s.* 6*d.* per day, for every day he worked; but with this condition, that for every day he played, he should forfeit 1*s.* Now it so happened, that upon the whole he had 2*l.* 14*s.* to receive. How many of the days did he work? Ans. 24.

3. A and B began to play together with equal sums of money: A first won 20 guineas, but afterwards lost back $\frac{2}{3}$ of what he then had; after which, B had 4 times as much as A. What sum did each begin with? Ans. 100 guineas.

4. Two persons, A and B, have both the same income. A saves $\frac{1}{5}$ of his; but B, by spending 50*l.* per annum more than A, at the end of 4 years finds himself 100*l* in debt. What does each receive and spend per annum?

Ans. They receive 125*l.* per annum; also A spends 100*l.*, and B spends 150*l.* per annum.

* For since, by the supposition, $r : s :: x - a : x - b$, therefore by division, $r - s : s :: b - a : x - b$, which is the 2d rule.

PRACTICAL QUESTIONS IN ARITHMETIC.

1. The swiftest velocity of a cannon-ball is about 2000 feet in a second of time. Then in what time, at that rate, would such a ball move from the earth to the sun, admitting the distance to be 100 millions of miles, and the year to contain 365 days 6 hours? Ans. $8\frac{4808}{13149}$ years.

2. What is the ratio of the velocity of light to that of a cannon-ball, which issues from the gun with a velocity of 1500 feet per second; light passing from the sun to the earth in $8\frac{1}{3}$ minutes? Ans. the ratio of 704000 to 1.

3. The slow or parade-step being 70 paces per minute at 28 inches each pace, it is required to determine at what rate per hour that movement is? Ans. $1\frac{113}{132}$ miles.

4. The quick-time or step in marching, being 2 paces per second, or 120 per minute, at 28 inches each, at what rate per hour does a troop march on a route, and how long will they be in arriving at a garrison 20 miles distant, allowing a halt of one hour by the way to refresh?

Ans. The rate is $3\frac{2}{11}$ miles an hour, and the time $7\frac{2}{7}$ hours, or 7 hours $17\frac{1}{7}$ min.

5. A wall was to be built 700 yards long in 29 days. Now, after 12 men had been employed on it for 11 days, it was found that they had completed only 220 yards of the wall. It is required to determine how many men must be added to the former, that the whole number of them may just finish the wall in the time proposed, at the same rate of working? Ans. 4 men to be added.

6. Determine how far 500 millions of guineas will reach, when laid down in a straight line touching one another; supposing each guinea to be an inch in diameter, as it is very nearly? Ans. 7891 miles, 728 yds., 2 ft. 8 in.

7. Two persons, A and B, being on opposite sides of a wood, which is 536 yards about, begin to go round it, both the same way, at the same instant of time; A goes at the rate of 11 yards per minute, and B 34 yards in 3 minutes; the question is, how many times will the wood be gone round before the quicker overtake the slower? Ans. 17 times.

8. A can do a piece of work alone in 12 days, and B alone in 14; in what time will they both together perform a like quantity of work? Ans. $6\frac{6}{13}$ days.

9. A person who is possessed of a $\frac{3}{5}$ share of a copper-mine, sold $\frac{3}{4}$ of his interest in it for 1800*l.*; what was the reputed value of the whole at the same rate? Ans. 4000*l.*

10. A person, after spending 20*l.* more than $\frac{1}{4}$ of his yearly income, had then remaining 30*l.* more than the half of it; what was his income? Ans. 200*l.*

11. The hour and minute-hands of a clock are exactly together at 12 o'clock; when are they next together? Ans. $1\frac{1}{11}$ hr., or 1 hr. $5\frac{5}{11}$ min.

12. If a gentleman, whose annual income is 1500*l.*, spend 20 guineas a-week; whether will he save or run in debt, and how much in the year? Ans. Save 408*l.*

13. A person bought 180 oranges at 2 a penny, and 180 more at 3 a penny; after which he sold them out again at 5 for 2 pence; did he gain or lose by the bargain? Ans. He lost 6 pence.

14. If a quantity of provisions serves 1500 men 12 weeks, at the rate of 20 ounces a-day for each man; how many men will the same provisions maintain for 20 weeks, at the rate of 8 ounces a-day for each man? Ans. 2250 men.

15. In the latitude of London, the distance round the earth, measured on the parallel of latitude, is about 15,550 miles; now, as the earth turns round in 23 hours 56 minutes, at what rate per hour is the city of London carried from west to east? Ans. $649\frac{259}{359}$ miles an hour.

16. A father left his son a fortune, $\frac{1}{4}$ of which he ran through in 8 months; $\frac{3}{7}$ of the remainder lasted him 12 months longer; after which he had 820*l.* left. What sum did the father bequeath his son? Ans. 1913*l.* 6*s.* 8*d.*

17. If 1000 men, besieged in a town, with provisions for 5 weeks, allowing each man 16 ounces a-day, be reinforced with 500 men more; and supposing that they cannot be relieved till the end of 8 weeks; how many ounces a-day must each man have that the provision may last that time? Ans. $6\frac{2}{3}$ ounces.

18. A younger brother received 8400*l.*, which was just $\frac{7}{9}$ of his elder brother's fortune. What was the father worth at his death? Ans. 19,200*l.*

19. A person looking on his watch, was asked what was the time of the day, who answered, "It is between 5 and 6;" but a more particular answer being required, he said "that the hour and minute-hands were then exactly together." What was the time? Ans. $27\frac{3}{11}$ min. past 5.

20. If 20 men perform a piece of work in 12 days, how many men will accomplish another, thrice as large, in one-fifth of the time? Ans. 300.

21. A father devised $\frac{7}{18}$ of his estate to one of his sons, and $\frac{7}{18}$ of the residue to another, and the surplus to his relict for life. The children's legacies were found to be 514*l.* 6*s.* 8*d.* different. What money did he leave the widow the use of? Ans. 1270*l.* 1*s.* $9\frac{1}{49}$*d.*

22. A person making his will, gave to one child $\frac{13}{20}$ of his estate, and the rest to another. When these legacies came to be paid, the one turned out to be 1200*l.* more than the other. What did the testator die worth? Ans. 4000*l.*

23. Two persons, A and B, travel between London and Exeter. A leaves Exeter at 8 o'clock in the morning, and walks at the rate of 3 miles an hour, without intermission; and B sets out from London at 4 o'clock the same evening, and walks for Exeter at the rate of 4 miles an hour constantly. Now, supposing the distance between the two cities to be 130 miles, where will they meet? Ans. $69\frac{3}{7}$ miles from Exeter.

24. One hundred eggs being placed on the ground, in a straight line, at the distance of a yard from each other; how far will a person travel who shall bring them one by one to a basket, which is placed at one yard from the first egg? Ans. 10,100 yards, or 5 miles and 1300 yards.

25. The clocks of Italy go on to 24 hours; then how many strokes do they strike in one complete revolution of the index? Ans. 300.

26. One Sessa, an Indian, having invented the game of chess, showed it to his prince, who was so delighted with it, that he promised him any reward he should ask; on which Sessa requested that he might be allowed one grain of wheat for the first square on the chess-board, 2 for the second; 4 for the third, and so on, doubling continually to 64, the number of squares. Now, supposing a pint to contain 7680 of these grains, and one quarter or 8 bushels to be worth 27*s.* 6*d.*, it is required to compute the value of all the corn.

Ans. 6450468216285*l.* 17*s.* $3\frac{1}{4}$*d.* $\frac{32757}{32768}$.

27. A person increased his estate annually by 100*l.* more than the $\frac{1}{4}$ part of it; and at the end of 4 years found that his estate amounted to 10342*l.* 3*s.* 9*d.* What had he at first? Ans. 4000*l.*

28. Paid 1012*l.* 10*s.* for a principal of 750*l.*, taken in 7 years before; at what rate per cent. per annum did I pay interest? Ans. 5*l.* per cent.

29. Divide 1000*l.* among A, B, C; so as to give A 120*l.* more, and B 95*l.* less than C. Ans. A 445*l.*, B 230*l.*, C 325*l.*

30. A person being asked the hour of the day, said, the time past noon is equal to $\frac{4}{5}$ths of the time till midnight. What was the time?

Ans. 20 min. past 5.

31. Suppose that I have $\frac{3}{16}$ of a ship, whose whole worth is 1200*l.*; what part of her have I left after selling $\frac{2}{5}$ of $\frac{4}{9}$ of my share, and what is it worth?

Ans. $\frac{37}{240}$; worth 185*l.*

32. What number is that, from which if there be taken $\frac{2}{7}$ of $\frac{3}{8}$, and to the remainder be added $\frac{8}{15}$ of $\frac{5}{16}$; the sum will be 10? Ans. $9\frac{79}{84}$.

33. There is a number which, if multiplied by $\frac{2}{3}$ of $\frac{4}{5}$ of $1\frac{1}{2}$, will produce 1. What is the square of that number? Ans. $1\frac{9}{16}$.

34. What length must be cut off a board, $8\frac{1}{2}$ inches broad, to contain a square foot, or as much as 12 inches in length, and 12 in breadth?

Ans. $16\frac{16}{17}$ inches.

35. What sum of money will amount to 138*l.* 2*s.* 6*d.* in 15 months, at 5 per cent. per annum simple interest? Ans. 130*l.*

36. A father divided his fortune among his three sons, A, B, C, giving A 4 as often as B 3, and C 5 as often as B 6; what was the whole legacy, supposing A's share was 4000*l.* Ans. 9500*l.*

37. A young hare starts 40 yards before a greyhound, and is not perceived by him till she has been up 40 seconds; she scuds away at the rate of 10 miles an hour, and the dog, on view, makes after her at the rate of 18. How long will the course hold, and what ground will be run over, counting from the outsetting of the dog? Ans. $60\frac{5}{22}$ sec., and 530 yds. run.

38. Divide 9360*l.* among A, B, and C, in such a manner that A's share may be to B's as 7 to 6, and B's to C's as 4 to $3\frac{1}{3}$.

Ans. A's share 3640*l.*; B's 3120*l.*; and C's 2600*l.*

39. If $\frac{2}{3}$ of a steam-ship be purchased for 15,360*l.* 13*s.* 4*d.*, how much will be gained per cent. by selling half the vessel for 12,902*l.* 19*s.* $2\frac{2}{3}$*d.*?

Ans. 12*l.* per cent.

40. Find the cube root of ·068 to eight places of decimals, contracting the work for the last four figures. Ans. ·40816551.

41. Suppose 2*l.* and $\frac{1}{8}$ of $\frac{1}{3}$ of a pound will purchase 3 yards and $\frac{2}{3}$ of $\frac{3}{5}$ of a yard of cloth; how much may be purchased by 9 shillings and $\frac{3}{8}$ of a shilling?

Ans. $\frac{3}{4}$ of a yard.

42. Divide 43*l.* 12*s.* 9*d.* among 7 men, 9 women, and 3 boys, and give a woman $\frac{3}{5}$ of a man's share, and a boy $\frac{6}{7}$ of a woman's.

Ans. A boy's share 1*l.* 12*s.* $2\frac{1}{4}$ $\frac{12}{61}$*d.*
A woman's 1 17 $6\frac{1}{2}$ $\frac{89}{122}$.
A man's . 3 2 $7\frac{1}{4}$ $\frac{67}{122}$.

43. A workman was hired for 24 days, at 4*s.* 6*d.* per day, for every day he worked; but for every day he was absent he was to forfeit 1*s.* 6*d.* How many days did he work when the balance due to him was 3*l.* 18*s.*; and also how many days was he absent, when he had to receive only one day's wages? To be done without position.

Ans. 19 days, in the former case; and $17\frac{1}{4}$ days, in the latter case.

44. The interest of a certain sum for 12 years and 9 months, at 4*l.* per cent. simple interest, was found to be 185*l.* more than the interest of the same sum for $6\frac{1}{4}$ years, at 5*l.* per cent. Find the sum without the aid of the rule of position? Ans. 1000*l.*

ALGEBRA.

DEFINITIONS AND NOTATION.

1. ALGEBRA is that department of Mathematics which enables us, by the aid of certain symbols, to abridge and generalize the reasoning employed in the solution of all questions relating to numbers.

These questions are of two kinds:—

The *Theorem*, whose object is to demonstrate certain properties and relations which exist in numbers which are known and given.

The *Problem*, whose object is to discover certain numbers which are unknown by means of other numbers which are known, and which bear a relation to the unknown numbers, indicated by the conditions of the problem.

2. The principal symbols employed in algebra are the following:—

I. The letters of the alphabet, a, b, c, &c., which are employed to denote the numbers which are the object of our reasonings.

II. The sign $+$ which is named *plus*, and is employed to denote the addition of two or more numbers.

Thus $12 + 30$ signifies 12 *plus* 30, or, 12 *augmented by* 30. In like manner $a + b$ signifies *a plus b*, or, the number designated by *a augmented by* the number designated by b.

III. The sign — which is named *minus*, and is employed to denote the subtraction of one number from another.

Thus $54 - 23$ signifies 54 *minus* 23, or, 54 *diminished by* 23. In like manner $a - b$ signifies *a minus b*, or, the number designated by *a diminished by* the number designated by b.

The sign $\sim$ is sometimes employed to denote *the difference* of two numbers, when it is not known which is the greater. Thus $a \sim b$ signifies *the difference of a* and b, when it is not known whether the number designated by a be less or greater than the number designated by b.

IV. The sign $\times$ which is named *into*, and is employed to denote the multiplication of two or more numbers.

Thus 72×26 signifies 72 *into* 26, or, 72 *multiplied by* 26. In like manner, $a \times b$ signifies *a into b*, or, *a multiplied by b*; and $a \times b \times c$ signifies the continued product of the numbers designated by a, b, c; and so on for any number of factors.

The process of multiplication is also frequently indicated by placing a point between the successive factors; thus, $a \,.\, b \,.\, c \,.\, d$ signifies the same thing as $a \times b \times c \times d$.

In general, however, when numbers are represented by letters, their multiplication is indicated by writing the letters in succession, without the interposition of any sign. Thus $a\,b$ signifies the same thing as $a\,.\,b$, or $a \times b$; and $a\,b\,c\,d$ is equivalent to $a\,.\,b\,.\,c\,.\,d$, or $a \times b \times c \times d$.

It must be remarked, that the notation $a\,.\,b$ or $a\,b$ can be employed only when the numbers are designated by letters; if, for example, we wished to represent the product of the numbers 5 and 6 in this manner, 5.6 would be confounded with an integer followed by a decimal fraction, and 56 would signify the number *fifty-six*, according to the common system of notation.

For the sake of brevity, however, the multiplication of numbers is sometimes expressed by placing a point between them in cases where no ambiguity can arise from the use of this symbol. Thus, 1 . 2 . 3 . 4, may represent the continued product of the numbers, 1, 2, 3, 4; and $\frac{2}{3}\,.\,\frac{7}{9}\,.\,\frac{6}{11}$ may represent the product of $\frac{2}{3}$, $\frac{7}{9}$, and $\frac{6}{11}$.

V. The sign $\div$ which is named *by*, and when placed between two numbers is employed to denote that the former is to be divided by the latter.

Thus $24 \div 6$ signifies 24 *by* 6, or, 24 *divided by* 6. In like manner $a \div b$ signifies a *by* b, or, a *divided by* b.

In general, however, the division of two numbers is indicated by writing the dividend above the divisor, and drawing a line between them. Thus $24 \div 6$ and $a \div b$ are usually written $\frac{24}{6}$ and $\frac{a}{b}$.

VI. The sign $=$ which is named *is equal to*, and when placed between two numbers denotes that they are equal to each other.

Thus $56 + 6 = 62$ signifies that the sum of 56 and 6 *is equal to* 62. In like manner, $a = b$ signifies that a *is equal to* b, and $a + b = c - d$ signifies that a *plus* b *is equal to* c *minus* d, or, that the sum of the numbers designated by a and b *is equal to* the difference of the numbers designated by c and d.

VII. The sign $\angle$ which is named *is unequal to*, and when placed between two numbers denotes that one of them is greater than the other, the opening of the sign being turned towards the greater number.

Thus $a \,7\, b$ signifies that a *is greater than* b, and $a \angle b$ signifies that a *is less than* b.

VIII. The *coefficient* is a sign which is employed to denote that a number designated by a letter, or some combination of letters, is added to itself a certain number of times.

Thus instead of writing $a + a + a + a + a$, which represents 5 a's added together, we write $5\,a$. In like manner $10\,a\,b$ will signify the same thing as $a\,b + a\,b + a\,b + a\,b + a\,b + a\,b + a\,b + a\,b + a\,b + a\,b$, or *ten times* the product of a and b.

The *coefficient*, then, is a number written to the left of another number, represented by one or more letters, and denotes the number of times hat the given letter, or combination of letters, is to be repeated.

When no coefficient is expressed, the coefficient 1 is always understood; thus $1\,a$ and a signify the same thing.

IX. The *exponent* or *index* is a sign which is employed to denote that a number designated by a letter is multiplied by itself a certain number of times.

Thus instead of writing $a \times a \times a \times a \times a$, or $a\,a\,a\,a\,a$, which represents five a's multiplied together, we write a^5, where 5 is called the *exponent* or *index* of a. Similarly $b \times b \times b \times b \times b \times b \times b \times b \times b \times b$, or $b\,.\,b\,.\,b\,.\,b\,.\,b\,.\,b\,.\,b\,.\,b\,.\,b\,.\,b$, or $b\,b\,b\,b\,b\,b\,b\,b\,b\,b$; or the continued product of 10 b's is written more briefly b^{10}, where 10 is the *exponent* or *index* of b.

The *exponent* or *index* of a number is, therefore, a number written a little above a letter to the right, and denotes the number of times which the number designated by the letter enters as a factor into a product. When no exponent is expressed, the exponent 1 is always understood; thus a^1 and a signify the same thing.

The products thus formed by the successive multiplication of the same number by itself, are in general called the *powers* of that number. Thus a is *the first power* of a; $a \times a = a\,a = a^2$ is *the second power* of a, or *the square* of a; $a\,a\,a = a^3$ is *the third power*, or *cube* of a; $a\,a\,a\,a\,a = a^5$ is *the fifth power* of a, and $a\,a\,a\,a\,\ldots\ldots$ to n factors $= a^n$, is the nth power of a, or the power of a designated by the number n.

X. The *square root* of any expression is that quantity which, when multiplied by itself will produce the proposed expression, and, in numbers, is generally denoted by the symbol $\sqrt{\ }$, which is called the *radical sign*. Thus the square root of 9 is $\sqrt{9} = 3$, and $\sqrt{a^2} = a$, is the square root of a^2; for in the former case $3 \times 3 = 9$, and in the latter $a \times a = a^2$.

XI. The *cube root* of any expression is that quantity which, when multiplied twice by itself, will produce the proposed expression. The *fourth*, or *biquadrate root* of any expression is that quantity which, when multiplied three times by itself, produces the given expression; and the nth root of any expression is that quantity which, multiplied $(n-1)$ times by itself, produces the proposed expression. Thus the *cube root* of 8 is 2; for $2 \times 2 \times 2 = 8$, the *fourth root* of a^4 is a; for $a\,.\,a\,.\,a\,.\,a = a^4$, and the nth root of $x^n\,y^n$ is $x\,y$; for $x\,y \times x\,y \times x\,y \,\ldots\ldots$ to n factors $= x\,.\,x\,.\,x\,.\,x \,\ldots\ldots$ to n factors $\times\, y\,.\,y\,.\,y\,.\,y \,\ldots\ldots$ to n factors $= x^n\,y^n$.

The roots of expressions are frequently designated by fractional or decimal exponents, the figure in the numerator of the fractional exponent denoting the power to which the expression is to be raised or involved, and the figure in the denominator denoting the root to be extracted or evolved. Thus the symbol of operation for the *square root* of a is either $\sqrt{a}$ or $a^{\frac{1}{2}}$; for the *cube root* it is $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$; for the fourth root $\sqrt[4]{a}$, or $a^{\frac{1}{4}}$; and $\sqrt[n]{a}$, or $a^{\frac{1}{n}}$, denotes the nth root of a. Also $\sqrt[6]{a^5}$, or $a^{\frac{5}{6}}$, denotes the *sixth root* of the *fifth power* of a; and $a^{\frac{m}{n}}$, or $\sqrt[n]{a^m}$, signifies the nth *root* of the mth *power* of a.

XII. A *rational quantity* is that which has no radical sign, or fractional exponent annexed to it, as $3\,m\,n$, or $5x^2\,y^2$.

XIII. An *irrational quantity* is that which has no exact root, and is expressed by means of the radical sign $\sqrt{\ }$, or a fractional exponent, as $\sqrt{2}$ $\sqrt[3]{a^2}$, or $x^{\frac{1}{2}}\,y^{\frac{2}{3}}$.

XIV. The *reciprocal* of any quantity is unity divided by that quantity; thus the reciprocals of a^2, x^3, y^5, $z^{\frac{1}{2}}$, are respectively $\frac{1}{a^2}, \frac{1}{x^3}, \frac{1}{y^5}, \frac{1}{z^{\frac{1}{2}}}$; but the following notation is generally used, as being more commodious: thus the fractions $\frac{1}{a^2}, \frac{1}{x^3}, \frac{1}{y^5}, \frac{1}{z^{\frac{1}{2}}}$, are expressed by a^{-2}, x^{-3}, y^{-5}, $z^{-\frac{1}{2}}$.

XV. The following characters are used to connect several quantities together, viz.:—

vinculum or *bar* $\overline{}$
parentheses ( )
braces or *brackets* $\{ \quad \}$.

Thus $\overline{m+n}\,.\,x$, or $(m+n)\,x$ signifies that the quantity denoted by $m+n$ is to be multiplied by x, and $\left\{\frac{m}{n}+\frac{p}{q}\right\}.\left\{\frac{m}{n}-\frac{p}{q}\right\}$ signifies that $\frac{m}{n}+\frac{p}{q}$ is to be multiplied by $\frac{m}{n}-\frac{p}{q}$.

XVI. The signs $\therefore$ *therefore* or *consequently*, and $\because$ *because*, are used to avoid the too frequent repetition of these words.

XVII. Every number written in algebraic language, that is, by aid of algebraic symbols, is called an *algebraic quantity*, or, a *literal quantity*, or, an *algebraic expression*.

Thus $3\,a$ is the algebraic expression for three times the number a; $5\,a^2$ is the algebraic expression for five times the square of the number a; $7\,a^5\,b^3$ is the algebraic expression for seven times the fifth power of a multiplied by the cube of b.

$3\,a^2-6\,b^3\,c^4$ is the algebraic expression for the difference between three times the square of a and six times the cube of b multiplied by the fourth power of c.

$2\,a-3\,b^2\,c^3+4\,d^4\,e^5\,f^6$ is the algebraic expression for twice a, diminished by three times the square of b multiplied by the cube of c and augmented by four times the fourth power of d multiplied by the product of the fifth power of e and the sixth power of f.

XVIII. An algebraic quantity, which is not combined with any other by the sign of addition or subtraction, is called a *monomial*, or, a *quantity of one term*, or simply, a *term*. Thus, $3\,a^2$, $4\,b^2$, $6\,c$, are *monomials*.

An algebraic expression, which is composed of several terms, separated from each other by the signs + or — is called generally a *polynomial*. Thus, $3\,a^2+4\,b^2-6\,c+d$, is a polynomial.

A polynomial, consisting of two terms only, is usually called a *binomial*, when consisting of three terms, a trinomial. Thus, $a+b$, $3\,b^2\,c-x\,z$, are binomials, and $a+b-c$, $3\,m^2\,n^5-6\,p^3\,r+9\,d$, are trinomials.

XIX. The *numerical value* of an algebraic expression is the number which results from giving particular values to the letters which compose the expression, and performing the arithmetical operations indicated by the algebraic symbols. This numerical value will, of course, depend upon the particular values assigned to the letters. Thus the numerical value of $2\,a^3$ is 54 when we make $a=3$, for the cube of 3 is 27, and twice 27 is 54. The numerical value of the same expression will be 250 if we make $a=5$; for the cube of 5 is 125, and twice 125 is 250.

The numerical value of a polynomial undergoes no change, however we may transpose the order of the terms, provided we preserve the proper sign of each. Thus the polynomials $4\,a^3-3\,a^2\,b+5\,a\,c^2$, $4\,a^3+5\,a\,c^2-3\,a^2\,b$, $5\,a\,c^2-3\,a^2\,b+4\,a^3$, have all the same numerical value. This follows manifestly from the nature of arithmetical addition and subtraction.

XX. Of the different terms which compose a polynomial, some are preceded by the sign +, others by the sign —. The former are called *additive*, or *positive* terms, the latter, *subtractive*, or *negative* terms.

The first term of a polynomial is not in general preceded by any sign; in that case the sign $+$ is always understood.

Terms composed of the same letters, affected with the same exponents, are called *similar terms.*

Thus, $7ab$ and $3ab$ are similar terms, so are $6a^2c$ and $7a^2c$; also, $10ab^3c^4d$ and $2ab^3c^4d$; for they are composed of the same letters, and these letters in each are affected with the same exponents. On the other hand, $8ab^3c$ and $3a^2b^3c$ are not similar terms, for although composed of the same letters, these letters are not affected with the same exponents in each.

Examples of the numeral values of algebraic expressions:—

Let $a = 4$, $b = 3$, $c = 2$; then will

(1) $a + b - c = 4 + 3 - 2 = 7 - 2 = 5$

(2) $a^2 + ab + b^2 = 4^2 + 4 \times 3 + 3^2 = 16 + 12 + 9 = 37$

(3) $ac - ab + bc = 4 \times 2 - 4 \times 3 + 3 \times 2 = 8 - 12 + 6 = 2$

(4) $\dfrac{a^2 + b^2 - c^2}{ab - ac + bc} = \dfrac{4^2 + 3^2 - 2^2}{4 \times 3 - 4 \times 2 + 3 \times 2} = \dfrac{16 + 9 - 4}{12 - 8 + 6} = \dfrac{21}{10}$

(5) $\sqrt{(a+b)c} - \sqrt[3]{(a-b)c^3} = \sqrt{(4+3)\times 2} - \sqrt[3]{(4-3)\times 2^3} = \sqrt{14} - \sqrt[3]{8}$
$= 3{\cdot}7416574 - 2 = 1{\cdot}7416574$

(6) $\dfrac{a+b}{a-c} + \dfrac{a-c}{b+c} - \dfrac{a-b}{a+b} = \dfrac{7}{2} + \dfrac{2}{5} - \dfrac{1}{7} = \dfrac{263}{70}.$

ADDITION.

1. Addition is the collecting of several similar quantities into one term or sum, and the connecting of dissimilar quantities by their respective signs.

The rule of addition may be divided into two cases:—

(1) When the quantities are similar, and have the same signs.

(2) When the quantities are similar, and have different signs.

Case I.

2. *When the quantities are similar, and have the same signs.*

Add the coefficients; affix the letter or letters of the similar terms, and prefix the common sign $+$ or $-$.*

Thus $a+2a+3a+4a+5a=(1+2+3+4+5)a=15a$

$(-a)+(-2a)+(-3a)+(-4a)= -(1+2+3+4)a= -10a$

$(2a+3b)+(4a+5b)=(2a+4a)+(3b+5b)=6a+8b.$

* The truth of this rule is evident; for suppose $3a$ and $5a$ are to be added together; then by the definition of a coefficient we have

$$5a = a + a + a + a + a$$
$$3a = a + a + a$$
Hence $5a + 3a = a + a + a + a + a + a + a + a = 8a.$
Similarly, $-5a = (-a) + (-a) + (-a) + (-a) + (-a)$
$$-3a = (-a) + (-a) + (-a).$$
Hence $-5a + (-3a) = (-a) + (-a) + (-a) + (-a) + (-a) + (-a) + (-a) + (-a)$
$$= 8(\quad) = -8a.$$

EXAMPLES.

(1)	(2)	(3)	(4)	(5)
$3a$	abc	$9axy$	$-5bx$	$\sqrt{a+x}$
$7a$	$2abc$	$3axy$	$-2bx$	$2\sqrt{a+x}$
$2a$	$7abc$	$7axy$	$-bx$	$5\sqrt{a+x}$
a	$3abc$	$5axy$	$-bx$	$\sqrt{a+x}$
$6a$	abc	axy	$-4bx$	$7\sqrt{a+x}$
$8a$	$5abc$	$5axy$	$-10bx$	$4\sqrt{a+x}$
$27a$	$19abc$			

(6)	(7)	(8)
$3a^2+b^2$	$2x^2-xy$	$20(a^2-b^2)^{\frac{1}{2}}-15\sqrt{x^2-y^2}$
$2a^2+3b^2$	$4x^2-7xy$	$\sqrt{a^2-b^2}-7\sqrt{x^2-y^2}$
$6a^2+5b^2$	$3x^2-4xy$	$12\sqrt{a^2-b^2}-\sqrt{x^2-y^2}$
a^2+7b^2	x^2-xy	$4(a^2-b^2)^{\frac{1}{2}}-3(x^2-y^2)^{\frac{1}{2}}$
a^2+6b^2	$8x^2-7xy$	$2(a^2-b^2)^{\frac{1}{2}}-5(x^2-y^2)^{\frac{1}{2}}$

CASE II.

8. *When the quantities are similar, and have different signs.*

Collect into one sum the coefficients affected with the sign $+$, and also those affected with the sign $-$; to the difference of these sums affix the common literal quantity, and prefix the sign $+$ or $-$, according as the sum of the $+$ or $-$ coefficients is the greater.*

Thus $a-2a+3a-4a+5a=(1+3+5)a-(2+4)a=9a-6a=3a$

And $3x+4y-2x+3y=(3-2)x+(4+3)y=x+7y$.

EXAMPLES.

(1)	(2)	(3)
$a+b$	$xy-ab$	$\sqrt{x^2+y^2}-m^2+n^2-2mn$
$-2a+3b$	$2xy+3ab$	$-2\sqrt{x^2+y^2}+3m^2-3n^2+5mn$
$3a-4b$	$-5xy+7ab$	$-5\sqrt{x^2+y^2}-4m^2+5n^2-7mn$
$-5a+6b$	$-xy-3ab$	$2(x^2+y^2)^{\frac{1}{2}}+12m^2-2\frac{1}{2}n^2+mn$
$7a-b$	$8xy-9ab$	$8(x^2+y^2)^{\frac{1}{2}}-8m^2-\frac{1}{2}n^2-6mn$
$4a+5b$		

* The truth of this will be obvious; for to add $5a$ and $-3a$ together, we have

$$5a=a+a+a+a+a$$
$$-3a=(-a)+(-a)+(-a)$$

Hence $5a+(-3a)=a+a+a+a+a+(-a)+(-a)+(-a)$.
$$=a+a=2a.$$

Similarly $2a+(-5a)=a+a+(-a)+(-a)+(-a)+(-a)+(-a)$
$$=+(-a)+(-a)+(-a)$$
$$=3(-a)=-3a.$$

(4)

$$\begin{array}{rrr} 5ax^{\frac{1}{2}} & -\sqrt[3]{x+y} & +\ (a-b) \\ -\ 7a\sqrt{x} & +2\,(x+y)^{\frac{1}{3}} & -\ 3(a-b) \\ 12a\sqrt{x} & -3\sqrt[3]{x+y} & +12(a-b) \\ -\ 3a\sqrt{x} & -4\sqrt[3]{x+y} & -\ (a-b) \\ -\ \ ax^{\frac{1}{2}} & +\ (x+y)^{\frac{1}{3}} & -\ 3(a-b) \end{array}$$

(5)

$$\begin{array}{rr} 2\sqrt{xy+xz+yz} & +\sqrt[4]{ax+by} \\ -\ 5\sqrt{xy+xz+yz} & -3\,(ax+by)^{\frac{1}{4}} \\ 12\ (xy+xz+yz)^{\frac{1}{2}} & +5\,(ax+by)^{\frac{1}{4}} \\ -\ 3\sqrt{xy+xz+yz} & -2\sqrt[4]{ax+by} \\ (xy+xz+yz)^{\frac{1}{2}} & +\ (ax+by)^{\frac{1}{4}} \end{array}$$

(6)

$$\begin{array}{r} a+b+c+d+e-f \\ a+b+c+d-e+f \\ a+b+c-d+e+f \\ a+b-c+d+e+f \\ a-b+c+d+e+f \\ -a+b+c+d-e+f \end{array}$$

(7)

$$\begin{array}{rr} 4(a+b)\sqrt{x^2-y^2} & -\ 2(a-b)\sqrt{x^2+y^2} \\ -\ 3(a+b)\sqrt{x^2-y^2} & +\ \ (a-b)\sqrt{x^2+y^2} \\ -\ \ (a+b)\ (x^2-y^2)^{\frac{1}{2}} & +\ 3(a-b)\ (x^2+y^2)^{\frac{1}{2}} \\ 6(a+b)\ (x^2-y^2)^{\frac{1}{2}} & -\ \ (a-b)\ (x^2+y^2)^{\frac{1}{2}} \\ 10(a+b)\sqrt{x^2-y^2} & -\ 5(a-b)\ (x^2+y^2)^{\frac{1}{2}} \\ -\ 2(a+b)\ (x^2-y^2)^{\frac{1}{2}} & +\ 4(a-b)\sqrt{x^2+y^2} \end{array}$$

4. Dissimilar quantities can only be collected by writing them in succession, and prefixing to each its respective sign. Thus $9\,x\,y$, $-\,5\,c\,d$, and $3\,a\,b$, are dissimilar quantities, and their sum is $9\,x\,y + 3\,a\,b - 5\,c\,d$. In like manner $2\,a\,b$, $3\,a\,b^2$, $4\,a\,b^3$ are dissimilar quantities, and their sum is $2\,a\,b + 3\,a\,b^2 + 4\,a\,b^3$; which, however, admits of another form of expression, as will be explained in the rule of Division. When several polynomials, containing both similar and dissimilar quantities, are to be collected into one polynomial, the process of addition will be much facilitated by writing all the similar terms under each other in vertical columns.

EXAMPLES.

(1.) Add together $a\,x + 2\,b\,y + c\,z$; $\sqrt{x} + \sqrt{y} + \sqrt{z}$; $3\,y^{\frac{1}{2}} - 2\,x^{\frac{1}{2}} + 3\,z^{\frac{1}{2}}$, $4\,c\,z - 3\,a\,x - 2\,b\,y$; $2\,a\,x - 4\sqrt{y} - 2\,z^{\frac{1}{2}}$.

$$\begin{array}{l} a\,x + 2\,b\,y + c\,z + \sqrt{x} + \sqrt{y} + \sqrt{z} \\ -\,3\,a\,x - 2\,b\,y + 4\,c\,z - 2\,x^{\frac{1}{2}} + 3\,y^{\frac{1}{2}} + 3\,z^{\frac{1}{2}} \\ 2\,a\,x \qquad\qquad\qquad\qquad -4\sqrt{y} - 2\,z^{\frac{1}{2}} \\ \hline \qquad\qquad 5\,c\,z - \sqrt{x} + 2\,\sqrt{z} = \text{sum required.} \end{array}$$

(2.) Add together,
$4\,a^2\,b + 3\,c^3\,d - 9\,m^2 n$; $4\,m^2 n + a\,b^2 + 5\,c^3\,d + 7\,a^2\,b$; $6\,m^2\,n - 5\,c^3\,d + 4\,m\,n$; $-\,8\,a\,b^2$; $7\,m\,n^2 + 6\,c^3\,d - 5\,m^2\,n - 6\,a^2\,b$; $7\,c^3\,d - 10\,a\,b^2 - 8\,m^2\,n - 10\,d^4$; and $12\,a^2\,b - 6\,a\,b^2 + 2\,c^3\,d + m\,n$.

Arranging the similar terms in vertical columns, we have

$$\begin{array}{rrrrrrr}
4a^2b+ & 3c^3d- & 9m^2n & & & & \\
7a^2b+ & 5c^3d+ & 4m^2n+ & ab^2 & & & \\
- & 5c^3d+ & 6m^2n- & 8ab^2 & +4mn^2 & & \\
-6a^2b+ & 6c^3d- & 5m^2n & & +7mn^2 & & \\
+ & 7c^3d- & 8m^2n- & 10ab^2 & & -10d^4 & \\
12a^2b+ & 2c^3d & & -6ab^2 & & & +mn \\
\hline
\end{array}$$

$$17a^2b+18c^3d-12m^2n-23ab^2+11mn^2-10d^4+mn=\text{sum}$$

(3.) Add $11bc+4ad-8ac+5cd$; $8ac+7bc-2ad+4mn$; $2cd-3ab+5ac+an$; and $9an-2bc-2ad+5cd$ together.

(4.) Add together

$$\begin{array}{l}
2ab^2+3ac^2-8cx^2+9b^2x-8hy^2-10hy \\
5a^3-4ab^2-7bx^2-b^2x-4hy^2-15hy \\
5hy-hy^2+11x+14b^3-22ac^2-10x^2 \\
19ac^2-8b^2x+9x^2+6hy+\tfrac{?}{}hy^2+2ab^2
\end{array}$$

(5.) Add together $a^3-b^3+3a^2b-5ab^2$; $3a^3-4a^2b+3b^3-3ab^2$; $a^3+b^3+3a^2b$; $2a^3-4b^3-5ab^2$; $6a^2b+10ab^2$, and $-6a^3-7a^2b+4ab^2+2b^3$.

(6.) Add $\sqrt{x^2+y^2}-\sqrt{x^2-y^2}-5xy$; $-3(x^2-y^2)^{\frac{1}{2}}+8xy-2(x^2+y^2)^{\frac{1}{2}}$; $2\sqrt{x^2+y^2}-3xy-5\sqrt{x^2-y^2}$; $7xy+10\sqrt{x^2-y^2}-12\sqrt{x^2+y^2}$, and $xy+\sqrt{x^2-y^2}+\sqrt{x^2+y^2}$ together.

Answers.

(3.) $16bc+5ac+12cd+4mn-3ab+10an$.

(4.) $5a^3+14b^3-8cx^2-7bx^2-x^2+11x-9hy^2-2hy^2-5hy-9hy$.

(5.) $a^3+a^2b+ab^2+b^3$.

(6.) $2\sqrt{x^2-y^2}-10\sqrt{x^2+y^2}+8xy$.

5. When the coefficients are *literal* instead of *numeral,* that is, denoted by letters instead of numbers, their sum may be found by the rules for the addition of similar and dissimilar terms; and the sum thus found being enclosed in a parenthesis, and prefixed to the common literal quantity, will express the sum required.

Examples.

(1)

$$\begin{array}{l}
ax+by+cz \\
bx+cy+az \\
cx+ay+bz \\
\hline
\left.\begin{array}{l}(a+b+c)x \\ +(b+c+a)y \\ +(c+a+b)z\end{array}\right\}=\text{sum.}
\end{array}$$

(2)

$$\begin{array}{l}
3ax+(a+b)(x+y)+2mnz^2 \\
-ax-2(a+b)(x+y)-5mnz^2 \\
4mnz^2+5(a+b)(x+y)+10ax \\
2pqz^2+(p+q)(x+y)+2px \\
\hline
\left.\begin{array}{l}(12a+2p)x+\{4(a+b)+p+q\}(x+y) \\ +(mn+2pq)z^2\end{array}\right\}=\text{sum.}
\end{array}$$

(3)	(4)
$(a-b)\sqrt{x} + (m-n)\sqrt{y} + \sqrt{2}$	$(m+n)\, y^2-(a-b)x^2+axy$
$(a+c)\, x^{\frac{1}{2}} - (m-n)y^{\frac{1}{2}} + 2\sqrt{2}$	$(n-p)\, y^2-(2a+b)x^2-bxy$
$(b-c)\sqrt{x} + 3(m-n)\sqrt{y} - 3\sqrt{2}$	$(p-2n)y^2-(c-3a)x^2+cxy$
$(c-a)\sqrt{x} - 5(m-n)\sqrt{y} - 6\sqrt{2}$	$(q-m)y^2-(c+2d)x^2-dxy$

(5) Add $a\,x^2 + b\,y + c$ to $d\,x^2 + h\,y + k$.

(6) Add together $x^2 + x\,y + y^2$; $a\,x^2 - a\,x\,y + a\,y^2$; and $-b\,y^2 + b\,x\,y + b\,x^2$

(7) Add $\frac{1}{2}(x+y)$ and $\frac{1}{2}(x-y)$. Also $\frac{x^2 + x\,y + y^2}{2}$ and $\frac{x^2 - x\,y + y^2}{2}$.

(8) What is the sum of $(a+b)\,x + (c-d)\,y - x\sqrt{2}$; $(a-b)\,x + (3\,c + 2\,d)\,y + 5\,x\sqrt{2}$; $2\,b\,x + 3\,d\,y - 2\,x\sqrt{2}$; and $-3\,b\,x - d\,y - 4\,x\sqrt{2}$.

ANSWERS.

(3) $(a+c)\sqrt{x} - 2\,(m-n)\,\sqrt{y} - 6\sqrt{2}$.
(4) $q\,y^2 - (2\,c + 2\,d)\,x^2 + (a - b + c - d)\,x\,y$.
(5) $(a+d)\,x^2 + (b+h)\,y + c + k$.
(6) $(1 + a + b)\,x^2 + (1 - a + b)\,x\,y + (1 + a - b)\,y^2$.
(7) First part x. Second part $x^2 + y^2$.
(8) $(2\,a - b)\,x + (4\,c + 3\,d)\,y - 2\,x\sqrt{2}$.

SUBTRACTION.

6. THE subtraction of monomials is indicated by placing the sign — between the quantity to be subtracted and that from which it is to be taken. Thus $a-b$ signifies that the quantity denoted by b is to be subtracted from that denoted by a; and if $2\,x\,y$ is to be subtracted from $x^2 + y^2$, the result is represented by $x^2 + y^2 - 2\,x\,y$.

Place the quantity to be subtracted under that from which it is to be taken; change the signs of all the terms in the lower line from + to — and from — to +, or else conceive them to be changed, and then proceed as directed in Addition.

It is evident, that if all the terms of the quantity to be subtracted are affected with the sign +, we must take away, in succession, all the parts or terms of the quantity to be subtracted; and this is indicated by affecting all

its terms with the sign —. Also, if $c - d$ is to be subtracted from $a + b$, then c taken from $a + b$ is expressed by $a + b - c$; but if $c - d$, which is less than c by the quantity d, be taken from $a + b$, the former difference, $a + b - c$, will obviously be too small, and will require the addition of d to make up the deficiency; and therefore $c - d$ taken from $a + b$ is expressed by $a + b - c + d$, which is equivalent to the addition of $-c + d$ to $a + b$. Hence the reason for the change of the signs in the quantity to be subtracted. Or thus: Since $c - d$ is to be subtracted from $a + b$; then, if c be subtracted, we shall have subtracted too much by d; hence the remainder $a + b - c$ is too small by d; and therefore, to make up the defect, the quantity d must be added.

Examples.

(1)

$$\begin{array}{ll} \text{From} & 4a+3b-2c+8d \\ \text{Take} & a+2b+c+5d \\ \hline \text{Rem.} & 3a+b-3c+3d \end{array}$$

(2)

$$\begin{array}{ll} \text{From} & 12xy+3y^2-17x^2+3\sqrt{2} \\ \text{Take} & -5xy+7y^2-19x^2+2\sqrt{2} \\ \hline \text{Rem.} & 17xy-4y^2+2x^2+\sqrt{2} \end{array}$$

(3)

$$\begin{array}{r} 32a+3b \\ 5a+17b \\ \hline \\ \hline \end{array}$$

(4)

$$\begin{array}{r} 28ax^3-16a^2x^2+25a^3x-13a^4 \\ 18ax^3+20a^2x^2-24a^3x-7a^4 \\ \hline \\ \hline \end{array}$$

(5)

$$\begin{array}{r} 2(a+b)+3(a-x) \\ (a+b)-3(a-x) \\ \hline \\ \hline \end{array}$$

(6)

$$\begin{array}{r} 6aby-3yx+4zx \\ -2aby+6zx+2yx \\ \hline \\ \hline \end{array}$$

(7)

$$\begin{array}{l} \sqrt{x^2-y^2}+4(x+y)-3\sqrt{a+x} \\ 3(x+y)-2(x^2-y^2)^{\frac{1}{2}}+3(a+x)^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

(8)

$$\begin{array}{l} x^2+2xy+y^2 \\ x^2-2xy+y^2 \\ \hline \\ \hline \end{array}$$

(9)

$$\begin{array}{l} x^2-2xy+y^2+(x^2-y^2)+(2xy-y^2) \\ x^2+2xy-y^2+(x^2+y^2)-2(2xy-y^2) \\ \hline \\ \hline \end{array}$$

(10)

$$\begin{array}{l} 2a^2+ax+x^2-12a^2x+20ax^2-4x^3+6a^2x^2-10ax^3 \\ a^2-3ax+2x^2-16a^2x+12ax^2-12ax^3-4x^3+2a^2x^2 \\ \hline \\ \hline \end{array}$$

(11)

$$4y^2-4yx+x^2-2a(x+y)+6\sqrt{a^2-x^2}-8\sqrt[3]{b^2-y^2}$$
$$4x^2-4xy+y^2-4a(x+y)-10\sqrt[3]{b^2-y^2}+4\sqrt{a^2-x^2}$$

7. In order to *indicate* the subtraction of a polynomial, without actually performing the operation, we have simply to enclose the polynomial to be subtracted within *brackets* or *parentheses*, and prefix the sign —. Thus, $2a^3-3a^2b+4ab^2-(a^3+b^3+ab^2)$ signifies that the quantity $a^3+b^3+ab^2$ is to be subtracted from $2a^3-3a^2b+4ab^2$. When the operation is actually performed, we have by the rule

$$2a^3-3a^2b+4ab^2-(a^3+b^3+ab^2)=2a^3-3a^2b+4ab^2-a^3-b^3-ab^2$$
$$=a^3-3a^2b+3ab^2-b^3.$$

8. According to this principle, we may make polynomials undergo several transformations, which are of great utility in various algebraic calculations. Thus,

$$a^3-3a^2b+3ab^2-b^3=a^3-(3a^2b-3ab^2+b^3)$$
$$=a^3-b^3-(3a^2b-3ab^2)$$
$$=a^3+3ab^2-(3a^2b+b^3)$$
$$=-(-a^3+3a^2b-3a^2b+b^3)$$

And $x^2-2xy+y^2=x^2-(2xy-y^2)=y^2-(2xy-x^2)$.

Examples of Quantities with Literal Coefficients.

(1)

From $ax^2+byx+cy^2$
Take $dx^2-hxy+ky^2$

Rem. $(a-d)x^2+(b+h)xy+(c-k)y^2$.

(2)

From $(a+b)\sqrt{x^2+y^2}+(a+c)\ (a+x)^3$
Take $(a-b)\sqrt{x^2+y^2}+\quad c\ (a+x)^3$

Rem. $2b\sqrt{x^2+y^2}+a\,(a+x)^3$.

(3) From $m^2n^2x^2-2mnpqx+p^2q^2$ take $p^2q^2x^2-2pqmnx+m^2n^2$.
(4) From $a(x+y)-bxy+c(x-y)$ take $4(x+y)+(a+b)xy-7(x-y)$.
(5) From $(a+b)(x+y)-(c-d)(x-y)+h^2$ take $(a-b)(x+y)+(c+d)(x-y)+k^2$.
(6) From $(2a-5b)\sqrt{x+y}+(a-b)xy-cz^2$ take $3bxy-(5+c)z^2-(3a-b)(x+y)^{\frac{1}{2}}$.
(7) From $2x-y+(y-2x)-(x-2y)$ take $y-2x-(2y-x)+(x+2y)$.
(8) To what is $a+b+c-(a-b)-(b-c)-(-b)$ equal?

Answers.

(3) $(m^2n^2-p^2q^2)x^2+p^2q^2-m^2n^2$ or $(m^2n^2-p^2q^2)x^2-(m^2n^2-p^2q^2)$.
(4) $(a-4)(x+y)-(a+2b)xy+(c+7)(x-y)$.
(5) $2b(x+y)-2c(x-y)+h^2-k^2$.
(6) $(5a-6b)\sqrt{x+y}+(a-4b)xy+5z^2$.
(7) $y-x$.
(8) $2b+2c$.

MULTIPLICATION.

9. Multiplication is usually divided into three cases:—

(1) When both multiplicand and multiplier are simple quantities.

(2) When the multiplicand is a compound, and the multiplier a simple uantity.

(3) When both multiplicand and multiplier are compound quantities.

Case I.

10. *When both multiplicand and multiplier are simple quantities.*

To the product of the coefficients affix that of the letters.

Thus, to multiply $5\,a\,x$ by $4\,a\,x\,y$, we have

$$5 \times 4 = 20;\ a\,x \times a\,x\,y = a^2 x^2 y;$$

$$\therefore\ 5\,a\,x \times 4\,a\,x\,y = 20 \times a^2 x^2 y = 20\,a^2 x^2 y = \text{product.}$$

Rule of Signs in Multiplication.*

The product of quantities with like signs, is affected with the sign $+$; the product of quantities with unlike signs, is affected with the sign $-$;

or

$+$ multiplied by $+$ and $-$ multiplied by $-$ give $+$;

$+$ multiplied by $-$ and $-$ multiplied by $+$ give $-$;

or

like signs produce $+$ and unlike signs $-$.

The truth of this may be shown in the following manner:—

(1) Let it be required to multiply $+a$ by $+b$.

Here a is to be taken as often as there are units in b, and the sum of any number of quantities affected with the sign $+$, being $+$, the product $a\,b$ must be affected with the sign $+$, and is therefore $+\,a\,b$.

(2) Multiply $+a$ by $-b$, or $-a$ by $+b$.

In the former case $-b$ is to be taken as often as there are units in a, and in the latter $-a$ is to be taken as often as there are units in b; but the sum of any number of quantities affected with the sign $-$ is also $-$; hence in either case the product $a\,b$ must be affected with the sign $-$, and is therefore $-\,a\,b$.

* Let N represent either a number or any quantity whatever, and put

$$a = +\mathrm{N};\ b = -\mathrm{N}$$

Then, since $a = +a$, and $b = +b$, we shall have

$$+a = +\mathrm{N};\ +b = -\mathrm{N}$$
$$-a = -\mathrm{N};\ -b = +\mathrm{N}.$$

Now, if in these four last equations we substitute the values of a and b from the first two equations, we have

$$+(+\mathrm{N}) = +\mathrm{N};\ +(-\mathrm{N}) = -\mathrm{N}$$
$$-(+\mathrm{N}) = -\mathrm{N};\ -(-\mathrm{N}) = +\mathrm{N}.$$

Now, in each of these formulas, the sign of the second number is what is named the *product* of the two signs of the first number; hence the truth of the rule of signs.

(3) Multiply $-a$ by $-b$.

Since by the last case $+a$ multiplied by $-b$ produces $-ab$; and since $-a$ multiplied by $-b$ cannot produce the same product as $+a$ multiplied by $-b$, it is evident that the product of $-a$ and $-b$ can only be $+ab$.

11. *Powers* of the same quantity are multiplied by simply adding their indices; for since by the definition of a power

$$a^5 = aaaaa;\ a^7 = aaaaaaa$$

$$\therefore a^5 \times a^7 = aaaaa \times aaaaaaa = aaaaaaaaaaaa = a^{12}$$

Also $a^m = aaa \ldots.$ to m factors; $a^n = aaa \ldots.$ to n factors

$$\therefore a^m \times a^n = aaa \ldots. \text{ to } m \text{ factors} \times aaa \ldots. \text{ to } n \text{ factors}$$
$$= aaaaaa \ldots.. \text{ to } (m+n) \text{ factors}$$
$$= a^{m+n}.$$

It is proved in the same manner that $a^m \times a^n \times a^h \times a^k = a^{m+n+h+k}$

EXAMPLES.

(1.) $4a^2b^2cd \times 3abc^2d^2 = 12a^3b^3c^3d^3$.
(2.) $12\sqrt{ay} \times 4bx = 48bx\sqrt{ay}$.
(3.) $5\frac{1}{2}x^2y^3z^4 \times 6xy^4z^3 = 33x^3y^7z^7$.
(4.) $13a^2b^3x^3y \times -5abxy^3 = -65a^3b^4x^4y^4$.
(5.) $-5x^my^n \times -4x^ny^m = +20x^{m+n}y^{m+n}$.
(6.) $-20a^pb^q \times 5a^mb^nc^r = -100a^{m+p}b^{n+q}c^r$.

CASE II.

12. *When the multiplicand is a compound, and the multiplier a simple quantity.*

Multiply each term of the multiplicand by the multiplier, beginning at the left hand; and these partial products being connected by their respective signs, will give the complete product.

EXAMPLES.

(1.) Multiply $a^2 + ab + b^2$
By $4a$

Product $4a^3 + 4a^2b + 4ab^2$.

(2.) Multiply $a^2 - 2ab + b^2$
By $3xy$

Product $3a^2xy - 6abxy + 3b^2xy$.

(3.) Multiply $5mn + 3m^2 - 2n^2$ by $12abn$.
(4.) Multiply $3ax - 5by + 7xy$ by $-7abxy$.
(5.) Multiply $-15a^2b + 3ab^2 - 12b^3$ by $-5ab$.
(6.) Multiply $ax^3 - bx^2 + cx - d$ by $-x^5$.
(7.) Multiply $\sqrt{a+b} + \sqrt{x^2 - y^2} - 3xy$ by $-2\sqrt{x}$
(8.) Multiply $a^mx^n + b^my^n - c^ny^m - d^nx^m$ by x^my^n.

Case III.

13. *When both multiplicand and multiplier are compound quantities.*

Multiply each term of the multiplicand, in succession, by each term of the multiplier, and the sum of these partial products will give the complete product

Examples.

(1)

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

(2)

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2-b^2 \end{array}$$

(3)

$$\begin{array}{l} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array}$$

(4)

$$\begin{array}{l} ab+cd \\ ab-cd \\ \hline a^2b^2+abcd \\ \quad -abcd-c^2d^2 \\ \hline a^2b^2-c^2d^2 \end{array}$$

(5)

$$\begin{array}{l} a^2+2ab+b^2 \\ a^2-b^2 \\ \hline a^4+2a^3b+a^2b^2 \\ \quad -a^2b^2-2ab^3-b^4 \\ \hline a^4+2a^3b-2ab^3-b^4 \end{array}$$

(6) Multiply $4a^3-5a^2b-8ab^2+2b^3$ by $2a^2-3ab-4b^2$.

$$\begin{array}{l} 4a^3-5a^2b-8ab^2+2b^3 \\ 2a^2-3ab-4b^2 \\ \hline 8a^5-10a^4b-16a^3b^2+4a^2b^3 \\ \quad -12a^4b+15a^3b^2+24a^2b^3-6ab^4 \\ \quad\quad -16a^3b^2+20a^2b^3+32ab^4-8b^5 \\ \hline 8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5 = \text{product.} \end{array}$$

(7) Multiply $a'b-ab'$ by $h'k-hk'$.

$$\begin{array}{l} a'b-ab' \\ h'k-hk' \\ \hline a'bh'k-ab'h'k \\ \quad -a'bhk'+ab'hk' \\ \hline a'bh'k-ab'h'k-a'bhk'+ab'hk' = \text{product.} \end{array}$$

(8) Multiply $x^m+x^{m-1}y+x^{m-2}y^2+x^{m-3}y^3+$, &c., by $x+y$.

$$
\begin{array}{l}
x^m+x^{m-1}y+x^{m-2}y^2+x^{m-3}+y^3+\ldots\ldots \\
x+y \\
\hline
x^{m+1}+x^my+x^{m-1}y^2+x^{m-2}y^3+\ldots\ldots \\
\qquad +x^my+x^{m-1}y^2+x^{m-2}y^3+\ldots\ldots \\
\hline
x^{m+1}+2x^my+2x^{m-1}y^2+2x^{m-2}y^3+\ldots\ldots \\
\hline
\end{array}
$$

(9) Multiply x^2+y^2 by x^2-y^2.
(10) Multiply $x^2+2xy+y^2$ by $x-y$.
(11) Multiply $5a^4-2a^3b+4a^2b^2$ by $a^3-4a^2b+2b^3$.
(12) Multiply $x^4+2x^3+3x^2+2x+1$ by x^2-2x+1.
(13) Multiply $\frac{5}{2}x^2+3ax-\frac{7}{3}a^2$ by $2x^2-ax-\frac{1}{2}a^2$.
(14) Multiply $a^2+2ab+b^2$ by $a^2-2ab+b^2$.
(15) Multiply x^2+xy+y^2 by x^2-xy+y^2.
(16) Multiply $x^2+y^2+z^2-xy-xz-yz$ by $x+y+z$.
(17) Multiply together $x-a$, $x-b$, and $x-c$.

Answers.

(9) x^4-y^4.
(10) $x^3+x^2y-xy^2-y^3$.
(11) $5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$.
(12) x^6-2x^3+1.
(13) $5x^4+\frac{7}{2}ax^3-\frac{107}{12}a^2x^2+\frac{5}{6}a^3x+\frac{7}{6}a^4$.
(14) $a^4-2a^2b^2+b^4$.
(15) $x^4+x^2y^2+y^4$.
(16) $x^3+y^3+z^3-3xyz$.
(17) $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.

Multiplication by detached Coefficients.

14. In many cases the powers of the quantity or quantities in the multiplication of polynomials may be omitted, and the operation performed by the coefficients alone; for the same powers occupy the same vertical columns, when the polynomials are arranged according to the successive powers of the letters; and these successive powers, generally increasing or decreasing by a common difference, are readily supplied in the final product.

Examples.

(1.) Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$.

$$
\begin{array}{ll}
\text{Coefficients of multiplicand} & 1+1+1+1 \\
\text{————— multiplier} & 1-1 \\
\cline{2-2}
 & 1+1+1+1 \\
 & \quad -1-1-1-1 \\
\cline{2-2}
 & 1+0+0+0-1
\end{array}
$$

Since $x^3 \times x = x^4$, the highest power of x is 4, and decreases successively by unity, while that of y increases by unity; hence the product is

$$x^4 + 0 \cdot x^3 y + 0 \cdot x^2 y^2 + 0 \cdot x y^3 - y^4 = x^4 - y^4 = \text{product.}$$

(2.) Multiply $3a^2 + 4ax - 5x^2$ by $2a^2 - 6ax + 4x^2$.

$$\begin{array}{l}
3 + 4 - 5 \\
2 - 6 + 4 \\
\hline
6 + 8 - 10 \\
\quad - 18 - 24 + 30 \\
\qquad\quad + 12 + 16 - 20 \\
\\
6 - 10 - 22 + 46 - 20
\end{array}$$

$\therefore$ Product $= 6a^4 - 10a^3x - 22a^2x^2 + 46ax^3 - 20x^4$.

(3.) Multiply $2a^3 - 3ab^2 + 5b^3$ by $2a^2 - 5b^2$

Here the coefficients of a^2 in the multiplicand, and a in the multiplier, are each zero; hence,

$$\begin{array}{l}
2 + 0 - 3 + 5 \\
2 + 0 - 5 \\
\\
4 + 0 - 6 + 10 \\
\qquad - 10 + 0 + 15 - 25 \\
\\
4 + 0 - 16 + 10 + 15 - 25
\end{array}$$

Hence $4a^5 - 16a^3b^2 + 10a^2b^3 + 15ab^4 - 25b^5 =$ product.

The coefficient of a^4 being zero in the product, causes that term to disappear.

(4.) Multiply $x^3 - 3x^2 + 3x - 1$ by $x^2 - 2x + 1$

(5.) Multiply $y^2 - ya + \frac{1}{4}a^2$ by $y^2 + ya - \frac{1}{4}a^2$

(6.) Multiply $ax - bx^2 + cx^3$ by $1 - x + x^2 - x^3 + x^4$.

ANSWERS.

(4.) $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1$.

(5.) $y^4 - a^2y^2 + \frac{1}{2}a^3y - \frac{1}{16}a^4$.

(6.) $$\begin{array}{r|r|r|r|r|l}
ax - a & x^2 + a & x^3 - a & x^4 + a & x^5 - b & x^6 + cx^7 \\
-b & b & -b & +b & -c & \\
 & c & -c & +c & &
\end{array}$$

Or, $ax - (a+b)x^2 + (a+b+c)x^3 - (a+b+c)x^4 + (a+b+c)x^5 - (b+c)x^6 + cx^7$.

DIVISION.

15. THE object of algebraic division is to discover one of the factors of a given product, the other factor being given; and as multiplication is divided into three cases, so in like manner division is also divided into the three following cases.

(1) When both dividend and divisor are monomials.

(2) When the dividend is a polynomial, and the divisor a monomial.

(3) When both dividend and divisor are polynomials.

Case I.

16. *When both dividend and divisor are monomials.*

Write the divisor under the dividend, in the form of a fraction; cancel like quantities in both divisor and dividend, and suppress the greatest factor common to the two coefficients.

17. Powers of the same quantity are divided by subtracting the exponent of the divisor from that of the dividend, and writing the remainder as the exponent of the quotient.

Thus $a^7 = aaaaaaa$; $a^4 = aaaa$

$$\therefore \frac{a^7}{a^4} = \frac{aaaaaaa}{aaaa} = aaa = a^3 = a^{7-4}$$

Generally $a^m = aaaa \ldots\ldots$ to m factors; $a^n = aaa \ldots\ldots$ to n factors
$b^p = bbbb \ldots\ldots$ to p factors; $b^q = bbb \ldots\ldots$ to q factors

$$\therefore \frac{a^m b^p}{a^n b^q} = \frac{aaa \ldots \text{ to } m \text{ factors} \times bbb \ldots \text{ to } p \text{ factors}}{aaa \ldots \text{ to } n \text{ factors} \times bbb \ldots \text{ to } q \text{ factors}}$$
$$= aaa \ldots \text{ to } (m-n) \text{ factors} \times bbb \ldots \text{ to } (p-q) \text{ factors}$$
$$= a^{m-n}\, b^{p-q}.$$

From this reasoning it follows that *every quantity whose exponent is* 0, *is equal to* 1.

$$\text{For } \frac{a^m}{a^m} = a^{m-m} = a^0; \text{ but } \frac{a^m}{a^m} = 1.$$
$$\therefore a^0 = 1.$$
$$\text{Again, } \frac{a^3}{a^5} = \frac{aaa}{aaaaa} = \frac{1}{aa} = \frac{1}{a^2}.$$

But we may subtract 5, the greater exponent, from 3, the less, and affect the difference with the sign — ; hence

$$\frac{a^3}{a^5} = a^{3-5} = a^{-2}; \text{ but } \frac{a^3}{a^5} = \frac{1}{a^2}$$
$$\frac{1}{a^2} = a^{-2}$$

$$\text{Similarly, } \frac{1}{a+x} = (a+x)^{-1}; \quad \frac{1}{(x+y)^2} = (x+y)^{-2}; \quad a^x = 1$$

$$\text{And } \frac{1}{(x^2+y^2)^3(x^2-y^2)^{\frac{1}{2}}} = (x^2+y^2)^{-3}(x^2-y^2)^{-\frac{1}{2}}; \text{ and so on.}$$

For more information on *negative exponents*, see a subsequent article.

18. In multiplication, the product of two terms, having the same sign, is affected with the sign +; and the product of two terms, having different signs, is affected with the sign —; hence we may conclude,

(1.) That if the term of the dividend have the sign +, and that of the divisor the sign +, the resulting term of the quotient must have the sign +.

(2.) That if the term of the dividend have the sign +, and that of the divisor the sign —, the resulting term of the quotient must have the sign —.

(3.) That if the term of the dividend have the sign —, and that of the divisor the sign +, the resulting term of the quotient must have the sign —.

(4.) That if the term of the dividend have the sign —, and that of the divisor the sign —, the resulting term of the quotient must have the sign +

RULE OF SIGNS IN DIVISION

$+$ divided by $+$, and $-$ divided by $-$ give $+$
$-$ divided by $+$, and $+$ divided by $-$ give $-$;
or,

$$\frac{+ab}{+a} = +b; \frac{-ab}{-a} = +b; \frac{-ab}{+a} = -b; \frac{+ab}{-a} = -$$

EXAMPLES.

(1.) Divide $48a^3 b^3 c^2 d$ by $12ab^2 c$.

$$\frac{48a^3 b^3 c^2 d}{12ab^2 c} = \frac{48a\,a\,a\,b\,b\,b\,c\,c\,d}{12a\,b\,b\,c} = 4a\,a\,b\,c\,d = 4a^2\,b\,c\,d.$$

(2.) $\frac{150a^5 b^8 c\,d^3}{30a^3 b^5 d^2} = 5a^{5-3}\,b^{8-5}\,c\,d^{3-2} = 5a^2\,b^3\,c\,d.$

(3.) $\frac{-16a^2 b^2 c^2}{-4a\,b\,c} = 4a^{2-1}\,b^{2-1}\,c^{2-1} = 4a\,b\,c.$

(4.) $\frac{15a^{2m} x^{3n} y^{4n}}{3a^m x^{2n} y^{5n}} = 5a^{2m-m}\,x^{3n-2n}\,y^{4n-5n} = 5a^m\,x^n\,y^{-n}.$

(5.) $\frac{-48a^m b^n}{6a^p b^q} = -8a^{m-p}\,b^{n-q}.$

(6.) $\frac{-63a^3 b^4 c^5 d^6 x^2 y z}{-7a^2 b\,c\,d^3 x^4 y^5 z^6} = +9a\;b^3 c^4 d^3 x^{-2} y^{-4} z^{-5}.$

CASE II.

19. *When the dividend is a polynomial, and the divisor a monomial.*

Divide each of the terms of the dividend separately by the divisor, and connect the quotients with their respective signs.

EXAMPLES.

(1.) Divide $6a^2 x^4 y^6 - 12a^3 x^3 y^6 + 15a^4 x^5 y^3$ by $3a^2 x^2 y^2$.

$$\frac{6a^2 x^4 y^6 - 12a^3 x^3 y^6 + 15a^4 x^5 y^3}{3a^2 x^2 y^2} = 2x^2 y^4 - 4a\,x\,y^4 + 5a^2 x^3 y.$$

(2.) Divide $15a^2 b\,c - 20a\,c\,y^2 + 5\,c\,d^2$ by $-5a\,b\,c$.

Ans. $-3a + 4\frac{y^2}{b} - \frac{d^2}{a\,b}.$

(3.) Divide $x^{n+1} - x^{n+2} + x^{n+3} - x^{n+4}$ by x^n. Ans. $x - x^2 + x^3 - x^4.$

(4.) Divide $5(a+b)^3 - 10(a+b)^2 + 15(a+b)$ by $-5(a+b)$.

Ans. $-(a+b)^2 + 2(a+b) - 3.$

(5.) Divide $12a^4 y^6 - 16a^5 y^5 + 20a^6 y^4 - 28a^7 y^3$ by $-4a^4 y^3$.

Ans. $-3y^3 + 4a\,y^2 - 5a^2 y + 7a^3.$

CASE III.

20. *When both dividend and divisor are polynomials.*

1. Arrange the dividend and divisor according to the powers of the same letter in both.

2. Divide the first term of the dividend by the first term of the divisor, and the result will be the first term in the quotient, by which multiply all the terms in the divisor, and subtract the product from the dividend.

3. Then to the remainder annex as many of the remaining terms of the dividend as are necessary, and find the next term in the quotient as before.

EXAMPLES.

(1.) Divide $a^4-4a^3x+6a^2x^2-4ax^3+x^4$ by $a^2-2ax+x^2$.

$$a^2-2ax+x^2)\ a^4-4a^3x+6a^2x^2-4ax^3+x^4\ (a^2-2ax+x^2$$
$$\underline{a^4-2a^3x+a^2x^2}$$
$$-2a^3x+5a^2x^2-4ax^3$$
$$\underline{-2a^3x+4a^2x^2-2ax^3}$$
$$a^2x^2-2ax^3+x^4$$
$$\underline{a^2x^2-2ax^3+x^4}$$

Arranging the terms according to the descending powers of x, we have

$$x^2-2ax+a^2)\ x^4-4ax^3+6a^2x^2-4a^3x+a^4\ (x^2-2ax+a^2$$
$$\underline{x^4-2ax^3+a^2x^2}$$
$$-2ax^3+5a^2x^2-4a^3x$$
$$\underline{-2ax^3+4a^2x^2-2a^3x}$$
$$a^2x^2-2a^3x+a^4$$
$$\underline{a^2x^2-2a^3x+a^4.}$$

(2.) Divide $x^4+x^2y^2+y^4$ by x^2+xy+y^2.

$$x^2+xy+y^2)\ x^4+x^2y^2+y^4\ (x^2-xy+y^2$$
$$\underline{x^4+x^3y+x^2y^2}$$
$$-x^3y+y^4$$
$$\underline{-x^3y-x^2y^2-xy^3}$$
$$x^2y^2+xy^3+y^4$$
$$\underline{x^2y^2+xy^3+y^4.}$$

(3.) Divide $a^5-a^3b^2+2a^2b^3-ab^4+b^5$ by a^2-ab+b^2.

$$a^2-ab+b^2)\ a^5-a^3b^2+2a^2b^3-ab^4+b^5\ \left(a^3+a^2b-ab^2+\frac{b^5}{a^2-ab+b^2}.\right.$$
$$\underline{a^5-a^4b+a^3b^2}$$
$$a^4b-2a^3b^2+2a^2b^3$$
$$\underline{a^4b-a^3b^2+a^2b^3}$$
$$-a^3b^2+a^2b^3-ab^4$$
$$\underline{-a^3b^2+a^2b^3-ab^4}$$
$$\underline{*\quad *\quad *\quad +b^5.}$$

Arranging the terms according to powers of b, we get

$$b^2-ab+a^2)\ b^5-ab^4+2a^2b^3-a^3b^2+a^5\ \left(b^3+a^2b+\frac{-a^4b+a^5}{b^2-ab+a^2}.\right.$$
$$\underline{b^5-ab^4+a^2b^3}$$
$$a^2b^3-a^3b^2+a^5$$
$$\underline{a^2b^3-a^3b^2+a^4b}$$
$$-a^4b+a^5$$

The results we have obtained in these two arrangements are apparently different; but their equivalence will be established as follows:—

(1) $(a^2-ab+b^2)\,(a^3+a^2b-ab^2) = a^5-a^3b^2+2a^2b^3-ab^4$

Add remainder $= +b^5$

Proof . . . . $a^5-a^3b^2+2a^2b^3-ab^4+b^5$

(2) $(b^2-ab+a^2)\,(b^3+a^2b) = b^5-ab^4+2a^2b^3-a^3b^2+a^4b$

Add remainder $= -a^4b+a^5$

Proof . . . . $b^5-ab^4+2a^2b^3-a^3b^2+a^5$.

Examples for Practice.

(1.) Divide $a^2-2ab+b^2$ by $a-b$.
(2.) Divide $a^2+4ax+4x^2$ by $a+2x$.
(3.) Divide $12x^4-192$ by $3x-6$.
(4.) Divide $6x^6-6y^6$ by $2x^2-2y^2$.
(5.) Divide $a^6-3a^4b^2+3a^2b^4-b^6$ by $a^3-3a^2b+3ab^2-b^3$.
(6.) Divide $x^3+5x^2y+5xy^2+y^3$ by $x^2+4xy+y^2$.
(7.) Divide x^5-y^5 by $x-y$.
(8.) Divide a^4-b^4 by $a^3+a^2b+ab^2+b^3$.
(9.) Divide $x^3-9x^2+27x-27$ by $x-3$.
(10.) Divide x^4+y^4 by $x+y$.
(11.) Divide $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.
(12.) Divide $\frac{1}{4}x^3+x^2+\frac{3}{8}x+\frac{3}{4}$ by $\frac{1}{4}x+1$.

Answers.

(1.) $a-b$.
(2.) $a+2x$.
(3.) $4x^3+8x^2+16x+32$.
(4.) $3x^4+3x^2y^2+3y^4$.
(5.) $a^3+3a^2b+3ab^2+b^3$.
(6.) $x+y$.
(7.) $x^4+x^3y+x^2y^2+xy^3+y^4$.
(8.) $a-b$.
(9.) x^2-6x+9.
(10.) $x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}$.
(11.) $24x^2-2ax-35a^2$
(12.) $x^2+\frac{3}{4}$.

Examples with literal Exponents.

(1.) Divide $2a^{3n}-6a^{2n}b^n+6a^nb^{2n}-2b^{3n}$ by a^n-b^n.

$$
\begin{array}{l}
a^n-b^n)\ 2a^{3n}-6a^{2n}b^n+6a^nb^{2n}-2b^{3n}\ (2a^{2n}-4a^nb^n+2b^{2n} \\
\qquad\quad\ \underline{2a^{3n}-2a^{2n}b^n} \\
\qquad\qquad -4a^{2n}b^n+6a^nb^{2n} \\
\qquad\qquad \underline{-4a^{2n}b^n+4a^nb^{2n}} \\
\qquad\qquad\qquad\quad 2a^nb^{2n}-2b^{3n} \\
\qquad\qquad\qquad\quad \underline{2a^nb^{2n}-2b^{3n}}.
\end{array}
$$

(2.) Divide $x^{m+1}+x^m y+x y^m+y^{m+1}$ by x^m+y^m

(3.) Divide a^n-x^n by $a-x$.

(4.) Divide $x^{4n}+x^{2n}y^{2n}+y^{4n}$ by $x^{2n}+x^n y^n+y^{2n}$.

(5.) Divide $a^{m+n}b^n-4a^{m+n-1}b^{2n}-27a^{m+n-2}b^{3n}+42a^{m+n-3}b^{4n}$ by $a^n b^n-7a^{n-1}b^{2n}$.

Answers.

(2.) $x+y$.

(3.) $a^{n-1}+a^{n-2}x+a^{n-3}x^2+\dfrac{a^{n-3}x^3-x^n}{a-x}$.

(4.) $x^{2n}-x^n y^n+y^{2n}$.

(5.) $a^m+3a^{m-1}b^n-6a^{m-2}b^{2n}$.

Examples with literal Coefficients.

(1.) Divide $ax^5+ax^4+bx^4+ax^3+bx^3+cx^3+ax^2+bx^2+cx^2+bx+cx+c$ by ax^2+bx+c.

Arrange the terms of the dividend in the following manner, in order to keep the operation within the breadth of the page.

$$ax^2+bx+c)\ ax^5+\left.\begin{matrix}a\\b\\\end{matrix}\right|x^4+\left.\begin{matrix}a\\b\\c\end{matrix}\right|x^3+\left.\begin{matrix}a\\b\\c\end{matrix}\right|x^2+\left.\begin{matrix}b\\c\\\end{matrix}\right|x+c\ (x^3+x^2+x+1.$$

$$ax^5+bx^4+cx^3$$

$$ax^4+\left.\begin{matrix}a\\b\\\end{matrix}\right|x^3+\left.\begin{matrix}a\\b\\c\end{matrix}\right|x^2$$

$$ax^4+bx^3+cx^2$$

$$ax^3+\left.\begin{matrix}a\\b\end{matrix}\right|x^2+\left.\begin{matrix}b\\c\end{matrix}\right|x$$

$$ax^3+bx^2+cx$$

$$ax^2+bx+c$$

$$ax^2+bx+c$$

(2.) Divide x^3+ax^2+bx+c by $x-r$.

$$x-r)\ x^3+ax^2+bx+c\ (x^2+(r+a)x+(r^2+ar+b)$$

$$x^3-rx^2$$

$$(r+a)x^2+bx$$

$$(r+a)x^2-(r^2+ar)x$$

$$(r^2+ar+b)x+c$$

$$(r^2+ar+b)x-(r^3+ar^2+br)$$

$$r^3+ar^2+br+c \text{ Remainder.}$$

In the preceding and similar examples, the remainder differs only from the dividend in having r instead of x.

(3.) Divide $x^3-a\,x^2+b\,x-c$ by $x-r$.
(4.) Divide $x^3-(a+b+c)x^2+(a\,b+b\,c+c\,a)\,x-a\,b\,c$ by $x-c$.
(5.) Divide $x^3-(a+2)\,x^2+(2a+b)\,x-2b$ by $x-2$.
(6.) Divide $11\,a^2\,b-19\,a\,b\,c+10\,a^3-15\,a^2\,c+3\,a\,b^2+15\,b\,c^2-5\,b^2\,c$ by $5a^2+3a\,b-5b\,c$.
(7.) Divide $x^3-(a+b+d)\,x^2+(a\,d+b\,d+c)\,x-c\,d$ by $x^2-(a+b)\,x+c$.

ANSWERS.

(3.) $x^2+(r-a)\,x\ +(r^2-a\,r+b)$, and remainder is $r^3-a\,r^2+b\,r-c$.
(4.) $x^2-(a+b)\,x+a\,b$. (5.) $x^2-a\,x+b$.
(6.) $2a+(b-3c)$. (7.) $x-d$.

21. In those cases in which the division does not terminate, and the quotient may be continued to an unlimited number of terms; then the quotient is termed *an infinite series*, and the successive terms of the quotient are generally regulated by a law, which in most cases is readily discoverable.

EXAMPLES.

(1.) Divide 1 by $1-x$.

$$
\begin{array}{l}
1-x)\ 1 \qquad (1+x+x^2+x^3+x^4+x^5+\ \ldots\ldots \\
\qquad\ \ 1-x \\
\hline
\qquad\quad +x \\
\qquad\quad +x-x^2 \\
\hline
\qquad\qquad +x^2 \\
\qquad\qquad +x^2-x^3 \\
\hline
\qquad\qquad\quad +x^3 \\
\hline
\end{array}
$$

The quotient in this case is called an infinite series, and the law of formation of this series is that any term in the quotient is the product of the immediately preceding term by x.

(2.) Divide 1 by $1+x$. Ans. $1-x+x^2-x^3+x^4-\ldots.$
(3.) Divide $1+x$ by $1-x$. Ans. $1+2x+2x^2+2x^3+2x^4+\ldots.$
(4.) Divide 1 by $x+1$. Ans. $x^{-1}-x^{-2}+x^{-3}-x^{-4}+x^{-5}-\ldots$
(5.) Divide $x-a$ by $x-b$.
Ans. $1-(a-b)\,x^{-1}-(a-b)\,bx^{-2}-(a-b)\,b^2x^{-3}-$
(6.) Divide 1 by $1-2x+x^2$. Ans. $1+2x+3x^2+4x^3+5x^4+\ldots$

22. When a polynomial is the product of two or more factors, it is often requisite to resolve it into the factors of which it is composed, and merely to indicate the multiplication. This can frequently be done by inspection, and by the aid of the following formulas:—

$$(x+a)\,(x+b) = x^2+(a+b)\,x+a\,b \ \ldots\ldots (1)$$
$$(x+a)\,(x-b) = x^2+(a-b)\,x-a\,b \ \ldots\ldots (2)$$
$$(x-a)\,(x+b) = x^2-(a-b)\,x-a\,b \ \ldots\ldots (3)$$
$$(x-a)\,(x-b) = x^2-(a+b)\,x+a\,b \ \ldots\ldots (4)$$

$$(a+b)\,(a-b) = a^2-b^2 \ldots\ldots\ldots\ldots (5)$$
$$(n+1)\,(n+1) = n^2+2n+1 \ldots\ldots\ldots\ldots (6)$$
$$(n-1)\,(n-1) = n^2-2n+1 \ldots\ldots\ldots\ldots (7)$$

EXAMPLES.

(1.) Resolve $a\,x^2+b\,x^2-c\,x^2$ into its component factors.
Here $a\,x^2+b\,x^2-c\,x^2=x^2\,(a+b-c)$.

(2.) Transform the expression $n^3+2\,n^2+n$ into factors
Here $n^3+2n^2+n = n\,(n^2+2n+1)$
$= n\,(n+1)\,(n+1)$ by (6)
$= n\,(n+1)^2$.

(3.) Decompose the expression x^2-x-72 into two factors.
By inspecting formula (3) we have $-1=-9+8$, and $-72=-9\times 8$;
hence $x^2-x-72 = (x-9)\,(x+8)$.

(4.) Decompose $5a^2\,b\,c+10a\,b^2\,c+15a\,b\,c^2$ into two factors.
(5.) Transform $3m^4\,n^6-6m^3\,n^5\,p+3m^2\,n^4\,p^2$ into factors.
(6.) Transform $3b^3\,c-3b\,c^3$ into factors.
(7.) Decompose $x^2+8x+15$ into two factors.
(8.) Decompose x^3-2x^2-15x into three factors.
(9.) Decompose x^2-x-30 into factors.
(10.) Transform $a^2-b^2+2b\,c-c^2$ into two factors.
(11.) Transform a^2x-x^3 into factors.

ANSWERS.

(4.) $5a\,b\,c\,(a+2b+3c)$.
(5.) $3m^2\,n^4\,(m\,n-p)^2$.
(6.) $3b\,c\,(b+c)\,(b-c)$.
(7.) $(x+3)\,(x+5)$.
(8.) $x\,(x+3)\,(x-5)$.
(9.) $(x-5)\,(x+6)$.
(10.) $(a+b-c)\,(a-b+c)$.
(11.) $x\,(a+x)\,(a-x)$.

23. By the usual process of division we might obtain the quotient of a^n-b^n divided by $a-b$, when any particular number is substituted for n; but we shall here prove generally that a^n-b^n is always exactly divisible by $a-b$, and exhibit the quotient.

It is required to divide a^n-b^n by $a-b$.

$$a-b)\;a^n-b^n\;\left(a^{n-1}+\frac{b\,(a^{n-1}-b^{n-1})}{a-b}\right.$$
$$a^n-a^{n-1}\,b$$
$$a^{n-1}\,b-b^n$$
$$\text{or } b\,(a^{n-1}-b^{n-1})$$

Hence $$\frac{a^n-b^n}{a-b} = a^{n-1}+\frac{b\,(a^{n-1}-b^{n-1})}{a-b} \ldots\ldots\ldots (1).$$

Now it appears from this result, that a^n-b^n will be exactly divisible by $a-b$, if $a^{n-1}-b^{n-1}$ be divisible by $a-b$; that is, if the difference of the same powers of two quantities is divisible by their difference; then the difference of the powers of the next higher degree is also divisible by that difference.

But a^2-b^2 is exactly divisible by $a-b$, and we have

$$\frac{a^2-b^2}{a-b} = a+b \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \;(2).$$

And since a^2-b^2 is divisible by $a-b$, it appears from what has been just proved, that a^3-b^3 must be exactly divisible by $a-b$; and hence, by putting 3 for n in formula (1), we get

$$\begin{aligned} \frac{a^3-b^3}{a-b} &= a^2+b.\,\frac{a^2-b^2}{a-b} \\ &= a^2+b.\,(a+b) \text{ by } (2) \\ &= a^2+ab+b^2 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \;(3). \end{aligned}$$

Again, a^4-b^4 must be exactly divisible by $a-b$, since a^3-b^3 is divisible by $a-b$; hence, by writing 4 for n in formula (1), we have

$$\begin{aligned} \frac{a^4-b^4}{a-b} &= a^3+b.\,\frac{a^3-b^3}{a-b} \\ &= a^3+b\,(a^2+a\,b+b^2) \text{ by } (3) \\ &= a^3+a^2 b+a\,b^2+b^3 \;.\;.\;.\;.\;.\;.\;.\;.\;. \;(4). \end{aligned}$$

Hence, generally, a^n-b^n will always be exactly divisible by $a-b$, and give the quotient

$$\frac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}\,b+a^{n-3}\,b^2+ \;.\;.\;.\;.\;.\; a^2\,b^{n-3}+a\,b^{n-2}+b^{n-1} \;.\;.\;.\;. \;(5).$$

In a similar manner we find, when n is an *odd* number,

$$\frac{a^n+b^n}{a+b}=a^{n-1}-a^{n-2}\,b+a^{n-3}\,b^2- \;.\;.\;.\;.\;.\;. +a^2\,b^{n-3}-a\,b^{n-2}+b^{n-1} \;.\;.\;. \;(6).$$

And when n is an *even* number,

$$\frac{a^n-b^n}{a+b}=a^{n-1}-a^{n-2}\,b+a^{n-3}\,b^2- \;.\;.\;.\;.\;. -a^2\,b^{n-3}+a\,b^{n-2}-b^{n-1} \;.\;.\;.\;. \;(7).$$

By substituting particular numbers for n, in the formulas (5), (6), (7), we may deduce various algebraical formulas, several of which will be found in the following deductions from the rules of multiplication and division.

Useful Algebraic Formulas.

(1.) $a^2-b^2 = (a+b)\,(a-b)$.

(2.) $a^4-b^4 = (a^2+b^2)\,(a^2-b^2) = (a^2+b^2)\,(a+b)\,(a-b)$.

(3.) $a^3-b^3 = (a^2+a\,b+b^2)\,(a-b)$.

(4.) $a^3+b^3 = (a^2-a\,b+b^2)\,(a+b)$.

(5.) $a^6-b^6 = (a^3+b^3)\,(a^3-b^3) = (a^3+b^3)\,(a^2+a\,b+b^2)\,(a-b)$.

(6.) $a^6-b^6 = (a^3+b^3)\,(a^3-b^3) = (a^3-b^3)\,(a^2-a\,b+b^2)\,(a+b)$.

(7.) $a^6-b^6 = (a^3+b^3)\,(a^3-b^3) = (a^2-b^2)\,(a^4+a^2\,b^2+b^4)$.

(8.) $a^6-b^6 = (a+b)\,(a-b)\,(a^2+a\,b+b^2)\,(a^2-a\,b+$ [illegible]

(9.) $(a^2-b^2) \div (a-b) = a+b$.

(10.) $(a^3-b^3) \div (a-b) = a^2+a\,b+b^2$.

(11.) $(a^3+b^3) \div (a+b) = a^2-a\,b+b^2$.

(12.) $(a^4-b^4) \div (a+b) = a^3-a^2b+a\,b^2-b^3$.

(13.) $(a^5-b^5) \div (a-b) = a^4+a^3b+a^2\,b^2+a\,b^3+b^4$.

(14.) $(a^5+b^5) \div (a+b) = a^4-a^3b+a^2\,b^2-a\,b^3+b^4$.

(15.) $(a^6-b^6) \div (a^2-b^2)=a^4+a^2\,b^2+b^4$.

Division by detached Coefficients.

24. Arrange the terms of the divisor and dividend according to the successive powers of the letter or letters common to both; write down simply the coefficients with their respective signs, supplying the coefficients of the absent terms with zeros, and proceed as usual. Divide the highest power of the omitted letters in the dividend by that of the suppressed letters in the divisor, and the quotient will give the literal part of the first term in the quotient. The literal parts of the successive terms follow the same law of increase or decrease as those in the dividend. The coefficients prefixed to the literal parts will give the complete quotient, omitting those terms whose coefficients are zero.

Examples.

(1.) Divide $6a^4-96$ by $3a-6$.

$$\begin{array}{l} 3-6)\ 6+0+0+0-96\ (2+4+8+16 \\ \quad\quad\ \ 6-12 \\ \quad\quad\ \ \overline{\quad\quad\ 12} \\ \quad\quad\quad\quad 12-24 \\ \quad\quad\quad\quad \overline{\quad\quad\ 24} \\ \quad\quad\quad\quad\quad\quad 24-48 \\ \quad\quad\quad\quad\quad\quad \overline{\quad\quad 48-96} \\ \quad\quad\quad\quad\quad\quad\quad\quad 48-96 \\ \quad\quad\quad\quad\quad\quad\quad\quad \overline{\quad\quad\quad} \end{array}$$

But $a^4 \div a = a^3$, and the literal parts of the successive terms are therefore a^3, a^2, a^1, a^0, or a^3, a^2, a, 1; hence, $2a^3+4a^2+8a+16$=quotient.

(2.) Divide $8a^5-4a^4x-2a^3x^2+a^2x^3$ by $4a^2-x^2$.

$$\begin{array}{l} 4+0-1)\ 8-4-2+1\ (2-1 \\ \quad\quad\quad\ \ 8+0-2 \\ \\ \quad\quad\quad\ \ -4+0+1 \\ \quad\quad\quad\ \ -4-0+1 \\ \quad\quad\quad\ \ \overline{\quad\quad\quad\quad} \end{array}$$

Now, $a^5 \div a^2 = a^3$; hence a^3 and a^2x are the literal parts of the terms in the quotient, for there are only two coefficients in the quotient; therefore

$$2a^3-a^2x=\text{quotient required.}$$

(3.) Divide $x^4-3ax^3-8a^2x^2+18a^3x-8a^4$ by $x^2+2ax-2a^2$.

(4.) Divide $3y^3+3xy^2-4x^2y-4x^3$ by $x+y$.

(5.) Divide $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$ by $2a^2-3ax+4x^2$

(6.) Divide $a^6+4a^5-8a^4-25a^3+35a^2+21a-28$ by a^2+5a+4.

Answers.

(3.) $x^2-5ax+4a^2$

(4.) $-4x^2+3y^2$.

(5.) $5a^2-6ax-2x^2$.

(6.) $a^4-a^3-7a^2+14a-7$.

Synthetic Division.

25. In the common method of division, the several terms in the divisor are multiplied by the first term in the quotient, and the product subtracted from the dividend; but subtraction is performed by changing all the signs of the quantities to be subtracted, and then *adding* the several terms in the lower line to the similar terms in the higher. If, therefore, the signs of the terms in the divisor were changed, we should have to *add* the product of the divisor and quotient instead of subtracting it. And since the process would be the same for every step in the operation, the successive products of the divisor and the several terms in the quotient would all become additive. By this process, then, the second dividend would be *identically* the same as by the usual method; but the second term in the quotient is found by dividing the first term of the second dividend by the first term of the divisor; and since the sign of the first term in the divisor has been changed, it is obvious that the sign of the second term in the quotient will also be changed. To avoid this change of sign in the quotient, the sign of the first term in the divisor might remain unchanged, and then omit altogether the products of the first term in the divisor by the successive terms in the quotient; because in the usual method the first term in each successive dividend is cancelled by these products. Omitting, therefore, these products, the coefficients of the first term in any dividend will be the coefficient of the succeeding term in the quotient, the coefficient in the first term of the divisor being unity; for in all cases it can be made unity, by dividing both divisor and dividend by the coefficient of the first term in the divisor. This being the case, the coefficients in the quotient are respectively the coefficients of the first terms in the successive dividends. The operation, thus simplified, may however be further abridged by omitting the successive additions, except so much only as is necessary to show the first term in each dividend, which, as before remarked, is also the coefficient of the succeeding term in the quotient, and writing the products of the modified divisor, and the several terms of the quotient as they arise, diagonally, instead of horizontally, beginning at the upper line. Hence the following

Rule.*

(1). Divide the divisor and dividend by the coefficient of the first term in the divisor, which will make the leading coefficient of the divisor unity, and the first term of the quotient will be identical with that of the dividend.

(2). Change all the signs of the terms in the divisor, except the first, and multiply all the terms so changed by the term in the quotient, and place the products successively under the corresponding terms of the dividend, in a diagonal column, beginning at the upper line.

(3). Add the results in the second column, which will give the second term of the quotient; and multiply the changed terms in the divisor by this result, placing the products in a diagonal series, as before.

* The rule here given for *Synthetic Division* is due to the late W. G. Horner, Esq., of Bath, whose researches in science have issued in several elegant and useful processes, especially in the higher branches of algebra, and in the evolution of the roots of equation of all dimensions.

(4). Add the results in the third column, which will give the next term in the quotient, and multiply the changed terms in the divisor by this term in the quotient, placing the products as before.

(5). This process continued till the results become 0, or till the quotient is determined as far as necessary, will give the same series of terms as the usual mode of division when carried to an equivalent extent.

Examples.

(1.) Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$.

$$\begin{array}{r|rrrrrr}
1 & 1 & -5 & +10 & -10 & +5 & -1 \\
+2 & & +2 & -6 & +6 & -2 & \\
-1 & & & -1 & +3 & -3 & +1 \\
\hline
 & 1 & -3 & +3 & -1 & * & *
\end{array}$$

Hence $a^3-3a^2x+3ax^2-x^3$=quotient.

In this example the coefficients of the dividend are written horizontally, and those of the divisor vertically, with all the signs changed, except the first. Then $+2$ and -1, the changed terms in the divisor, are multiplied by 1, the first term of the dividend or quotient; and the products $+2$ and -1 are placed diagonally, under -5 and $+10$, the corresponding terms of the dividend. Then by adding the second column we have -3 for the second term in the quotient, and the changed terms $+2$ and -1 in the divisor, multiplied by -3, give -6 and $+3$, which are placed diagonally under $+10$ and -10. The sum of the third column is $+3$, the next term in the quotient, which multiplied into the changed terms of the divisor, gives $+6-3$, for the next diagonal column. The sum of the fourth column is -1, and by this we obtain the last diagonal column $-2+1$. The process here terminates, since the sums of the fifth and sixth columns are zero; and the quotient is completed by restoring the letters, as in detached coefficients.

Having made the coefficient of the first term in the divisor unity, that coefficient may be omitted entirely, since it is of no use whatever in continuing the operation here described.

(2.) Divide $x^6-5x^5+15x^4-24x^3+27x^2-13x+5$ by $x^4-2x^3+4x^2-2x+1$.

$$\begin{array}{r|rrrrrrr}
1 & 1 & -5 & +15 & -24 & +27 & -13 & +5 \\
+2 & & +2 & -6 & +10 & & & \\
-4 & & & -4 & +12 & -20 & & \\
+2 & & & & +2 & -6 & +10 & \\
-1 & & & & & -1 & +3 & -5 \\
\hline
 & 1 & -3 & +5 & 0 & 0 & 0 & 0
\end{array}$$

Hence x^2-3x+5=quotient required.

(3.) Divide $a^5+2a^4b+3a^3b^2-a^2b^3-2ab^4-3b^5$ by $a^2+2ab+3b^2$.

$$\begin{array}{r|rrrrrr}
 & 1 & +2 & +3 & -1 & -2 & -3 \\
-2 & & -2 & +0 & +0 & +2 & \\
-3 & & & -3 & +0 & +0 & +3 \\
\hline
 & 1 & +0 & +0 & -1 & &
\end{array}$$

Hence $a^3+0\cdot a^2b+0\cdot ab^2-b^3=a^3-b^3$=quotient.

(4.) Divide $1-x$ by $1+x$. Ans. $1-2x+2x^2-2x^3+$, &c.

(5.) Divide 1 by $1-x$. Ans. $1+x+x^2+x^3+$, &c.

(6.) Divide x^7-y^7 by $x-y$. Ans. $x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6$

(7.) Divide $a^6-3a^4x^2+3a^2x^4-x^6$ by $a^3-3a^2x+3ax^2-x^3$.

Ans. $a^3+3a^2x+3ax^2+x^3$

The Greatest Common Measure.

26. A *measure* of a quantity is any quantity that is contained in it exactly, or divides it without a remainder; and, on the other hand, a *multiple* of a quantity is any quantity that contains it exactly.

27. A *common measure*, of two or more quantities, is a quantity which is contained exactly in each of them.

28. The *greatest common measure*, of two or more quantities, is the *greatest* factor which is common to each of the quantities. Thus 5 is a measure of 15, and 15 is a multiple of 5; for 5 is contained in 15 exactly 3 times, and 15 contains 5 exactly 3 times; also $3x$ is a common measure of $12ax$ and $18bx$, and $6x$ is the greatest common measure of $12ax$ and $18bx$.

29. *To find the greatest common measure of two polynomials.*

Arrange the polynomials according to the powers of some letter, and divide that which contains the highest power of the letter by the other, as in division; then divide the last divisor by the remainder arising from the first division; consider the remainder that arises from this second division as a divisor, and the last divisor the corresponding dividend, and continue this process of division till the remainder is 0; then the last divisor is the greatest common measure.

Note 1. When the highest power of the leading quantity is the same in both polynomials, it is indifferent which of the polynomials is made the divisor, the only guide being the coefficients of the leading terms of the polynomials

Note 2. If the two polynomials have a simple common measure, it may be suppressed to simplify the process; but as it is a factor of the greatest common measure, it must be restored in the final divisor, and therefore the last divisor must be multiplied by the common factor at first rejected.

Note 3. If any divisor contains a factor, which is not a factor also of the dividend, that factor must be rejected before commencing the division, as such factor can form no part of the greatest common measure.

Note 4. If the coefficient of the leading term of any dividend be not divisible by that of the divisor, it may be rendered so by multiplying every term of the dividend by a proper factor, to make it divisible.

In order to prove the truth of this rule, we shall premise two lemmas.

Lemma 1. If a quantity measures another quantity, it will also measure any multiple of that quantity. Thus, if d measures a, it will also measure m times a, or ma; for let $a=hd$, then $ma=mhd$, and therefore d measures ma, the quotient being mh.

Lemma 2. If a quantity measures two other quantities, it will also measure both their sum and difference, or any multiples of them. For let $a = h\,d$, and

$b = k\,d$, then d measures both a and b; hence $a \pm b = h\,d \pm k d = d\,(h \pm k)$ and therefore d measures both $a+b$ and $a-b$, the quotient being $h+k$ in the former case, and $h-k$ in the latter; and by lemma 1, d measures any multiples of $a+b$ and $a-b$.

Now, let a and b be two polynomials, or the terms of a fraction, and let a divided by b leave a remainder c

b . . . . . . c . . . . d

c . . . . d leave no remainder, as is shown in the marginal scheme. Then we have, by the nature of division, these six equalities, viz.:

$$
\begin{array}{ll}
a - m\,b = c \;.\;.\;.\;.\; (1) & a = m\,b + c \;.\;.\;.\;.\; (4) \\
b - n\,c = d \;.\;.\;.\; (2) & b = n\,c + d \;.\;.\;.\;.\; (5) \\
c - p\,d = 0 \;.\;.\;.\;.\; (3) & c = p\,d \;.\;.\;.\;.\;.\; (6)
\end{array}
$$

$$
\begin{array}{l}
b)\ a\ (m \\
\underline{m\,b} \\
c)\ b\ (n \\
\underline{n\,c} \\
d\ )\ c\,(p \\
\underline{p\,d}
\end{array}
$$

Where the equalities marked (4), (5), (6), are not deduced from those marked (1), (2), (3), but from the consideration that the dividend is always equal to the product of the divisor and quotient, increased by the remainder.

Now, by (6) it is obvious that d measures c, since $c = p\,d$; hence (Lemma 1) d measures $n\,c$, and it likewise measures itself; therefore (Lemma 2) d measures $n\,c + d$, which by (5) is equal to b; hence again d, measuring b and c, measures $m\,b + c$, by the Lemmas 1 and 2.

$\therefore$ d measures a, which is equal to $m\,b + c$ by (4).

Hence d measures both the polynomials a and b, and is consequently a common measure of these polynomials; but d is also the greatest common measure of a and b; for if d' is a greater common measure of a and b than d is, it is obvious that by (1) d' measures $a - m\,b$, or c; and d' measuring both b and c, it measures $b - n\,c$, or d by (2); hence d' measures d, which is absurd, since no quantity measures a quantity less than itself; therefore d is the greatest common measure. Q. E. D.

Again, let $a = h\,a'$ and $b = h\,b'$; then the greatest common measure of a and b will be $h\,d'$, where d' is the greatest common measure of a' and b'. For let $a' = d'm$, and $b' = d'n$; then $a = h\,a' = h\,d'm$, and $b = hb' = h\,d'n$; then m and n contain no common factor, for d' is the greatest common measure of a' and b'; hence $h\,d'$ is obviously the greatest common measure of $h\,d'm$ and $h\,d'n$, or of a and b; and this proves the truth of Note 2.

Moreover, if any divisor contains a factor, which is not a factor also of the corresponding dividend, it must be rejected before commencing the division. For as in the marginal scheme, let c contain a factor h, which is not a factor also of a and b; then rejecting h, the remaining factor c' is employed as a new divisor, instead of c, and so on. For, by the nature of division, we have

$$
\begin{array}{ll}
a - mb = hc' \;.\;.\;.\;.\; (1) & a = mb + hc' \;.\;.\;.\;.\; (4) \\
b - nc' = kd' \;.\;.\;.\;.\; (2) & b = nc' + kd' \;.\;.\;.\;.\; (5) \\
c' - pd' = 0 \;.\;.\;.\;.\; (3) & c' = pd' \;.\;.\;.\;.\;.\; (6)
\end{array}
$$

$$
\begin{array}{l}
b)\ a\ (m \\
\underline{m\,b} \\
c = n\,c \\
c')\ b\ (n \\
\underline{n\,c'} \\
d = k\,d' \\
d')\ c'\ (p \\
p\,d'
\end{array}
$$

Now, by (6) we see that d' is a measure of c'; hence d', measuring $n\,c'$ and $k\,d'$, it also measures $n\,c'+k\,d'$, or b; therefore d', measuring c' and b, measures $m\,b+h\,c'$, or a; hence d' measures both a and b, and it is also the greatest common measure. For if d be a greater common measure than d', then d, measuring both a and b, measures $a-m\,b$, or $h\,c'$ by (1); but d does not measure h, and it must therefore measure c'; hence d measures $n\,c'$ and b; therefore it measures $b-n\,c'$, or $k\,d'$ by (2); but again, d does not measure k. and hence it must measure d'; but d is greater than d', and cannot therefore measure it; hence d' is the greatest common measure. This is the proof of note 3, and in very nearly the same manner it is proved, that if any dividend be multiplied by a factor, which is not a factor of the greatest common measure, in order to make the leading term of the dividend divisible by that of the divisor, the final divisor, or resulting greatest common measure, will remain unchanged.

Thus it appears that, to avoid the difficulty of operating with fractional quotients, we can always remove from the divisor, or introduce into the dividend any factor which may obstruct the exact division of the leading coefficient of the dividend by that of the divisor. These remarks will be fully exemplified in the subsequent examples; and as the process for finding the greatest common measure of any two polynomials is now very important, being employed in the general solution of equations, we have endeavoured to explain the reasons of the several steps in the process, with perspicuity and clearness, as far as our limits will permit.

30. If the greatest common measure of three quantities be required, find the greatest common measure of two of them, and then that of this measure and the remaining quantity will be the greatest common measure of all three. For let a, b, c, be the quantities, and let d be the greatest common measure of a and b, and d' the greatest common measure of c and d; then any measure of d will evidently measure a and b, and whatever measures c and d will also measure a, b, c; hence the *greatest* common measure of c and d is also the greatest common measure of a, b, c, and therefore d' is the greatest common measure of a, b, c. This reasoning may be extended to any number of quantities.

31. If the two polynomials be the terms of a fraction, as $\frac{a}{b}$, and d their greatest common measure, then we may put $a=d\,a'$, and $b=d\,b'$; hence $\frac{a}{b}=\frac{d\,a'}{d\,b'}=\frac{a'}{b'}$, and consequently, by dividing both numerator and denominator of a fraction by the greatest common measure of the terms of the fraction, the resulting fraction will be simplified to its utmost extent, and thus the proposed fraction will be reduced to its lowest terms.

Examples.

(1.) What is the greatest common measure of $4x^2y^3z^4$ and $8x^4y^3z^2$?

Here 4 is the greatest common measure of 4 and 8, and $x^2y^3z^2$ is that of the literal parts; hence $4x^2y^3z^2$ is the greatest common measure required.

(2.) Find the greatest common measure of $\frac{x^3+y^3}{x^2-y^2}$.

$$x^2-y^2)\ x^3+y^3\ (x$$
$$x^3-xy^2$$

$$xy^2+y^3=y^2\,(x+y);\ \text{rejecting the factor } y^2$$
$$x+y)\ x^2-y^2\ (x-y$$
$$x^2+xy$$
$$-xy-y^2$$
$$-xy-y^2$$

Hence $x+y$ is the greatest common measure sought, and

$$\frac{x^3+y^3}{x^2-y^2}=\frac{(x^3+y^3)\div(x+y)}{(x^2-y^2)\div(x+y)}=\frac{x^2-xy+y^2}{x-y}=\text{reduced fraction.}$$

(3.) Required the greatest common measure of the two polynomials

$$6a^3-6a^2y+2ay^2-2y^3\ \ldots\ldots\ (a)$$
$$12a^2-15ay+3y^2\ \ldots\ldots\ (b).$$

Here $6a^3-6a^2y+2ay^2-2y^3=2\,(3a^3-3a^2y+ay^2-y^3)$
$12a^2-15ay+3y^2=3\,(4a^2-5ay+y^2)$;

And therefore, by suppressing the factors 2 and 3, which have no common measure, we have to find the greatest common measure of

$$3a^3-3a^2y+ay^2-y^3 \text{ and } 4a^2-5ay+y^2.$$

$$4a^2-5ay+y^2)\ 3a^3-3a^2y+ay^2-y^3$$
$$4$$
$$12a^3-12a^2y+4ay^2-4y^3\ (3a$$
$$12a^3-15a^2y+3ay^2$$
$$3a^2y+ay^2-4y^3$$
$$4$$
$$12a^2y+4ay^2-16y^3\ (3y$$
$$12a^2y-15ay^2+3y^3$$
$$19ay^2-19y^3=19y^2\,(a-y)$$
$$\text{Or, } a-y)\ 4a^2-5ay+y^2\ (4a-y$$
$$4a^2-4ay$$
$$-ay+y^2$$
$$-ay+y^2.$$

Hence $a-y$ is the greatest common measure of the polynomials a and b.

(4.) Required the greatest common measure of the terms of the fraction

$$\frac{a^6-a^2x^4}{a^6+a^5x-a^4x^2-a^3x^3}.$$

Here a^2 is a simple factor of the numerator, and a^3 is a factor of the denominator; hence a^2 is the greatest common measure of these simple factors,

which must be reserved to be introduced into the greatest common measure of the other factors of the terms of the proposed fractions; viz.:

$$a^4-x^4 \text{ and } a^3+a^2x-ax^2-x^3.$$

$$\begin{array}{l} a^3+a^2x-ax^2-x^3)\ a^4-x^4 \qquad (a-x \\ \qquad\qquad\qquad\quad a^4+a^3x-a^2x^2-ax^3 \\ \hline \qquad\qquad\qquad\quad -a^3x+a^2x^2+ax^3-x^4 \\ \qquad\qquad\qquad\quad -a^3x-a^2x^2+ax^3+x^4 \\ \hline \qquad\qquad\qquad\qquad 2a^2x^2-2x^4=2x^2\,(a^2-x^2);\ \text{rejecting } 2x^2 \end{array}$$

$$\begin{array}{l} a^2-x^2)\ a^3+a^2x-ax^2-x^3\ (a+x \\ \qquad\quad a^3-ax^2 \\ \hline \qquad\qquad a^2x-x^3 \\ \qquad\qquad a^2x-x^3 \\ \hline \end{array}$$

Therefore $a^2\,(a^2-x^2)$ is the greatest common measure; and hence

$$\frac{a^6-a^2x^4}{a^6+a^5x-a^4x^2-a^3x^3}=\frac{(a^6-a^2x^4)\div a^2\,(a^2-x^2)}{(a^6+a^5x-a^4x^2-a^3x^3)\div a^2\,(a^2-x^2)}=\frac{a^2+x^2}{a^2+ax}.$$

Additional Examples.

(1.) Find the greatest common measure of $2a^2x^2$, $4x^2y^2$, and $6x^3y$.

(2.) Find the greatest common measure of the two polynomials $a^3-a^2b+3ab^2-3b^3$, and $a^2-5ab+4b^2$.

(3.) What is the greatest common measure of x^3-xy^2 and $x^2+2xy+y^2$?

(4.) Find the greatest common measure of x^8+y^8 and $x^{13}-y^{13}$.

(5.) Find the greatest common measure of the polynomials

$$(b-c)\,x^2-b\,(2b-c)\,x+b^3 \ \ldots\ldots \ (a)$$
$$(b+c)\,x^3-b\,(2b+c)\,x^2+b^3x \ \ldots\ldots \ (b).$$

(6.) Find the greatest common measure of the polynomials

$$x^4-8x^3+21x^2-20x+4 \ \ldots\ldots \ (a)$$
$$2x^3-12x^2+21x-10 \ \ldots\ldots \ (b).$$

Answers.

(1.)	$2x^2$	(4.)	$x-y$.
(2.)	$a-b$.	(5.)	$x-b$.
(3.)	$x+y$.	(6.)	$x-2$.

The least Common Multiple.

32. We have already defined a *multiple* of a quantity to be any quantity that contains it exactly; and a *common multiple* of two or more quantities is a quantity that contains each of them exactly.

The *least common multiple*, of two or more quantities, is therefore the least quantity that contains each of them exactly.

33. *To find the least common multiple of two quantities.*

Divide the product of the two proposed quantities by their greatest common measure, and the quotient is the least common multiple of these

quantities; or divide one of the quantities by their greatest common measure, and multiply the quotient by the other.

Let a and b be two quantities, d their greatest common measure; and m their least common multiple; then let

$$a = hd, \text{ and } b = kd;$$

and since d is the greatest common measure, h and k can have no common factor, and hence their least common multiple is hk; therefore hkd is the least common multiple of hd and kd; hence,

$$m = hkd = \frac{hkd^2}{d} = \frac{hd \times kd}{d} = \frac{a \times b}{d} = \frac{ab}{d} \qquad \text{Q. E. D.}$$

EXAMPLES.

(1.) Find the least common multiple of $2a^2x$ and $8a^3x^3$.

Here $m = \dfrac{ab}{d} = \dfrac{2a^2x \times 8a^3x^3}{2a^2x} = 8a^3x^3 =$ least common multiple.

(2.) Find the least common multiple of $4x^2(x^2-y^2)$ and $12x^3(x^3-y^3)$.

Here $d = 4x^2(x-y)$, and therefore we have

$$m = \frac{ab}{d} = \frac{4x^2(x^2-y^2) \times 12x^3(x^3-y^3)}{4x^2(x-y)} = 12x^3(x+y)(x^3-y^3);$$

$$\text{or } m = 12x^7+12x^6y-12x^4y^3-12x^3y^4.$$

(3.) Find the least common multiple of $x^2+2xy+y^2$ and x^3-xy^2.

Here $d = x+y$, and therefore we get

$$m = \frac{a}{d}.b = \frac{x^2+2xy+y^2}{x+y}.(x^3-xy^2)$$

$$= (x+y)(x^3-xy^2)$$

$$= x(x+y)(x^2-y^2) = \text{least common multiple.}$$

(4.) What is the least common multiple of $x^4-5x^3+9x^2-7x+2$, and x^4-6x^2+8x-3?

By the process for finding the greatest common measure, we find

$$d = x^3-3x^2+3x-1$$

$$\therefore m = \frac{x^4-5x^3+9x^2-7x+2}{x^3-3x^2+3x-1}(x^4-6x^2+8x-3)$$

$$= (x-2)(x^4-6x^2+8x-3)$$

$$= x^5-2x^4-6x^3+20x^2-19x+6, \text{ the least common multiple.}$$

(5.) Find the least common multiple of $a^2-2ab+b^2$, and a^4-b^4.

(6.) Find the least common multiple of a^2-b^2, and a^3+b^3.

(7.) Find the least common multiple of x^2-y^2, and x^3-y^3.

(8.) Find the least common multiple of y^2-8y+7, and y^2+7y-8.

ANSWERS.

(1.) $(a-b)(a^4-b^4)$. (3.) $(x+y)(x^3-y^3)$

(2.) $(a-b)(a^3+b^3)$. (4.) $y^3-57y+56$.

34. *Every common multiple of two quantities,* a *and* b, *is a multiple of* m, *the least common multiple.*

For let m' be a common multiple of a and b, then, because m' is greater than m; and if we suppose that m' is not a multiple of m, we have, as in the annexed scheme,

$$m) \; m' \; (h$$
$$hm$$
$$\overline{k = \text{remainder.}}$$

$$m' = hm + k \; . \; . \; . \; (1)$$

$$m' - hm = k \; . \; . \; . \; (2)$$

Now the remainder h is always less than m the divisor; hence, since a and b measure m and m', it is evident by (2) that a and b measure $m'-h\,m$, or h; therefore h is a common multiple of a and b, and it has been proved to be less than m, the least common multiple, which is absurd; hence m must measure m', or m' is a multiple of m.

35. *To find the least common multiple of three or more quantities.*

Let a, b, c, d, &c., be the proposed quantities;

find m the least common multiple of a and b

. . m' . . . . c and m

. . m'' . . . . d and m' .

&c. &c.

Then, since every multiple of a and b measures m, their least common multiple, the quantity sought, x, measures m; but x also measures c; therefore x measures both c and m, and thence it measures m'; but x measures d and m', and therefore must measure m''; hence x cannot be less than m'', and therefore m'' is the least common multiple.

Examples.

(1.) Find the least common multiple of $2a^2$, $4a^3 b^2$, and $6a\,b^3$.

Here taking $2a^2$ and $4a^3 b^2$, we find $d=2a^2$, and, therefore,

$$m = \frac{a\,b}{d} = \frac{2a^2 \times 4a^3 b^2}{2a^2} = 4a^3 b^2.$$

Again, taking m, or $4a^3 b^2$, and $6a\,b^3$, we find $d = 2a\,b^2$; hence

$$m' = \frac{c\,m}{d} = \frac{6a\,b^3 \times 4a^3 b^2}{2a\,b^2} = 12a^3 b^3 = \text{answer required.}$$

(2.) Find the least common multiple of $a-x$, a^2-x^2, and a^3-x^3.

Taking $a-x$ and a^2-x^2, we have $d=a-x$; and hence

$$m = \frac{a\,b}{d} = \frac{a-x}{a-x} \times (a^2-x^2) = a^2-x^2.$$

Again, taking a^2-x^2 and a^3-x^3, we find $d=a-x$; hence

$$m' = \frac{c\,m}{d} = \frac{(a^3-x^3)\,(a^2-x^2)}{a-x} = (a+x)\,(a^3-x^3) = \text{answer sought.}$$

(3.) Find the least common multiple of $15a^2 b^2$, $12a\,b^3$, and $6a^3 b$.

(4.) Find the least common multiple of $6a^2 x^2 (a-x)$, $8x^3 (a^2-x^2)$ and $12 (a-x)^2$.

(5.) Find the least common multiple of $x^3-x^2y-xy^2+y^3$, $x^3-x^2y+xy^2-y^3$; and x^4-y^4.

(6.) Find the least common multiple of $(a+b)^2$, (a^2-b^2), $(a-b)^2$, and $a^3+3a^2 b+3a\,b^2+b^3$.

Answers.

(3.) $60a^3 b^3$.

(4.) $24a^2 x^3 (a-x)$ ([illegible]).

(5.) $x^5-x\,y^4-x^4y+y^5$.

(6.) $(a+b)\,(a^2-b^2)^2$.

Of Algebraic Fractions.

36. Algebraic fractions differ in no respect from arithmetical fractions; and all the observations which we have made upon the latter, apply equally to the former. We shall therefore merely repeat the rules already deduced, adding a few examples of the application of each. It may be proper to remind the reader, that all our operations with regard to fractions were founded upon the three following principles:—

1. *In order to multiply a fraction by any number, we must multiply the numerator, or divide the denominator of the fraction by that number.*

2. *In order to divide a fraction by any number, we must divide the numerator, or multiply the denominator of the fraction by that number.*

3. *The value of a fraction is not changed, if we multiply or divide both the numerator and denominator by the same number.**

Reduction of Fractions.

1. *To reduce a fraction to its lowest terms.*

37. Rule.—*Divide both numerator and denominator by their greatest common measure, and the result will be the fraction in its lowest terms.*

When the numerator and denominator are, one or both of them, monomials, their greatest common factor is immediately detected by inspection; thus.

$$\frac{a^2bc}{5a^2b^2}=\frac{a^2b\times c}{a^2b\times 5b}=\frac{c}{5b} \text{ in its lowest terms.}$$

So also,

$$\frac{ax^2}{ax+x^2}=\frac{x\times ax}{x(a+x)}=\frac{ax}{a+x} \text{ in its lowest terms.}$$

If, however, both numerator and denominator are polynomials, we must have recourse to the method of finding the greatest common measure of two algebraic quantities, developed in a former article. Thus, let it be required to reduce the following fraction to its lowest terms:

$$\frac{6a^3-6a^2y+2ay^2-2y^3}{12a^2-15ay+3y^2}$$

* These principles will be obvious from the following considerations:—

1. If the numerator of a fraction be increased any number of times, the fraction itself will be increased as many times; and if the denominator be diminished any number of times, the fraction must still be increased as many times.

2. If the denominator of a fraction be increased any number of times, or the numerator diminished the same number of times, the fraction itself will in either case be diminished the same number of times.

3. If the numerator of a fraction be increased any number of times, the fraction is increased the same number of times; and if the denominator be increased as many times, the fraction is again diminished the same number of times, and must therefore have its original value.

The greatest common measure of the two terms of this fraction was found in page 114 to be $a-y$; therefore, dividing both numerator and denominator by this quantity, we obtain as our result the fraction in its lowest terms; or,

$$\frac{6a^2+2y^2}{12a-3y}.$$

In like manner, taking the fraction $\frac{4a^4-4a^2b^2+4ab^3-b^4}{6a^4+4a^3b-9a^2b^2-3ab^3+2b^4}$ the greatest common measure of the two terms is found to be $2a^2+2ab-b^2$; and dividing both numerator and denominator by this quantity, the reduced fraction is,

$$\frac{2a^2-2ab+b^2}{3a^2-ab-2b^2}$$

Examples for practice.

(1.) Reduce $\frac{2x^3-16x-6}{3x^3-24x-9}$ to its lowest terms.

(2.) Reduce $\frac{48x^3+36x^2-15}{24x^3-22x^2+17x-6}$ to its lowest terms.

(3.) Reduce $\frac{20x^4+x^2-1}{25x^4+5x^3-x-1}$ to its lowest terms.

(4.) Reduce $\frac{9x^5+2x^3+4x^2-x+1}{15x^4-2x^3+10x^2-x+2}$ to its lowest terms.

(5.) Reduce $\frac{4a^3cx-4a^3dx+24a^2bcx-24a^2bdx+36ab^2cx-36ab^2dx}{7abcx^3-7abdx^3+7ac^2x^3-7acdx^3-21b^2dx^3+21b^2cx^3+21bc^2x^3+21bcdx^3}$ to its lowest terms. Ans. $\frac{4a(a+3b)}{7x^2(b+c)}$

38. It frequently happens, however, that when the polynomials which form the numerator and denominator of a fraction which can be decomposed are not very complicated, we are enabled by a little practice to detect the factor and effect the reduction, without performing the operation of finding the greatest common measure, which is generally a tedious process. The results to which we called the attention of the reader, at the end of algebraic division (see page 107), will be found particularly useful in simplifications of this nature.

Thus for example:

(6.) $$\frac{3x^2y+3xy^2}{3x^2+6xy+3y^2}=\frac{3xy(x+y)}{3(x+y)^2}=\frac{3xy(x+y)}{3(x+y)(x+y)}=\frac{xy}{x+y}.$$

(7.) $$\frac{a^2-b^2}{a^2-2ab+b^2}=\frac{(a-b)(a+b)}{(a-b)^2}=\frac{a+b}{a-b}$$

(8.) $$\frac{5a^3+10a^2b+5ab^2}{8a^3+8a^2b}=\frac{5a(a^2+2ab+b^2)}{8a^2(a+b)}=\frac{5a(a+b)^2}{8a^2(a+b)}=\frac{5(a+b)}{8a}$$

(9.) $$\frac{a^3-x^3}{a^2-2ax+x^2}=\frac{(a^2+ax+x^2)(a-x)}{(a-x)^2}=\frac{a^2+ax+x^2}{a-x}$$

(10.) $\dfrac{ac+bd+ad+bc}{af+2bx+2ax+bf}=\dfrac{(a+b)c+(a+b)d}{(a+b)f+2x(a+b)}=\dfrac{c+d}{f+2x}$

(11.) $\dfrac{6ac+10bc+9ad+15bd}{6c^2+9cd-2c-3d}=\dfrac{3a(2c+3d)+5b(2c+3d)}{3c(2c+3d)-(2c+3d)}=\dfrac{3a+5b}{3c-1}$

(12.) $\dfrac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}=\dfrac{x^{m-1}(ax-bx^2)}{bx(a^2-b^2x^2)}=\dfrac{x^{m-1}(ax-bx^2)}{bx(a+bx)(a-bx)}=\dfrac{x^{m-1}}{b(a+bx)}$

II. *To reduce a mixed quantity to an improper fraction.*

39. RULE.—*Multiply the integral part by the denominator of the fraction, and to the product add the numerator with its proper sign; then the result placed over the denominator will give the improper fraction required.* Thus,

(1.) $\dfrac{a}{b}+1=\dfrac{a+b}{b}$

(2.) $1+\dfrac{a^2-x^2}{a^2+x^2}=\dfrac{a^2+x^2+a^2-x^2}{a^2+x^2}=\dfrac{2a^2}{a^2+x^2}$

(3.) $ab+cd+\dfrac{abc-c^2d-2cd^2}{c+2d}=\dfrac{abc+c^2d+2abd+2cd^2+abc-c^2d-2cd^2}{c+2d}$

$$=\frac{2abc+2abd}{c+2d}$$

$$=\frac{2ab(c+d)}{c+2d}$$

(4.) $1+\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{2bc+b^2+c^2-a^2}{2bc}=\dfrac{(b+c)^2-a^2}{2bc}$

40. It is to be remarked that when a fraction has the sign $-$, it signifies that the whole fraction is to be subtracted, and consequently the negative sign applies to the numerator alone; and when the numerator is a polynomial, the negative sign extends to every term of the polynomial; thus,

(5.) $1-\dfrac{b}{a}=\dfrac{a-b}{a}$

(6.) $c-\dfrac{ef}{d}=\dfrac{cd-ef}{d}$

(7.) $1-\dfrac{a^2-2ab+b^2}{a^2+b^2}=\dfrac{a^2+b^2-(a^2-2ab+b^2)}{a^2+b^2}=\dfrac{2ab}{a^2+b^2}$

(8.) $1-\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{2bc-(b^2+c^2-a^2)}{2bc}$

$$=\frac{a^2-(b^2-2bc+c^2)}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}$$

(9.) $x^2+2xy+y^2-\dfrac{x^3-3x^2y+3xy^2-y^3}{x+y}$

$$=\frac{x^3+3x^2y+3xy^2+y^3-(x^3-3x^2y+3xy^2-y^3)}{x+y}$$

$$=\frac{6x^2y+2y^3}{x+y}$$

$$=\frac{2y(3x^2+y^2)}{x+y}$$

(10) $mn-pq-\dfrac{2mn^2-2pqn}{m+n}=\dfrac{m^2n-mpq+mn^2-npq-(2mn^2-2pqn)}{m+n}$

$$=\frac{m^2n-mpq-mn^2+pqn}{m+n}$$

$$=\frac{mn(m-n)-pq(m-n)}{m+n}$$

$$=\frac{(mn-pq)(m-n)}{m+n}$$

III. *To reduce fractions to others equivalent, and having a common denominator.*

41. RULE.—*Multiply each of the numerators, separately, into all the denominators, except its own, for the new numerators, and all the denominators together for a common denominator.*

Thus: reduce $\dfrac{a}{b}$ and $\dfrac{c}{d}$ to equivalent fractions having a common denominator.

$a\times d$ is the new numerator of the first,
$c\times b$ is the new numerator of the second,
$b\times d$ is the common denominator;

Therefore the fractions required are $\dfrac{ad}{bd}$ and $\dfrac{bc}{bd}$.

Reduce $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}, \dfrac{g}{h}, \dfrac{k}{l}, \dfrac{m}{n}$, to a common denominator.

$\dfrac{adfhln}{bdfhln}, \dfrac{cbfhln}{bdfhln}, \dfrac{ebdhln}{bdfhln}, \dfrac{gbdfln}{bdfhln}, \dfrac{kbdfhn}{bdfhln}, \dfrac{mbdfhl}{bdfhln}$, are the fractions required.

Reduce $\dfrac{1+x}{1-x}, \dfrac{1+x^2}{1-x^2}, \dfrac{1+x^3}{1-x^3}$ to a common denominator.

$\dfrac{(1+x)(1-x^2)(1-x^3)}{(1-x)(1-x^2)(1-x^3)}, \dfrac{(1+x^2)(1-x)(1-x^3)}{(1-x)(1-x^2)(1-x^3)}, \dfrac{(1+x^3)(1-x)(1-x^2)}{(1-x)(1-x^2)(1-x^3)}$ are the fractions required.

ADDITION OF FRACTIONS.

42. RULE.—*Reduce the fractions to a common denominator, add the numerators together, and subscribe the common denominator.* Thus:

(1.) $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad}{bd}+\dfrac{bc}{bd}=\dfrac{ad+bc}{bd}$

(2.) $\frac{a}{b}+\frac{m}{n}+\frac{p}{q}+\frac{x}{y}=\frac{anqy}{bnqy}+\frac{mbqy}{bnqy}+\frac{pbny}{bnqy}+\frac{xbnq}{bnqy}$

$=\frac{anqy+mbqy+pbny+xbnq}{bnqy}$

(3.) $\frac{a}{bx}+\frac{c}{dx^2}+\frac{e}{fx^3}=\frac{adfx^5}{bdfx^6}+\frac{cbfx^4}{bdfx^6}+\frac{ebdx^3}{bdfx^6}$

$=\frac{adfx^5+bcfx^4+bdex^3}{bdfx^6}$

(4.) $\frac{1+x^2}{1-x^2}+\frac{1-x^2}{1+x^2}=\frac{(1+x^2)^2}{(1-x^2)(1+x^2)}+\frac{(1-x^2)^2}{(1-x^2)(1+x^2)}$

$=\frac{(1+x^2)^2+(1-x^2)^2}{(1-x^2)(1+x^2)}$

$=\frac{2(1+x^4)}{1-x^4}$

(5.) $\frac{1}{1+x}+\frac{1}{1-x}=\frac{1-x}{(1+x)(1-x)}+\frac{1+x}{(1+x)(1-x)}$

$=\frac{1-x+1+x}{(1+x)(1-x)}$

SUBTRACTION OF FRACTIONS.

43. RULE.—*Reduce the fractions to a common denominator, subtract the numerator or the sum of the numerators of the fractions to be subtracted, from the numerator or the sum of the numerators of the others, and subscribe the common denominator.*

(1) $\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$

(2) $\frac{a}{b}+\frac{m}{n}-\frac{p}{q}-\frac{x}{y}=\frac{anqy}{bnqy}+\frac{mbqy}{bnqy}-\frac{pbny}{bnqy}-\frac{xbnq}{bnqy}$

$=\frac{anqy+mbqy-pbny-xbnq}{bnqy}$

(3) $\frac{a}{bx}+\frac{c}{dx^2}-\frac{e}{fx^3}-\frac{g}{hx^4}=\frac{adfhx^9}{bdfhx^{10}}+\frac{bcfhx^8}{bdfhx^{10}}-\frac{bedhx^7}{bdfhx^{10}}-\frac{bdfgx^6}{bdfhx^{10}}$

$=\frac{adfhx^9+bcfhx^8-bedhx^7-bdfgx^6}{bdfhx^{10}}$

(4) $\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{(a+b)^2-(a-b)^2}{(a+b)(a-b)}$

$=\frac{4ab}{a^2-b^2}$

(5) $\frac{1+x^2}{1-x^2} - \frac{1-x^2}{1+x^2} = \frac{(1+x^2)^2}{(1-x^2)(1+x^2)} - \frac{(1-x^2)^2}{(1-x^2)(1+x^2)}$

$$= \frac{(1+x^2)^2 - (1-x^2)^2}{(1-x^2)(1+x^2)}$$

$$= \frac{4x^2}{1-x^4}$$

44. When the denominators of the fractions which it is required to reduce are expressed in numbers, the result will frequently be much simplified by finding the least common multiple of the denominators, and then reducing the fractions to their least common denominator, according to the method explained in Arithmetic.

Thus, if we are required to reduce the following fractions:

$$\frac{a-3x}{4} + \frac{3a-5x}{5} + \frac{3a-5x}{20}$$

The least common multiple of 4 and 5 is 20, the denominator of the third fraction; therefore the fractions, when reduced to their least common denominator, are

$$\frac{5a-15x}{20} + \frac{12a-20x}{20} + \frac{3a-5x}{20} = \frac{5a-15x+12a-20x+3a-5x}{20}$$

$$= \frac{20a-40x}{20}$$

$$= a-2x$$

So also,

$$x + \frac{27-9x}{4} - \frac{5x+2}{6} - \frac{61}{12} + \frac{2x+5}{3} + \frac{29+4x}{12} - \frac{5-37x}{12}$$

the least common multiple of 3, 4, 6 is 12, which will be the least common denominator, and the above fractions become

$$\frac{12x}{12} + \frac{81-27x}{12} - \frac{10x+4}{12} - \frac{61}{12} + \frac{8x+20}{12} + \frac{29+4x}{12} - \frac{5-37x}{12}$$

Or,

$$\frac{12x+81-27x-10x-4-61+8x+20+29+4x-5+37x}{12} = \frac{24x+60}{12}$$

$$= 2x+5$$

MULTIPLICATION OF FRACTIONS.

45. RULE.—*Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator.* Thus,

(1) $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

(2) $\frac{a}{b} \times \frac{m}{n} \times \frac{p}{q} \times \frac{x}{y} = \frac{ampx}{bnqy}$

(3) $\frac{a+b}{c+d} \times \frac{e-f}{g-h} \times \frac{k+l}{m-n} \times \frac{p-q}{r+s} = \frac{(a+b)(e-f)(k+l)(p-q)}{(c+d)(g-h)(m-n)(r+s)}$

(4) $\frac{a}{b\,x} \times \frac{b}{c\,x^2} \times \frac{c}{d\,x^3} \times \frac{d}{e\,x^4} \times \frac{e}{f\,x^5} = \frac{a\,b\,c\,d\,e}{b\,c\,d\,e\,f\,x^{15}} = \frac{a}{f\,x^1}$

DIVISION OF FRACTIONS.

46. RULE.—*Invert the divisor and proceed as in Multiplication.*

(1) $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

(2) $\frac{a+b}{c+d} \div \frac{e-f}{g-h} = \frac{a+b}{c+d} \times \frac{g-h}{e-f} = \frac{(a+b)\,(g-h)}{(c+d)\,(e-f)}$

(3) $\frac{1+x^2}{1-x^2} \div \frac{1-x^2}{1+x^2} = \frac{1+x^2}{1-x^2} \times \frac{1+x^2}{1-x^2} = \frac{(1+x^2)^2}{(1-x^2)^2}$

(4) $\frac{x^4-b^4}{x^2-2bx+b^2} \div \frac{x^2+b\,x}{x-b} = \frac{x^4-b^4}{x^2-2\,b\,x+b^2} \times \frac{x-b}{x^2+b\,x}$

$$= \frac{(x^4-b^4)\,(x-b)}{(x^2-2\,b\,x+b^2)\,(x^2+b\,x)}$$

$$= \frac{(x^2-b^2)\,(x^2+b^2)\,(x-b)}{(x-b)^2\,.\,x\,.\,(x+b)}$$

$$= \frac{(x+b)\,(x-b)\,(x^2+b^2)\,(x-b)}{x\,(x-b)\,(x-b)\,(x+b)}$$

$$= \frac{x^2+b^2}{x}$$

47. *Miscellaneous Examples in the operations performed in Algebraic Fractions*

(1) $\frac{3\,a}{4\,b} + \frac{5\,f}{8\,e} - \frac{x}{7\,y} = \frac{42\,a\,e\,y + 35\,b\,f\,y - 8\,b\,e\,x}{56\,b\,e\,y}$

(2) $\frac{2\,a}{3\,b\,c} + \frac{5\,d\,f}{8\,b^2\,c} - \frac{d\,e\,g}{6\,b^3\,c^3} = \frac{16\,a\,b\,c + 15\,c\,d\,f - 4\,d\,e\,g}{24\,b^3\,c^3}$

(3) $e - f - \frac{g^3}{2\,e\,f} + \frac{f^m}{3\,e\,g} = \frac{6\,e\,f\,g\,(e-f) - 3\,g^4 + 2\,f^{m+1}}{6\,e\,f\,g}$

(4) $\frac{a}{x^n} - \frac{c}{x^{n-1}} + \frac{d}{x^{n-r-s}} = \frac{a - c\,x + d\,x^{r+s}}{x^n}$

(5) $c + 2\,a\,b - 3\,a\,c - \frac{b^2\,c - 5\,a\,b^2\,c + a^2}{b^2 - b\,c} = \frac{2\,a\,b^3 - b\,c^2 + 3\,a\,b\,c^2 - a^2}{b^2 - b\,c}$

(6) $\frac{a+b}{2} + \frac{a-b}{2} = a$

(7) $\frac{a+b}{2} - \frac{a-b}{2} = b$

(8) $\frac{13a-5b}{4} - \frac{7a-2b}{6} - \frac{3a}{5} = \frac{89a-55b}{60}$

(9) $\frac{3a-4b}{7} - \frac{2a-b-c}{3} + \frac{15a-4c}{12} = \frac{85a-20b}{84}$

(10) $\frac{a}{b} + \frac{a-3b}{cd} + \frac{a^2-b^2-ab}{bcd} = \frac{acd-4b^2+a^2}{bcd}$

(11) $\frac{a^3}{(a+b)^3} - \frac{ab}{(a+b)^2} + \frac{b}{a+b} = \frac{a^3+ab^2+b^3}{(a+b)^3}$

(12) $\frac{3}{4(1-x)^2} + \frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x^2)} = \frac{1+x+x^2}{1-x-x^4+x^5}$

(13) $\frac{a^2+b^2}{a^2-b^2} \times \frac{a-b}{a+b} = \frac{a^2+b^2}{a^2+2ab+b^2}$

(14) $\frac{x^2-9x+20}{x^2-6x} \times \frac{x^2-13x+42}{x^2-5x} = \frac{x^2-11x+28}{x^2}$

(15) $\frac{x^2+3x+2}{x^2+2x+1} \times \frac{x^2+5x+4}{x^2+7x+12} = \frac{x+2}{x+3}$

(16) $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{e}{f}+\frac{g}{h}} = \frac{(ad+bc)fh}{(eh+fg)bd}$

(17) $\dfrac{\frac{a}{a+b}+\frac{b}{a-b}}{\frac{a}{a-b}-\frac{b}{a+b}} = 1$

(18) $\dfrac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}} = \frac{a+x^2}{2ax}$

(19) $\dfrac{1+\frac{n-1}{n+1}}{1-\frac{n-1}{n+1}} = n$

(20) $\frac{a^3-a^2x+ax^2-x^3}{a^5-a^4x+a^3x^2-a^2x^3+ax^4-x^5} = \frac{a^4-x^4}{a^6-x^6}$

$= \frac{a^2+x^2}{a^4+a^2x^2+x^4}$

ON THE FORMATION OF POWERS, AND THE EXRACTION OF ROOTS OF ALGEBRAIC QUANTITIES.

48. We begin by considering the case of monomials, and, in order to simplify the subject as much as possible, we shall first treat of the formation of the square and the extraction of the square root only, and then proceed to generalize our reasonings in such a manner as to embrace powers and roots of any degree whatsoever.

Definition.—The *square root* of any expression is that quantity which, when multiplied by itself, will produce the proposed expression. Thus the square root of a^2 is a, because a, when multiplied by itself, produces a^2; the square root of $(a+b)^2$ is $a+b$, because $a+b$, when multiplied by itself, produces $(a+b)^2$; in like manner 8 is the square root of 64, 12 of 144, and so on. The process of finding the square root of any quantity is called the *extraction of the square root.*

The extraction of the square root is *indicated* by prefixing the symbol $\sqrt{\ }$ to the quantity whose root is required. Thus $\sqrt{a^4}$ signifies that the square root of a^4 is to be extracted; $\sqrt{a^2+2ab+b^2}$ signifies that the square root of $a^2+2ab+b^2$ is to be extracted, &c.

In order to discover the method which we must pursue in order to extract the square root of a monomial, let us consider in what manner we form its square. According to the rule for the multiplication of monomials,

$$(5a^2b^3c)^2 = 5a^2b^3c \times 5a^2b^3c = 25a^4b^6c^2$$

So,

$$(9ab^2c^3d^4)^2 = 9ab^2c^3d^4 \times 9ab^2c^3d^4 = 81a^2b^4c^6d^8$$

And,

$$(Ax^my^nz^h \text{---})^2 = Ax^my^nz^h \text{---} \times Ax^my^nz^h \text{---} = A^2x^{2m}y^{2n}z^{2h} \,..$$

49. Hence it appears, that, in order to square a monomial, we must *square its coefficient, and multiply the exponents of each of the different letters by* 2. Therefore, in order to derive the square root of a monomial from its square, we must,

I. *Extract the square root of its coefficient according to the rules of Arithmetic.*

II. *Divide each of the exponents by* 2.

Thus we shall have,

$$\sqrt{64a^6b^4} = 8a^3b^2$$

This is manifestly the true result, for

$$(8a^3b^2)^2 = 8a^3b^2 \times 8a^3b^2 = 64a^6b^4$$

Similarly,

$$\sqrt{625a^2b^8c^6} = 25ab^4c^3$$

Here also,

$$(25ab^4c^3)^2 = 25ab^4c^3 \times 25ab^4c^3 = 625a^2b^8c^6$$

50. It appears from the preceding rule, that *a monomial cannot be the square of another monomial unless its coefficient be a square number, and the exponents of the different letters all even numbers.* Thus $98ab^4$ is not a perfect square, for 98 is not a square number, and the exponent of a is not an even number. In this case we introduce the quantity into our calculations, affected with the sign

/ , and it is written under the form $\sqrt{98\,a\,b^4}$. Expressions of this nature are called *Surds*, or *Radicals*, *of the Second degree.*

51. Such expressions can frequently be simplified by the application of the following principle:—*The square root of the product of two or more factors is equal to the product of the square roots of these factors.* Or, in algebraic language,

$$\sqrt{a\,b\,c\,d\,\text{-----}} = \sqrt{a} \times \sqrt{b} \times \sqrt{c} \times \sqrt{d}\,\text{-----}$$

In order to demonstrate this principle, let us remark, that, according to our definition of the square root of any expression, we have,

$$(\sqrt{a\,b\,c\,d\,\text{------------}})^2 = a\,b\,c\,d\,\text{-----}$$

Again,

$$(\sqrt{a} \times \sqrt{b} \times \sqrt{c} \times \sqrt{d}\,\text{---})^2 = (\sqrt{a})^2 \times (\sqrt{b})^2 \times (\sqrt{c})^2 \times (\sqrt{d})^2\,\text{---}$$
$$= a\,b\,c\,d\,\text{-----}$$

Hence, since the squares of the quantities $\sqrt{a\,b\,c\,d\,\text{-----}}$, and $\sqrt{a}\,.\,\sqrt{b}$ $\sqrt{c}\,.\,\sqrt{d}\,\text{---}$ are equal, the quantities themselves must be equal.

This being established, the expression given above $\sqrt{98\,a\,b^4}$ may be put under the form $\sqrt{49\,b^4} \times \sqrt{2\,a}$, but $\sqrt{49\,b^4}$ is by (Art. 49) $= 7\,b^2$; hence

$$\sqrt{98\,b^4\,a} = \sqrt{49\,b^4} \times \sqrt{2\,a} = 7\,b^2\sqrt{2\,a}$$

Similarly,

$$\sqrt{45\,a^2\,b^3\,c^2\,d} = \sqrt{9\,a^2\,b^2\,c^2 \times 5\,b\,d} = \sqrt{9\,a^2\,b^2\,c^2} \times \sqrt{5\,b\,d}$$
$$= 3\,a\,b\,c\,\sqrt{5\,b\,d}$$

So also,

$$\sqrt{864\,a^2\,b^5\,c^{11}} = \sqrt{144\,a^2\,b^4\,c^{10} \times 6\,b\,c} = \sqrt{144\,a^2\,b^4\,c^{10}} \times \sqrt{6\,b\,c}$$
$$= 12\,a\,b^2\,c^5\,\sqrt{6\,b\,c}$$

In general, therefore, in order to simplify a monomial radical of the second degree, *separate those factors which are perfect squares, extract their root* (Art. 49), *place the product of all these roots before the radical sign, under which are to be included all those factors which are not perfect squares.*

In the expressions, $7\,b^2\sqrt{2\,a}$, $3\,a\,b\,c\,\sqrt{5\,b\,d}$, $12\,a\,b^2\,c^5\sqrt{6\,b\,c}$, &c. the quantities $7\,b^2$, $3\,a\,b\,c$, $12\,a\,b^2\,c^5$, are called the *coefficients of the radical.*

52. We have not hitherto considered the sign with which the radical may be affected. But since, as will be seen hereafter, in the solution of problems we are led to consider monomials affected with the sign —, as well as the sign +, it is necessary that we should know how to treat such quantities. Now the square of a monomial being the product of the monomial by itself, it necessarily follows, that *whatever may be the sign of a monomial, its square must be affected with the sign +.* Thus, the square of $+\,5\,a^2\,b^3$, or of $-\,5\,a^2\,b^3$, is $+\,25\,a^4\,b^6$.

Hence we conclude, that *if a monomial be positive its square root may be either positive or negative.* Thus, $\sqrt{9\,a^4} = +\,3\,a^2$, or $-\,3\,a^2$, for either of these quantities, when multiplied by itself, produces $9\,a^4$; we therefore always affect the square root of a quantity with the double sign $\pm$, which is called *plus or minus.* Thus, $\sqrt{9\,a^4} = \pm\,3\,a^2$, $\sqrt{144\,a^2\,b^4\,c^6} = \pm\,12\,a\,b^2\,c^3$.

53. If the monomial be affected with a negative sign, the extraction of its square root is impossible, since we have just seen that the square of every quantity, whether positive or negative, is essentially positive. Thus, $\sqrt{-\,9}$, $\sqrt{-4\,a^2}$

$\sqrt{-5}$, are algebraic symbols which represent operations which it is impossible to execute. Quantities of this nature are called *imaginary*, or, *impossible*, *quantities*, and are symbols of absurdity which we frequently meet with in resolving quadratic equations.

By an extension of our principles, however, we perform the same operations upon quantities of this nature as upon ordinary surds. Thus, by (Art. 51).

$$\sqrt{-9} = \sqrt{9 \times -1} = \sqrt{9}.\sqrt{-1} = 3\sqrt{-1}.$$
$$\sqrt{-4a^2} = \sqrt{4a^2 \times -1} = \sqrt{4a^2}\sqrt{-1} = 2a\sqrt{-1}.$$
$$\sqrt{-8a^2b} = \sqrt{2 \times 4a^2 \times b \times -1} = \sqrt{4a^2} \times \sqrt{2b} \times \sqrt{-1} = 2a\sqrt{2b}\sqrt{-1}.$$

54. Let us now proceed to consider the formation of powers and extraction of roots of any degree in monomial algebraic quantities.

DEFINITION.—The *cube root* of any expression is that quantity which, multiplied twice by itself, will produce the proposed expression. The *fourth*, or, *biquadrate root* of any expression is that quantity which, multiplied three times by itself, will produce the proposed expression; and in general, the n^{th} root of any expression is that quantity which, multiplied $(n - 1)$ times by itself, will produce the proposed expression. Thus, the cube root of $a^3 b^3$ is $a b$, because $a b$, multiplied by itself twice, produces $a^3 b^3$; for the same reason, $(a + b)$ is the 6^{th} root of $(a + b)^6$, 2 is the seventh root of 128, &c.

55. Let it be required to form the fifth power of $2a^3b^2$.

$$\begin{aligned}(2a^3b^2)^5 &= 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \\ &= 32a^{15}b^{10}.\end{aligned}$$

Where we perceive, 1°. That the coefficient has been raised to the fifth power; 2°. That the exponent of each of the letters has been multiplied by 5.

In like manner,

$$\begin{aligned}(8a^2b^3c)^3 &= 8a^2b^3c \times 8a^2b^3c \times 8a^2b^3c \\ &= 8^3a^{2+2+2}b^{3+3+3}c^{1+1+1} \\ &= 512a^6b^9c^3.\end{aligned}$$

So also,

$$\begin{aligned}(2ab^2c^3d^4)^n &= 2ab^2c^3d^4 \times 2ab^2c^3d^4 \times \text{-----to } n \text{ factors} \\ &= 2^na^nb^{2n}c^{3n}d^{4n}.\end{aligned}$$

Hence we deduce the following general

RULE TO RAISE A MONOMIAL TO ANY POWER.

Raise the numerical coefficient to the given power, and multiply the exponents of each of the letters by the index of the power required.

And hence reciprocally we obtain a

RULE TO EXTRACT THE ROOT, OF ANY DEGREE, OF A MONOMIAL

1°. *Extract the root of the numerical coefficient according to the rules of arithmetic.*

2°. *Divide the exponent of each letter by the index of the required root.*
Thus,

$$\sqrt[3]{64\,a^9 b^3 c^6} = 4\,a^3 b\,c^2$$
$$\sqrt[4]{16\,a^8 b^{12} c^{16} d^4} = 2\,a^2 b^3 c^4 d.$$

56 According to this rule, we perceive that in order that a monomial may be a perfect power of that degree whose root is required, its coefficient must be a perfect power of that degree, and the exponent of each letter must be divisible by the index of the root.

When the monomial whose root is required is not a perfect power of the required degree, we can only *indicate* the operation by placing the sign, $\sqrt{\ }$ before the quantity, and writing within it the index of the root. Thus, if it be required to extract the cube root of $4\,a^2 b^5$, the operation will be indicated by writing the expression,

$$\sqrt[3]{4\,a^2 b^5}.$$

Expressions of this nature are called *surds*, or, *irrational quantities*, or, *radicals of the second, third*, or, n^{th} *degree*, according to the index of the root required.

57. We can frequently simplify these quantities by the application of the following principle, which is merely an extension of that already proved in (Art. 51).

The n^{th} root of the product of any number of factors is equal to the product of the n^{th} roots of the different factors. Or, in algebraic language,

$$\sqrt[n]{a\,b\,c\,d\,------} = \sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} \times \sqrt[n]{d} \times ----$$

Raise each of these expressions to the power of n, then

$$(\sqrt[n]{a\,b\,c\,d\,-----})^n = a\,b\,c\,d\,----$$

Again,

$$(\sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} \times \sqrt[n]{d}---)^n = (\sqrt[n]{a})^n \times (\sqrt[n]{b})^n \times (\sqrt[n]{c})^n (\sqrt[n]{d})^n ----$$
$$= a\,b\,c\,d\,----$$

Hence, since the n^{th} powers of the quantities $\sqrt[n]{a\,b\,c\,d}$, and $\sqrt[n]{a}\,.\,\sqrt[n]{b}\,.\,\sqrt[n]{c}\,.\,\sqrt[n]{d}\,----$, are equal, the quantities themselves must be equal.

This being established, let us take the expression $\sqrt[3]{54\,a^4 b^3 c^2}$, whose root cannot be exactly extracted, since 54 is not a perfect cube, and the exponents of a and c are not exactly divisible by 3.

We have,

$$(1.)\ \sqrt[3]{54\,a^4 b^3 c^2} = \sqrt[3]{27 \times 2 \times a^3 \times a \times b^3 \times c^2}$$
$$= \sqrt[3]{27} \times \sqrt[3]{a^3} \times \sqrt[3]{b^3} \times \sqrt[3]{2\,a\,c^2}$$

by the principle just proved,

$$= 3\,a\,b\sqrt[3]{2\,a\,c^2}$$

So also,

$$(2.)\ \sqrt[4]{48\,a^5 b^8 c^6} = \sqrt[4]{16 \times 3 \times a^4 \times a \times b^8 \times c^4 \times c^2}$$
$$= \sqrt[4]{16} \times \sqrt[4]{a^4} \times \sqrt[4]{b^8} \times \sqrt[4]{c^4} \times \sqrt[4]{3} \times \sqrt[4]{a} \times \sqrt[4]{c^2}$$
$$= 2\,a\,b^2 c\sqrt[4]{3\,a\,c^2}.$$

$$(3.)\ \sqrt[6]{192\,a^7 b\,c^{12}} = \sqrt[6]{64 \times 3 \times a^6 \times a \times b \times c^{12}}$$
$$= \sqrt[6]{64} \times \sqrt[6]{a^6} \times \sqrt[6]{c^{12}} \times \sqrt[6]{3} \times \sqrt[6]{a} \times \sqrt[6]{b}$$
$$= 2\,a\,c^2 \sqrt[6]{3\,a\,b}.$$

In the above expressions, the quantities $3ab$, $2ab^2c$, $2ac^2$, placed before the radical sign, are called the *coefficients of the radical.*

58. There is another principle which can frequently be employed with advantage in treating these quantities; this is,

The m^{th} power of the n^{th} power of any quantity is equal to the $m\,n^{th}$ power of that quantity. Or, in algebraic language,

$$\{a^n\}^m = a^{mn}.$$

For we have,

$$\begin{aligned}\{a^3\}^4 &= a^3 \times a^3 \times a^3 \times a^3 \\ &= a^{3+3+3+3} = a^{12}.\end{aligned}$$

And in general,

$$\begin{aligned}\{a^n\}^m &= a^n \times a^n \times a^n \times a^n \text{ - - - to } m \text{ factors,} \\ &= a^{n+n+n+n} \text{ - - - to } m \text{ terms,} \\ &= a^{mn}.\end{aligned}$$

And reciprocally,

The $m\,n^{th}$ root of any quantity is equal to the m^{th} root of the n^{th} root of that quantity. Or, in algebraic language,

$$\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}$$

For let $\sqrt[m]{\sqrt[n]{a}} = p,$

Raise the two quantities to the power of m, $\sqrt[n]{a} = p^m,$

Again, raise both to the power of n, $a = p^{mn},$

Extract the $m\,n^{th}$ root, $\sqrt[mn]{a} = p,$

But by supposition, $\sqrt[m]{\sqrt[n]{a}} = p,$

$$\therefore \quad \sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}$$

Hence, as often as the index of the root is a number composed of two or more factors, we may obtain the root required by extracting in succession the roots whose indices are the factors of that number. Thus,

(1.) $$\begin{aligned}\sqrt[6]{4a^2} &= \sqrt[3\times 2]{4a^2}, \\ &= \sqrt[3]{\sqrt[2]{4a^2}} \text{ by the above principle,} \\ &= \sqrt[3]{2a}\end{aligned}$$

(2.) $$\begin{aligned}\sqrt[4]{36a^2b^2} &= \sqrt{\sqrt{36a^2b^2}} \\ &= \sqrt{6ab}\end{aligned}$$

(3.) In general,

$$\begin{aligned}\sqrt[mn]{a^n} &= \sqrt[m]{\sqrt[n]{a^n}} \\ &= \sqrt[m]{a}\end{aligned}$$

That is to say, that *when the index of the radical is multiplied by a certain number n, and the quantity under the radical sign is an exact n^{th} power, we can*

without changing the value of the radical, divide its index by n, and extract the n^{th} root of the quantity under the sign.

Thus,

$$\begin{aligned}\sqrt[8]{25a^4b^2c^6} &= \sqrt[2\times4]{5^2a^4b^2c^{2\times3}} \\ &= \sqrt[4]{5a^2bc^3} \\ \sqrt[6]{27m^{15}n^9p^6} &= \sqrt[2\times3]{3^3m^{3\times5}n^{3\times3}p^{3\times2}} \\ &= \sqrt[2]{3m^5n^3p^2}\end{aligned}$$

59. This last proposition is the converse of another not less important, which consists in this, that *we may multiply the index of a radical by any number, provided we raise the quantity under the sign to the power whose degree is marked by that number*, or, in algebraic language,

$$\sqrt[n]{a} = \sqrt[mn]{a^m}$$

For in fact, a is the same thing as $\sqrt[m]{a^m}$, and therefore,

$$\begin{aligned}\sqrt[n]{a} &= \sqrt[n]{\sqrt[m]{a^m}} \\ &= \sqrt[mn]{a^m}\end{aligned}$$

60. By aid of this last principle, we can always reduce two or more radicals of different degrees to others which shall have the same index. Let it be required, for example, to reduce the two radicals $\sqrt[3]{2a}$ and $\sqrt[5]{3bc}$ to others which shall be equivalent, and have the same index. If we multiply 3 the index of the first, by 5 the index of the second, and at the same time, raise $2a$ to the 5th power; if, in like manner, we multiply 5 the index of the second, by 3 the index of the first, and at the same time raise $3bc$ to the 3d power, we shall not change the value of the two radicals, which will thus become

$$\begin{aligned}\sqrt[3]{2a} &= \sqrt[5\times3]{(2a)^5} = \sqrt[15]{32a^5} \\ \sqrt[5]{3bc} &= \sqrt[3\times5]{(3bc)^3} = \sqrt[15]{27b^3c^3}\end{aligned}$$

We shall thus have the following general

RULE.

In order to reduce two or more radicals to others which shall be equivalent and have the same index, multiply the index of each radical by the product of the indices of all the others, and raise the quantity under the sign to the power whose degree is marked by that product.

Thus, let it be required to reduce $\sqrt{2a}$, $\sqrt[3]{3b^2c^3}$, $\sqrt[5]{4d^4e^5f^6}$, to the same index,

$$\begin{aligned}\sqrt[2]{2a} &= \sqrt[3\times5\times2]{(2a)^{3\times5}} &&= \sqrt[30]{2^{15}a^{15}} \\ \sqrt[3]{3b^2c^3} &= \sqrt[2\times5\times3]{(3b^2c^3)^{2\times5}} &&= \sqrt[30]{3^{10}b^{20}c^{30}} \\ \sqrt[5]{4d^4e^5f^6} &= \sqrt[2\times3\times5]{(4d^4e^5f^6)^{2\times3}} &&= \sqrt[30]{4^6d^{24}e^{30}f^{36}}\end{aligned}$$

The above rule, which bears a great analogy to that given for the reduction of fractions to a common denominator, is susceptible of the same modifications. Let it be required, for example, to reduce the radicals, $\sqrt[4]{a}$, $\sqrt[6]{5b}$, $\sqrt[8]{2c}$, to the same index: since the least common multiple of the numbers 4, 6, 8, is 24, it will be sufficient to multiply the index of the first by 6, of the second by 4, and of the third by 3, raising the quantities under the radical in each case to the powers of 6, 4, 3, respectively,

$$\sqrt[4]{a} = \sqrt[24]{a^6},\ \sqrt[6]{5b} = \sqrt[24]{625b^4},\ \sqrt[8]{3c} = \sqrt[24]{27c^3}$$

61. Let us now proceed to execute upon radicals, the fundamental operations of arithmetic.

ADDITION AND SUBTRACTION OF RADICALS.

DEFINITION.—Radicals are said to be *similar* when they have the same index, and when the quantity under the radical sign is the same in each; thus, $3\sqrt{a}$, $12\,a\,c\sqrt{a}$, $15\,b\sqrt{a}$, are similar radicals, as are also, $4\,a^2 b\sqrt[4]{m\,n^2 p^3}$, $5\,l\sqrt[4]{m\,n^2 p^3}$, $25\,d\sqrt[4]{m\,n^2 p^3}$, &c.

This being premised, in order to add or subtract two similar radicals we have the following

RULE.

Add or subtract their coefficients, and place the sum or difference, as a coefficient, before the common radical. For example,

(1.) $3\sqrt[3]{b}+2\sqrt[3]{b} \quad = 5\sqrt[3]{b}$

(2.) $3\sqrt[3]{b}-2\sqrt[3]{b} \quad = \sqrt[3]{b}$

(3.) $3\,p\,q\sqrt[5]{m\,n}+4\,l\sqrt[5]{m\,n} = (3\,p\,q+4\,l)\sqrt[5]{m\,n}$

(4.) $9\,c\,d\sqrt{a}-4\,c\,d\sqrt{a} \quad = 5\,c\,d\sqrt{a}$

If the radicals are not similar, we can only *indicate* the addition or subtraction by interposing the signs $+$ or $-$.

It frequently happens, that two radicals, which do not at first appear similar, may become so by simplification; thus,

(5.) $\sqrt{48\,a\,b^2}+b\sqrt{75\,a} = \sqrt{3\times16\times a\times b^2}+b\sqrt{3\times25\times a}$
$= 4\,b\sqrt{3\,a}+5\,b\sqrt{3\,a}$
$= 9\,b\sqrt{3\,a}$

(6.) $2\sqrt{45}-3\sqrt{5} = 2\sqrt{5\times9}-3\sqrt{5}$
$= 3\sqrt{5}$

(7.) $\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3} = \sqrt[3]{8a^3(b+2a)}-\sqrt[3]{b^3(b+2a)}$
$= (2\,a-b)\sqrt[3]{2\,a+b}$

(8.) $3\sqrt[3]{4\,a^2}+2\sqrt[3]{2\,a} = 3\sqrt[3]{2\,a}+2\sqrt[3]{2\,a}$
$= 5\sqrt[3]{2\,a}$

MULTIPLICATION AND DIVISION OF RADICALS.

62. In the first place, with regard to radicals which have the same index, let it be required to multiply or divide $\sqrt[n]{a}$ by $\sqrt[n]{b}$, then we shall have

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{a\,b}, \text{and } \sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{\frac{a}{b}}$$

For if we raise $\sqrt[n]{a}\times\sqrt[n]{b}$, and $\sqrt[n]{a\,b}$, each to the power of n, we obtain the same result $a\,b$; hence these two expressions are equal.

In like manner, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\sqrt[n]{\frac{a}{b}}$ when raised to the nth power, give $\frac{a}{b}$; hence, the two expressions are equal. We shall thus have the following

RULE.

In order to multiply or divide two radicals which have the same index, multiply or divide the quantities under the sign by each other, and affect the result with the common radical sign. If there be any coefficients, we commence by multiplying or dividing them separately. Thus,

$$(1.)\quad 2a\sqrt[3]{\frac{(a^2+b^2)}{c}} \times -3a\sqrt[3]{\frac{(a^2+b^2)^2}{d}} = -6a^2\sqrt[3]{\frac{(a^2+b^2)^3}{cd}}$$
$$= -\frac{6a^2(a^2+b^2)}{\sqrt[3]{cd}}$$

$$(2.)\quad 3a\sqrt[4]{8a^2} \times 2b\sqrt[4]{4a^2c} = 6ab\sqrt[4]{32a^4c}$$
$$= 12a^2b\sqrt[4]{2c}$$

$$(3.)\quad 2a\sqrt{bc} \times 3b\sqrt{abc} \times a\sqrt{2a} = 6a^2b\sqrt{2a^2b^2c^2}$$
$$= 6a^3b^2c\sqrt{2}$$

$$(4.)\quad \frac{5a\sqrt{b}}{2b\sqrt{c}} = \frac{5a}{2b}\sqrt{\frac{b}{c}}$$

$$(5.)\quad \frac{25a^2b\sqrt{m^3n}}{5ab^2\sqrt{mn^2}} = \frac{25a^2b}{5ab^2}\sqrt{\frac{m^3n}{mn^2}}$$
$$= \frac{5a}{b}\sqrt{\frac{m^2}{n}}$$
$$5am\sqrt{\frac{1}{n}}$$

$$(6.)\quad \frac{\sqrt[3]{a^2b^2+b^4}}{\sqrt[3]{\frac{a^2-b^2}{8b}}} = \sqrt[3]{\frac{8b(a^2b^2+b^4)}{a^2-b^2}}$$
$$= 2b\sqrt[3]{\frac{a^2+b^2}{a^2-b^2}}$$

If the radicals have not the same index, we must reduce them to others having the same index, and then operate upon them as above; thus,

$$(7.)\quad 3a\sqrt[6]{b} \times 5b\sqrt[8]{2c} = 3a\sqrt[24]{b^4} \times 5b\sqrt[24]{8c^3}$$
$$= 15ab\sqrt[24]{8b^4c^3}$$

$$(8.)\quad \sqrt{5abc^3} \times \sqrt[3]{2a^2bc^2} = \sqrt[6]{125a^3b^3c^9} \times \sqrt[6]{4a^4b^2c^4}$$
$$= \sqrt[6]{500a^7b^5c^{13}}$$
$$= ac^2\sqrt[6]{500ab^5c}$$

FORMATION OF POWERS, AND EXTRACTION OF ROOTS OF RADICALS.

63. Let it be required to raise $\sqrt[m]{a}$ to the nth power; then,

$$(\sqrt[m]{a})^n = \sqrt[m]{a} \times \sqrt[m]{a} \times \sqrt[m]{a} \ - - - \ \text{to } n \text{ factors},$$

$$= \sqrt[m]{a^n} \text{ according to the rule for multiplication just established.}$$

Hence we have the following

RULE.

In order to raise a radical quantity to any given power, raise the quantity under the sign to that power, and affect the result with the radical sign with its original index. If there be any coefficient, we must raise the coefficient separately, to the required power. Thus,

$$(1.)\quad (\sqrt[4]{4a^3})^2 = \sqrt[4]{16a^6} = 2a\sqrt[4]{a^2}$$

$$(2.)\quad (3\sqrt[3]{2a})^5 = 3^5\sqrt[3]{32a^5} = 243\sqrt[3]{32a^5} = 486\,a\sqrt[3]{4a^2}$$

When the index of the radical is a multiple of the exponent of the power which we wish to form, the operation may be simplified.

Let it be required for example, to square $\sqrt[4]{2a}$, we have seen (Art. 58,) that $\sqrt[4]{2a} = \sqrt{\sqrt{2a}}$; but in order to square this quantity, it is sufficient to suppress the first radical sign, hence, $(\sqrt[4]{2a})^2 = \sqrt{2a}$. Again, let it be required to raise $\sqrt[10]{abc}$ to the power of 5; now, $\sqrt[10]{abc} = \sqrt[5]{\sqrt{abc}}$, but in order to raise this quantity to the power of 5, it is sufficient to suppress the first radical sign; hence, $(\sqrt[10]{abc})^5 = \sqrt{abc}$, and in general,

$$(\sqrt[mn]{a})^m = \sqrt[n]{a}$$

that is to say,

If the index of the radical be divisible by the index of the required power, we may divide the index of the radical by the index of the power, and leave the quantity under the sign unchanged.

64. With regard to the extraction of roots, by virtue of the principle established in (Art. 59), we shall manifestly have the following

RULE.

In order to extract any root of a radical quantity, multiply the index of the radical by the index of the root required, and leave the quantity under the sign unchanged. If there be any coefficient, we must extract its root separately. Thus,

$$(1.)\quad \sqrt[3]{\sqrt[4]{3c}} = \sqrt[12]{3c}$$

$$(2.)\quad \sqrt[2]{\sqrt[3]{5a}} = \sqrt[6]{5a}$$

$$(3.)\quad \sqrt[3]{8c^6\sqrt[5]{a^2b}} = 2c\sqrt[15]{a^2b}$$

If the quantity under the sign be a perfect power of the same degree as the root required, we may simplify. Thus,

$$(4.)\quad \sqrt[3]{\sqrt[4]{8a^3}} = \sqrt[4]{\sqrt[3]{8a^3}}\,{}^{*} = \sqrt[4]{2a}$$

$$(5.)\quad \sqrt[2]{\sqrt[9]{9a^2}} = \sqrt[9]{\sqrt[2]{9a^2}} = \sqrt[9]{3a}$$

Examples.

(1.) $\sqrt{24} + \sqrt{54} - \sqrt{6} = 4\sqrt{6}$

(2.) $\sqrt{12} + 2\sqrt{27} + 3\sqrt{75} + 9\sqrt{48} = 59\sqrt{3}$

(3.) $\sqrt[3]{81} - 2\sqrt[3]{24} + \sqrt{28} + 2\sqrt{63} = 8\sqrt{7} - \sqrt[3]{3}$

(4.) $\sqrt{45c^3} - \sqrt{80c^3} + \sqrt{5a^2c} = (a-c)\sqrt{5c}$

(5.) $\sqrt{18a^5b^3} + \sqrt{50a^3b^3} = (3a^2b + 5ab)\sqrt{2ab}$

(6.) $\sqrt[4]{2^{14}a^{13}b^5c} - \sqrt[4]{4\times 5^4a^5b^9c^5} + \sqrt[4]{4\times 6^4ab^5c}$
$= (8a^3b - 5ab^2c + 6b)\sqrt[4]{4abc}$

(7.) $\sqrt[3]{\dfrac{27a^5x}{2b}} - \sqrt[3]{\dfrac{a^2x}{2b}} = (3a-1)\sqrt[3]{\dfrac{a^2x}{2b}}$

(8.) $\sqrt[3]{54a^{m+6}b^3} - \sqrt[3]{16a^{m-3}b^6} + \sqrt[3]{2a^{4m+9}} + \sqrt[3]{2c^3a^m}$
$= \left(3a^2b - \dfrac{2b^2}{a} + a^{m+3} + c\right)\sqrt[3]{2a^m}$

(9.) $\dfrac{\sqrt[6]{3\times 2^3c^3f^4}}{\sqrt[6]{d^4g}} + \dfrac{\sqrt[6]{2^3g^{11}}}{\sqrt[6]{3^3c^3d^4f^2}} = \left\{\dfrac{f}{d} + \dfrac{g^2}{3cd}\right\}\sqrt[6]{\dfrac{3\times 2^3c^3d^2}{f\,g}}$

(10.) $x\sqrt[3]{\left(\dfrac{8a^4}{27b^3} + \dfrac{16a^3}{27b^2}\right)} = \dfrac{2ax}{3b}\sqrt[3]{a+2b}$

(11.) $\sqrt{3a^2c + 6abc + 3b^2c} = (a+b)\sqrt{3c}$

(12.) $\sqrt{4a^5b^2 - 20a^3b^3 + 25ab^4} = (2a^2 - 5b)\sqrt{ab^2}$

* It is manifest that, in general, $\sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}}$, for by (Art. 58) each of these expressions is $= \sqrt[mn]{a}$.

$$(13.)\quad \frac{\sqrt{a^2x-2ax^2+x^3}}{\sqrt{a^2+2ax+x^2}} = \frac{a-x}{a+x}\sqrt{x}$$

$$a+b\,\sqrt{a^2-2ab+b^2} = \frac{\sqrt{ac}}{a+b}$$

$$(15.)\quad \frac{a+b}{a-b}\cdot\sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{a+b}{a-b}}$$

$$(16.)\quad \sqrt[3]{2}\times\sqrt[6]{\frac{1}{3}}\times\sqrt[8]{3} = \sqrt[24]{\frac{256}{3}}$$

$$(17.)\quad \sqrt[5]{4}\times\sqrt[10]{3}\times\sqrt[15]{6} = \sqrt[30]{3981312}$$

$$(18.)\quad a\sqrt[m]{x}\times b\sqrt[n]{y}\times c\sqrt[p]{z} = abc\sqrt[mnp]{x^{np}y^{mp}z^{mn}}$$

$$(19.)\quad \sqrt[12]{\frac{a}{bc}}\times\sqrt[8]{\frac{a^m}{b}} = \sqrt[24]{\frac{a^{3m+2}}{b^5c^2}}$$

$$(20.)\quad \frac{ac}{b^3d^3}\sqrt[3]{\frac{bcd}{e}}\times\frac{\sqrt[6]{b^{10}d^7e}}{\sqrt[6]{a^2c^5}} = \frac{1}{bd}\sqrt[6]{\frac{a^4c^3}{d^3e}}$$

65. Let us now consider the sign with which the monomial may be affected. We have seen (Art. 52) that, whatever may be the sign of the monomial its square is always positive; and it is evident that, in like manner, every *even* power must be positive, whatever may be the sign of the original monomial, and that every *uneven power* will be affected with the same sign as the original monomial.

Thus $-a$ when raised to different powers in succession will give

$$-a,\ +a^2,\ -a^3,\ +a^4,\ -a^5,\ +a^6,\ -a^7,\ \text{\&c.}$$

And $+a$ in like manner will give

$$+a,\ +a^2,\ +a^3,\ +a^4,\ +a^5,\ +a^6,\ +a^7,\ \text{\&c.}$$

In fact, every even power $2n$ may be considered as the square of the n^{th} power, or $a^{2n}=(a^n)^2$, and must therefore be positive; and in like manner, every power of an uneven degree $(2n+1)$ may be considered as the product of the $2n^{\text{th}}$ power by the original monomial, and must therefore have the same sign with the monomial.

Hence it appears,

I. *That every root of an uneven degree of a monomial quantity, must be affected with the same sign as the quantity itself.* Thus:

$$\sqrt[3]{+8a^3}=2a;\quad \sqrt[3]{-8a^3}=-2a;\quad \sqrt[5]{-32a^{10}b^5}=-2a^2b$$

II. *That every root of an even degree of a positive monomial may be affected with the sign* $+$, *or the sign* $-$, *indifferently.* Thus:

$$\sqrt[4]{81a^4b^{12}}=\pm 3ab^3;\qquad \sqrt[6]{64a^{18}}=\pm 2a^3$$

III. *That every root of an even degree of a negative monomial is an impossible root.* For no quantity can be found which, when raised to an even power, can give a negative result. Thus $\sqrt[4]{-a}$, $\sqrt[8]{-c}$, . . . are symbols of operations which cannot be performed, and are called *impossible*, or, *imaginary quantities*, as $\sqrt{-a}$, $\sqrt{-b}$, in (Art. 58).

66. The different rules which have been established for the calculation of radicals, are exact so long as we treat of absolute numbers; but are subject to some modifications, when we consider expressions or symbols which are purely algebraical; such as the *imaginary expressions* just mentioned.

Let it be required, for example, to determine the product of $\sqrt{-a}$ by $\sqrt{-a}$; by the rule given in (Art. 62.)

$$\begin{aligned} \sqrt{-a} \times \sqrt{-a} &= \sqrt{-a \times -a} \\ &= \sqrt{+a^2} \end{aligned}$$

But $\sqrt{+a^2} = \pm a$, so that there is apparently a doubt as to the sign with which a ought to be affected, in order to answer the question. However, the true result is $-a$; because, in general, in order to square $\sqrt{m}$ it is sufficient to suppress the radical sign; but $\sqrt{-a} \times \sqrt{-a}$ is the same thing as $(\sqrt{-a})^2$, and consequently is equal to $-a$.

Next, let it be required to determine the product of $\sqrt{-a}$ by $\sqrt{-b}$; by the rule, (Art. 62.)

$$\begin{aligned} \sqrt{-a} \times \sqrt{-b} &= \sqrt{-a \times -b} \\ &= \sqrt{+ab} \\ &= \pm\sqrt{ab} \end{aligned}$$

The true result, however, is $-\sqrt{ab}$, so long as we suppose the radicals $\sqrt{-a}$, $\sqrt{-b}$ to each preceded by the sign $+$; for we have, according to (Art. 53.)

$$\begin{aligned} \sqrt{-a} &= \sqrt{a} \,.\, \sqrt{-1} \\ \sqrt{-b} &= \sqrt{b} \,.\, \sqrt{-1} \end{aligned}$$

Hence,

$$\begin{aligned} \sqrt{-a} \times \sqrt{-b} &= \sqrt{ab}\,(\sqrt{-1})^2 \\ &= \sqrt{ab} \times -1 \\ &= -\sqrt{ab} \end{aligned}$$

According to this principle, we shall find for the different powers of $\sqrt{-1}$ the following results:

$$\begin{aligned} \sqrt{-1} &= \sqrt{-1} \\ (\sqrt{-1})^2 &= -1 \\ (\sqrt{-1})^3 &= (\sqrt{-1})^2 \,.\, \sqrt{-1} \\ &= -\sqrt{-1} \\ (\sqrt{-1})^4 &= (\sqrt{-1})^2 \times (\sqrt{-1})^2 \\ &= -1 \times -1 \\ &= +1 \end{aligned}$$

Since the four following powers will be found by multiplying $+1$ by the first, the second, the third, and the fourth, we shall again find for the four new powers $+\sqrt{-1}$, -1, $-\sqrt{-1}$, $+1$; so that all the powers of $\sqrt{-1}$, will form a repeating cycle of four terms, being successively, $\sqrt{-1}$, -1, $-\sqrt{-1}$ $+1$.*

* This may be expressed in its most general form thus, if n be any whole number:

$$\begin{aligned} (a\sqrt{-1})^{4n} &= a^{4n} \times +1 = a^{4n} \\ (a\sqrt{-1})^{4n+1} &= a^{4n+1} \times +\sqrt{-1} = a^{4n+1} \,.\, \sqrt{-1} \\ (a\sqrt{-1})^{4n+2} &= a^{4n+2} \times -1 \quad -a^{4n+2} \\ (a\sqrt{-1})^{4n+3} &= a^{4n+3} \times -\sqrt{-1} \quad -a^{4n+2} \,.\, \sqrt{-1} \end{aligned}$$

Finally let it be required to determine the product of $\sqrt[4]{-a}$ by $\sqrt[4]{-b}$, which according to the rule would be $\sqrt[4]{+ab}$. To determine the true result, we must observe that,

$$\sqrt[4]{-a} \quad = \quad \sqrt[4]{a} \cdot \sqrt[4]{-1}$$
$$\sqrt[4]{-b} \quad = \quad \sqrt[4]{b} \cdot \sqrt[4]{-1}$$

And $\therefore$

$$\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{ab} \cdot (\sqrt[4]{-1})^2$$

But,

$$(\sqrt[4]{-1})^2 \quad = \quad \left(\sqrt{\sqrt{-1}}\right)^2$$
$$= \quad \sqrt{-1}$$

Hence,

$$\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{ab} \cdot \sqrt{-1}$$

The above principles will enable the student to operate upon these quantities without embarrassment.

THEORY OF FRACTIONAL AND NEGATIVE EXPONENTS.

67. This is the proper place to explain a species of notation which is found extremely useful in algebraic calculations.

I. Let it be required to extract the n^{th} root of a quantity such as a^m. We have seen by (Art. 55.) that, if m is a multiple of n, we must divide m, the index of the power, by n, the index of the root required. But if m is not divisible by n, in which case the extraction of the root is algebraically impossible, we may agree to *indicate* that operation, by indicating the division of the exponents. We shall thus have,

$$\sqrt[n]{a^m} \quad = \quad a^{\frac{m}{n}}.$$

the expression $a^{\frac{m}{n}}$ being understood to signify the n^{th} root of a^m, by a convention founded upon the rule for the extraction of roots of monomial quantities. According to this convention or definition, we shall have,

$$\sqrt[3]{a^2} \; = \; a^{\frac{2}{3}}; \qquad \sqrt[4]{a^7} \; = \; a^{\frac{7}{4}}$$

II. Let it be required to divide a^m by a^n. According to the rule in (Art. 17.) we must subtract the index of the divisor, from the index of the dividend; so that,

$$\frac{a^m}{a^n} \quad = \quad a^{m-n}$$

it is to be remarked, however, that here it is supposed, that $m \angle n$. But if $m > n$, in which case the division is algebraically impossible, we may agree to indicate the division by subtracting the index of the divisor, from the index of the dividend. Let p be the absolute difference of m and n; so that $n = m + p$, we shall then have

$$\frac{a^m}{a^n} \quad = \quad \frac{a^m}{a^{m+p}}$$
$$= \quad a^{m-(m+p)}$$
$$= \quad a^{-p}$$

But $\frac{a^m}{a^{m+p}}$ may also be put under the form $\frac{1}{a^p}$, by suppressing the factor a^m, common to both terms of the fraction; we shall then have

$$a^{-p} = \frac{1}{a^p}$$

The expression a^{-p} is then the symbol of a division which cannot be executed; and the true value of the expression is unity divided by the same letter a, affected with the exponent p, taken positively. According to this convention, we shall have,

$$a^{-3} = \frac{1}{a^3}; \qquad a^{-7} = \frac{1}{a^7}; \text{ \&c.}$$

III. By combining the two last conventions, we arrive at a third notation, which is the *negative and fractional exponent*.

Let it be required to extract the n^{th} root of $\frac{1}{a^m}$.

In the first place, $\frac{1}{a^m} = a^{-m}$; hence $\sqrt[n]{\frac{1}{a^m}} = \sqrt[n]{a^{-m}} = a^{-\frac{m}{n}}$ substituting the fractional exponent for the ordinary sign of the radical.

As in words, a^m is usually enunciated *a to the power of m*, m being a positive integer; so by analogy, $a^{\frac{m}{n}}$, a^{-m}, $a^{-\frac{m}{n}}$, are usually enunciated, *a to the power of m by n*, *a to the power of minus m*, and, *a to the power of minus m by n.*

All that has been hitherto said, with regard to fractional and negative exponents, must be considered as a mere matter of definition; in short, that by a *convention* among algebraists, $a^{\frac{m}{n}}$ is understood to mean the same thing as $\sqrt[n]{a^m}$, a^{-m} to be the same as $\frac{1}{a^m}$, and $a^{-\frac{m}{n}}$ as $\sqrt[n]{\frac{1}{a^m}}$. We shall now proceed *to prove*, that the rules already established for the multiplication, division, formation of powers, and extraction of roots, of quantities affected with positive integral exponents, are applicable without any modification, when the exponents are fractional or negative. We shall examine the different cases in succession.

68. MULTIPLICATION. Let it be required to multiply $a^{\frac{3}{5}}$ by $a^{\frac{2}{3}}$; then it is asserted, that it will be sufficient to add the two exponents, and that,

$$a^{\frac{3}{5}} \times a^{\frac{2}{3}} = a^{\frac{3}{5}+\frac{2}{3}}$$
$$= a^{\frac{19}{15}}$$

For by our definition,

$$a^{\frac{3}{5}} = \sqrt[5]{a^3}$$

And,

$$a^{\frac{2}{3}} = \sqrt[3]{a^2}$$

$$\therefore a^{\frac{3}{5}} \times a^{\frac{2}{3}} = \sqrt[5]{a^3} \times \sqrt[3]{a^2}$$
$$= \sqrt[15]{a^{19}}$$
$$= a^{\frac{19}{15}} \text{ by definition in (Art. 67. I.)}$$

Again, let it be required to multiply $a^{-\frac{3}{4}}$ by $a^{\frac{5}{6}}$, then it is asserted that

$$\begin{aligned} a^{-\frac{3}{4}} \times a^{\frac{5}{6}} &= a^{-\frac{3}{4}+\frac{5}{6}} \\ &= a^{-\frac{9}{12}+\frac{10}{12}} \\ &= a^{\frac{1}{12}}. \end{aligned}$$

For,

$$a^{-\frac{3}{4}} = \sqrt[4]{\frac{1}{a^3}}, \text{ and } a^{\frac{5}{6}} = \sqrt[6]{a^5},$$

$$\begin{aligned} \therefore a^{-\frac{3}{4}} \times a^{\frac{5}{6}} &= \sqrt[4]{\frac{1}{a^3}} \times \sqrt[6]{a^5} \\ &= \sqrt[12]{\frac{1}{a^9}} \times \sqrt[12]{a^{10}} \\ &= \sqrt[12]{\frac{a^{10}}{a^9}} \\ &= \sqrt[12]{a} \\ &= a^{\frac{1}{12}} \text{ by definition in (Art. 67. I.)} \end{aligned}$$

Generally, let it be required to multiply $a^{-\frac{m}{n}}$ by $a^{\frac{p}{q}}$, then

$$\begin{aligned} a^{-\frac{m}{n}} \times a^{\frac{p}{q}} &= a^{-\frac{m}{n}+\frac{p}{q}} \\ &= a^{\frac{np-mq}{nq}} \end{aligned}$$

For,

$$a^{-\frac{m}{n}} = \sqrt[n]{\frac{1}{a^m}} \text{ and } a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

$$\begin{aligned} a^{-\frac{m}{n}} \times a^{\frac{p}{q}} &= \sqrt[n]{\frac{1}{a^m}} \times \sqrt[q]{a^p} \\ &= \sqrt[nq]{a^{np-mq}} \\ &= a^{\frac{np-mq}{nq}} \text{ by definition.} \end{aligned}$$

69. Hence we have the following general

RULE OF EXPONENTS IN MULTIPLICATION.

In order to multiply quantities expressed by the same letter, add the exponents of that letter, whatever may be the nature of the exponents.

This is the same rule as was established in (Art. 11), for quantities affected with integral and positive exponents. According to this rule we shall find

$$\begin{aligned} a^{\frac{3}{4}} b^{-\frac{3}{2}} c^{-1} \times a^2 b^{\frac{2}{3}} c^{\frac{3}{5}} &= a^{\frac{11}{4}} b^{-\frac{5}{6}} c^{-\frac{2}{5}} \\ 3a^{-2} b^{\frac{2}{3}} \times 2a^{-\frac{4}{5}} b^{\frac{1}{2}} c^2 &= 6a^{-\frac{14}{5}} b^{\frac{7}{6}} c^2. \end{aligned}$$

70. Division. Let it be required to divide $a^{\frac{3}{2}}$ by $a^{\frac{1}{4}}$; then it is asserted, that it will be sufficient to subtract the index of the divisor from the index of the dividend, and that we shall thus have

$$\frac{a^{\frac{3}{2}}}{a^{\frac{1}{4}}} = a^{\frac{3}{2}-\frac{1}{4}}$$
$$= a^{\frac{5}{4}}$$

For,

$$a^{\frac{3}{2}} = \sqrt[2]{a^3}, \text{ and } a^{\frac{1}{4}} = \sqrt[4]{a}$$

$$\therefore \frac{a^{\frac{3}{2}}}{a^{\frac{1}{4}}} = \frac{\sqrt[2]{a^3}}{\sqrt[4]{a}}$$
$$= \sqrt[4]{\frac{a^6}{a}} \text{ by (Art. 62).}$$
$$= \sqrt[4]{a^5}$$
$$= a^{\frac{5}{4}} \text{ by definition.}$$

In like manner, we can prove that

$$\frac{a^{\frac{5}{4}}}{a^{-\frac{3}{2}}} = a^{\frac{5}{4}-(-\frac{3}{2})}$$
$$= a^{\frac{11}{4}}.$$

Generally, let it be required to divide $a^{\frac{m}{n}}$ by $a^{\frac{p}{q}}$.

Then,

$$a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}}.$$
$$= a^{\frac{mq-np}{nq}}$$

For,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, \text{ and } a^{\frac{p}{q}} = \sqrt[q]{a^p},$$

$$\therefore a^{\frac{m}{n}} \div a^{\frac{p}{q}} = \frac{\sqrt[n]{a^m}}{\sqrt[q]{a^p}}$$
$$= \sqrt[nq]{\frac{a^{mq}}{a^{np}}}$$
$$= \sqrt[nq]{a^{mq-np}}$$
$$= a^{\frac{mq-np}{nq}} \text{ by definition.}$$

71. Hence we have the following general

RULE OF EXPONENTS IN DIVISION.

In order to divide quantities expressed by the same letter, subtract the exponent of the divisor from the exponent of the dividend, whatever may be the nature of the exponents.

This is the same rule as that established in (Art. 17), for quantities affected with integral and positive exponents. According to this rule we have

$$a^{\frac{2}{3}} \div a^{-\frac{3}{4}} = a^{\frac{2}{3}-(-\frac{3}{4})}$$
$$= a^{\frac{17}{12}}$$
$$a^{\frac{3}{4}} \div a^{\frac{4}{5}} = a^{-\frac{1}{20}}$$
$$a^{\frac{2}{5}} b^{\frac{3}{4}} \div a^{-\frac{1}{2}} b^{\frac{7}{8}} = a^{\frac{9}{10}} b^{-\frac{1}{8}}$$

72. FORMATION OF POWERS. In order to raise a monomial to any power, the rule given in the case of positive and integral exponents was, to multiply the index of the quantity by the index of the power sought. We have now to prove that this holds good, whatever may be the nature of the exponent.

Let it be required to raise $a^{\frac{5}{7}}$ to the 4th power.

Then,

$$\left(a^{\frac{5}{7}}\right)^4 = a^{\frac{5}{7} \times 4}$$
$$= a^{\frac{20}{7}},$$

For,

$$a^{\frac{5}{7}} = \sqrt[7]{a^5}, \text{ and } \left(a^{\frac{5}{7}}\right)^4 = \left(\sqrt[7]{a^5}\right)^4,$$

But,

$$\left(\sqrt[7]{a^5}\right)^4 = \sqrt[7]{a^{20}}, \text{ by (Art. 63.)}$$
$$= a^{\frac{20}{7}}$$

Generally, let it be required to raise $a^{\frac{m}{n}}$ to the power of p.

Then,

$$\left(a^{\frac{m}{n}}\right)^p = a^{\frac{m}{n} \times p}$$
$$= a^{\frac{mp}{n}}$$

For,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, \text{ and } \left(a^{\frac{m}{n}}\right)^p = \left(\sqrt[n]{a^m}\right)^p,$$

But,

$$\left(\sqrt[n]{a^m}\right)^p = \sqrt[n]{a^{mp}}$$
$$= a^{\frac{mp}{n}}.$$

The demonstration will manifestly be precisely the same if we suppose one or both of the indices to be negative.

73. Hence we have the following general

RULE FOR RAISING A MONOMIAL TO ANY POWER.

Multiply the exponent of the monomial by the exponent of the power required, whatever may be the nature of the exponents.

This is the same rule as that established in (Art. 55) for quantities affected with positive integral exponents. According to this rule we have

$$\left(a^{\frac{3}{4}}\right)^5 = a^{\frac{3}{4} \times 5}$$
$$= a^{\frac{15}{4}}$$

$$(a^{\frac{2}{3}})^3 = a^{\frac{2}{3}\times 3}$$
$$= a^2$$
$$(2a^{-\frac{1}{2}}b^{\frac{3}{4}})^6 = 2^6\, a^{-\frac{1}{2}\times 6}\, b^{\frac{3}{4}\times 6}$$
$$= 64\, a^{-3}\, b^{\frac{9}{2}}.$$

74. **Extraction of Roots.** In order to extract the n^{th} root of any quantity according to the rule in (Art. 55), we must divide the exponent of each letter by the index n of the root. Let us examine the case of fractional exponents.

Let it be required to extract the cube root of $a^{\frac{5}{3}}$.

Then,

$$\sqrt[3]{a^{\frac{5}{3}}} = a^{\frac{5}{3}\div 3}$$
$$= a^{\frac{5}{9}}$$

For,

$$a^{\frac{5}{3}} = \sqrt[3]{a^5}, \text{ and } \therefore \sqrt[3]{a^{\frac{5}{3}}} = \sqrt[3]{\sqrt[3]{a}}$$

But,

$$\sqrt[3]{\sqrt[3]{a^5}} = \sqrt[9]{a^5}.$$
$$= a^{\frac{5}{9}}, \text{ by definition.}$$

Generally, let it be required to extract the p^{th} root of $a^{\frac{m}{n}}$.

Then,

$$\sqrt[p]{a^{\frac{m}{n}}} = a^{\frac{m}{np}}$$

For,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, \text{ and } \therefore \sqrt[p]{a^{\frac{m}{n}}} = \sqrt[p]{\sqrt[n]{a^m}}$$

But,

$$\sqrt[p]{\sqrt[n]{a^m}} = \sqrt[np]{a^m}, \text{ (by Art. 58.)}$$
$$= a^{\frac{m}{np}}, \text{ by definition.}$$

75. Hence we have the following

RULE FOR THE EXTRACTION OF ANY ROOT OF AN ALGEBRAIC MONOMIAL.

Divide the exponent of the monomial by the exponent of the root required, whatever may be the nature of the exponents. Thus,

$$\sqrt[5]{a^{\frac{3}{4}}} = a^{\frac{3}{4}\div 5}$$
$$= a^{\frac{3}{20}}$$
$$\sqrt[3]{a^{-\frac{2}{5}}} = a^{-\frac{2}{5}\div 3}$$
$$= a^{-\frac{2}{15}}$$

$$\sqrt[7]{a^{\frac{3}{5}} b^{-2}} = a^{\frac{3}{5} \div 7} b^{-2 \div 7}$$
$$= a^{\frac{3}{35}} b^{-\frac{2}{7}}.$$

76. We shall close this discussion by an operation which includes the demonstration of every possible variety of the two preceding rules.

Let it be required to raise $a^{\frac{m}{n}}$ to the power of $-\frac{r}{s}$; we must prove that

$$(a^{\frac{m}{n}})^{-\frac{r}{s}} = a^{\frac{m}{n} \times -\frac{r}{s}}$$
$$= a^{-\frac{mr}{ns}}.$$

If we recur to the origin of this notation, we find that

$$(a^{\frac{m}{n}})^{-\frac{r}{s}} = \sqrt[s]{\frac{1}{(a^{\frac{m}{n}})^r}}$$
$$= \sqrt[s]{\frac{1}{(\sqrt[n]{a^m})^r}}$$
$$= \sqrt[s]{\frac{1}{\sqrt[n]{a^{mr}}}}$$
$$= \sqrt[s]{\sqrt[n]{\frac{1}{a^{mr}}}}$$
$$= \sqrt[ns]{\frac{1}{a^{mr}}}$$
$$= \sqrt[ns]{a^{-mr}}, \text{ by definition.}$$
$$= a^{-\frac{mr}{ns}}$$

77. The notation above explained can be extended to polynomials, by including them within brackets, in the same manner as was explained in the case of integral exponents.

Thus, $(x + a)^{\frac{1}{2}}$ signifies the same thing as $\sqrt{x + a}$, or, *the square root of* $x + a$.

So, $(x + a)^{-\frac{1}{2}}$ is equivalent to $\frac{1}{\sqrt{x + a}}$, or, *unity divided by the square root of* $x + a$.

In like manner, $(x + a + b)^{\frac{3}{4}}$ will be the same as $\sqrt[4]{(x + a + b)^3}$, or, *the fourth root of the third power of the quantity* $x + a + b$, and $(x + a + b)^{-\frac{3}{4}}$ will be unity divided by the last mentioned quantity. Since the exponent unity is always understood, when no other index is expressed $(x + a)^{-1}$ is the same as $\frac{1}{x + a}$, and so on.—The same rules which have been established for the treatment of monomials affected with exponents will also manifestly apply to polynomials under the same restrictions.

Examples.

(1.) $a^{-\frac{3}{4}} \times a^{-\frac{7}{8}} = a^{-\frac{13}{8}} = \frac{1}{a\sqrt[8]{a^5}}$

(2.) $a^{-\frac{3}{4}} b^{-2} \times a^{\frac{5}{6}} b^{\frac{3}{2}} c = a^{\frac{1}{12}} b^{-\frac{1}{2}} c = \frac{c}{b}\sqrt[12]{\frac{a}{b^6}}$

(3.) $\dfrac{a}{b^{\frac{1}{2}}c^{\frac{3}{4}}} \times \dfrac{a^{\frac{7}{8}}b}{c^{-\frac{1}{2}}} \quad = a^{\frac{15}{8}}b^{\frac{1}{2}}c^{-\frac{1}{4}} = a^{2}\sqrt[8]{\dfrac{b^{4}}{a\,c}}$

(4.) $a^{-\frac{m}{n}} \div a^{-\frac{p}{q}} \quad = a^{\frac{p}{q}-\frac{m}{n}} = a^{\frac{np-mq}{nq}}$

(5.) $c\,a^{\frac{3}{4}} \div d\,a^{\frac{5}{6}} \quad = \dfrac{c}{d}\cdot a^{-\frac{1}{12}}$

(6.) $a^{\frac{3}{5}}b^{\frac{1}{2}} \div a^{-\frac{7}{5}}b^{-\frac{1}{4}}c = \dfrac{a^{2}\cdot b^{\frac{3}{4}}}{c}$

(7.) $\dfrac{a^{-\frac{9}{2}}b^{\frac{2}{3}}}{c^{\frac{1}{6}}d^{3}} \div \dfrac{a^{-\frac{29}{4}}d^{\frac{11}{3}}}{b^{\frac{8}{5}}} = \dfrac{a^{\frac{11}{4}}b^{\frac{34}{15}}}{c^{\frac{1}{6}}d^{\frac{20}{3}}}$

(8.) $\left(a^{\frac{3}{4}}b^{\frac{2}{3}}\right)^{\frac{1}{3}} \quad = \quad a^{\frac{1}{4}}b^{\frac{2}{9}}$

(9.) $\left(a^{2}b^{-\frac{1}{2}}c^{-\frac{2}{5}}\right)^{-\frac{1}{4}} = a^{-\frac{1}{2}}b^{\frac{1}{8}}c^{\frac{1}{10}}$

(10.) $\left\{\dfrac{c^{2}d}{(a+b)^{\frac{3}{2}}}\right\}^{-\frac{1}{3}} = \dfrac{c^{-\frac{2}{3}}d^{-\frac{1}{3}}}{(a+b)^{-\frac{1}{2}}}$

(11.) $\left(a^{\frac{5}{2}}+a^{2}b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}\right)\times\left(a^{\frac{1}{2}}-b^{\frac{1}{3}}\right)=a^{3}-b^{2}$

(12.) $\left(x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}\right)\times\left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right) = x^{\frac{3}{4}}-y^{\frac{3}{4}}$

(13.) $\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\times\left(x^{-\frac{1}{2}}+y^{-\frac{1}{2}}\right) = x^{\frac{1}{2}}y^{-\frac{1}{2}}+2+x^{-\frac{1}{2}}y^{\frac{1}{2}}$

(14.) $\dfrac{a^{3}-b^{3}}{a^{\frac{3}{4}}+b^{\frac{3}{4}}} = a^{\frac{9}{4}}-a^{\frac{3}{2}}b^{\frac{3}{4}}+a^{\frac{3}{4}}b^{\frac{3}{2}}-b^{\frac{9}{4}}$

(15.) $\dfrac{a^{\frac{7}{3}}-a^{2}b^{-\frac{2}{3}}-a^{\frac{1}{3}}b+b^{\frac{1}{3}}}{a^{\frac{1}{3}}-b^{-\frac{2}{3}}} \quad = a^{2}-b$

78. Having thus discussed the formation of powers, and the extraction of roots, in monomial quantities, we shall now direct our attention to polynomials; and in the first place, let it be required to determine the square of $x+a$; then,

$$(x+a)^{2} = (x+a)\times(x+a)$$
$$= x^{2}+2xa+a^{2}$$ by rules of multiplication.

Next let it be required to form the square of a trinomial $(x+a+b)$. Let us represent, for a moment, the two terms, $x+a$, by the single letter z.

Then,

$$(x+a+b)^2 = (z+b)^2$$
$$= z^2 + 2zb + b^2$$

But,

$$z^2 = (x+a)^2$$
$$= x^2 + 2xa + a^2$$

And,

$$2zb = 2b(x+a)$$
$$= 2xb + 2ab$$

Therefore, substituting for z^2 and $2zb$ their values, we find,

$$(x+a+b)^2 = x^2+a^2+b^2+2xa+2xb+2ab$$

Hence it appears, that *the square of a trinomial is composed of the sum of the squares of all the terms, together with twice the sum of the products of all the terms multiplied together two and two.*

We shall now prove, that this law of formation extends to all polynomials, whatever may be the number of terms. In order to demonstrate this, let us suppose that it is true for a polynomial consisting of n terms, and then endeavour to ascertain whether it will hold good for a polynomial composed of $(n+1)$ terms.

Let $x+a+b+c+ \cdots\cdots +k+l$ be a polynomial consisting of $n+1$ terms, and let us represent the sum of the n first terms, by the single letter z · then,

$$(x+a+b+c+ \cdots +k+l) = (z+l)$$

and, $\therefore (x+a+b+c+ \cdots +k+l)^2 = (z+l)^2$

$$= z^2 + 2zl + l^2$$

or putting for z its value, $= (x+a+b+c+ \cdots +k)^2 + 2(x+a+b+c+ \cdots +k)l + l^2$

But, the first part of this expression, being the square of a polynomial consisting of n terms, is, by hypothesis, composed of the sum of the squares of all the terms, together with twice the sum of the products of all the terms multiplied two and two: the second part of the above expression is equal to twice the sum of the products of all the n first terms of the proposed polynomial, multiplied by the $(n+1)^{th}$ term l: and the third part is the square of the $(n+1)^{th}$ term l.

Hence, if the *law of formation* already enounced, holds good for a polynomial composed of n terms, it will hold good for a polynomial composed of $(n+1)$ terms.

But we have seen above that it does hold good for a polynomial composed of *three* terms; therefore it must hold for a polynomial composed of *four* terms,

and therefore for a polynomial of *five* terms, and so on in succession. Therefore, the law is general, and we have the following

RULE FOR THE FORMATION OF THE SQUARE OF A POLYNOMIAL.

The square of any polynomial is composed of the sum of the squares of all the terms together, with twice the sum of the products of all the terms multiplied together two and two. According to this rule, we shall have,

(1.) $(a+b+c+d+e)^2=a^2+b^2+c^2+d^2+e^2+2ab+2ac+2ad+2ae+2bc$
$+2bd+2be+2cd+2ce+2de.$

(2.) $(a-b-c+d)^2 \quad =a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd.$

If any of the terms of the proposed polynomial be affected with exponents or coefficients, we must square these monomials according to the rules already established.

(3.) $(2a-4b^2c^3)^2 \quad =4a^2+16b^4c^6-16ab^2c^3$

(4.) $(3a^2-2ab+4b^2)^2 \quad =9a^4+4a^2b^2+16b^4-12a^3b$
$+24a^2b^2-16ab^3$
$=9a^4-12a^3b+28a^2b^2-16ab^3+16b^4$ arranging according to powers of a, and reducing.

(5.) $(5a^2b-4abc+6bc^2-3a^2c)^2=25a^4b^2+16a^2b^2c^2+36b^2c^4+9a^4c^2$
$-40a^3b^2c+60a^2b^2c^2-30a^4bc$
$-48ab^2c^3+24a^3bc^2-36a^2bc^3.$
$=25a^4b^2-40a^3b^2c+76a^2b^2c^2-48ab^2c^3$
$+36b^2c^4-30a^4bc+24a^3bc^2$
$-36a^2bc^3+9a^4c^2.$

79. Let us now pass on to the extraction of the square root of algebraic quantities.

Let P be the polynomial whose root is required, and let R represent the root which for the moment we suppose to be determined; let us also suppose the two polynomials P and R to be arranged according to the powers of some one of the letters which they contain; a, for example.

If we reflect upon the law of the formation of the square, it will be seen, that the two first terms of the polynomial P, when thus arranged, will enable us at once to determine the two first terms of the root sought; for,

1°. The square of the first term of R must involve a, affected with an exponent greater than any that is to be found in the other terms which compose the square of R.

2°. Twice the product of the first term of R by the second, must contain a, affected with an exponent greater than any to be found in the succeeding terms.

Hence, the two terms of which we speak not being susceptible of reduction with any other terms, will necessarily form the two terms of P, affected with the highest exponent of a, and the one immediately inferior. It follows from this, that if P be a perfect square,

I. The first term must be a perfect square; and the square root of this term, when extracted according to the rule for monomials (Art. 49), is the first term of R.

II. The second term must be divisible by twice the first term of R thus found, and the quotient will be the second term of R.

III. In order to obtain the remaining terms of R, *square the two terms of* R *already determined*, and subtract the result from P; we thus obtain a new polynomial P′, which contains twice the product of the first term of R by the third term, together with a series of other terms. But twice the product of the first term of R by the third, must contain a, affected with an exponent greater than any that is to be found in the succeeding terms, and hence this double product must form the first term of P′.

IV. The first term of P′ must be divisible by twice the first term of R, and the quotient will be the third term of R.

V. In order to obtain the remaining terms of R, square the three terms of the root already determined, and subtract the result from the original polynomial P; * we thus obtain a new polynomial P″, concerning which we may reason precisely in the same manner as for P′, and continuing to repeat the operation until we find no remainder, we shall arrive at the root required.

The above observations may be collected and embodied in the following

RULE FOR THE EXTRACTION OF THE SQUARE ROOT OF ALGEBRAIC POLYNOMIALS.

1°. *Arrange the polynomial according to the powers of some one letter.*

2°. *Extract the square root of the first term according to the rule for monomials, and the result will be the first term of the root required.*

3°. *Square the first term of the root thus determined, and subtract it from the original polynomial.*

4°. *Double the first term of the root, and divide by it the first term of the remainder, and annex the result (which will be the second term of the root) with its proper sign to the divisor.*

5°. *Multiply the whole of this divisor by the second term of the root, and subtract the product from the first remainder.*

6°. *Divide this second remainder by twice the two first terms of the root already found, and annex the result (which will be the third term of the root) with its proper sign to the divisor.*

7°. *Multiply the whole of this divisor by the third term of the root, and subtract the product from the second remainder; continue the operation in this manner until the whole root is ascertained.*

The above process will be readily understood by attending to the following examples:—

* In practice this operation is dispensed with by following the precepts 5°, 7°, in the following rule, which evidently come to the same thing.

Example 1.

Extract the square root of $10x^4 - 10x^3 - 12x^5 + 5x^2 + 9x^6 - 2x + 1$.

On arranging according to the powers of x,

$$
\begin{array}{r|l}
 & 9x^6 - 12x^5 + 10x^4 - 10x^3 + 5x^2 - 2x + 1 \;\big|\; 3x^3 - 2x^2 + x - 1 \\
 & 9x^6 \\
\hline
6x^3 - 2x^2 & -12x^5 + 10x^4 - 10x^3 + 5x^2 - 2x + 1 \\
 & -12x^5 + 4x^4 \\
\hline
6x^3 - 4x^2 + x & 6x^4 - 10x^3 + 5x^2 - 2x + 1 \\
 & 6x^4 - 4x^3 + x^2 \\
\hline
6x^3 - 4x^2 + 2x - 1 & -6x^3 + 4x^2 - 2x + 1 \\
 & -6x^3 + 4x^2 - 2x + 1 \\
\hline
 & 0.
\end{array}
$$

Having arranged the polynomial according to powers of x, we first extract the square root of $9x^6$ the first term, this gives $3x^3$ for the first term of the root required; this we place on the right hand of the polynomial as in division: squaring this quantity, and subtracting it from the whole polynomial, we obtain for a first remainder, $-12x^5 + 10x^4 - 10x^3 + 5x^2 - 2x + 1$; we now double $3x^3$ and place it as a divisor on the left of this remainder, and dividing by it $-12x^5$, the first term of the remainder, we obtain the quotient $-2x^2$ (the second term of the root sought), which we annex with its proper sign to the double root $6x^3$; multiplying the whole of this quantity, $6x^3 - 2x^2$, by $-2x^2$, and subtracting the product from the first remainder, we obtain for a second remainder, $6x^4 - 10x^3 + 5x^2 - 2x + 1$. Next doubling $3x^3 - 2x^2$, the two terms of the root thus found, and dividing $6x^4$, the first term of the new remainder, by $6x^3$, the first term of the double root, we obtain x for a quotient, (which is the third term of the root sought,) and annex it to the double root $6x^3 - 4x^2$, multiplying the whole of this quantity $6x^3 - 4x^2 + x$ by x, and subtracting the product from the second remainder, we obtain a third remainder $-6x^3 + 4x^2 - 2x + 1$, we now double $3x^3 - 2x^2 + x$, the three terms of the root already found, and dividing $-6x^3$, the first term of the new remainder, by $6x^3$, the first term of the double root, we obtain -1 for the quotient, (which is the fourth term of the root sought,) and annex it to the double root $6x^3 - 4x^2 + 2x$, multiplying the whole of this quantity $6x^3 - 4x^2 + 2x - 1$ by -1, and subtracting it from the third remainder, we find 0 for a new remainder, which shows that the root required is

$$3x^3 - 2x^2 + x - 1$$

Example 2.

Extract the square root of

$$25a^4b^2 - 40a^3b^2c + 76a^2b^2c^2 - 48ab^2c^3 + 36b^2c^4 - 30a^4bc + 24a^3bc^2 - 36a^2bc^3 + 9a^4c^2 \;\Big|\; 5a^2b - 4abc + 6bc^2 - 3a^2c$$

$$25a^4b^2$$

$$10a^2b - 4abc \;\Big|\; -40a^3b^2c + 76a^2b^2c^2 - \ldots$$

$$-40a^3b^2c + 16a^2b^2c^2$$

$$10a^2b - 8abc + 6bc^2 \;\Big|\; 60a^2b^2c^2 - 48ab^2c^3 + 36b^2c^4 - \ldots$$

$$60a^2b^2c^2 - 48ab^2c^3 + 36b^2c^4$$

$$10a^2b - 8abc + 12bc^2 - 3a^2c \;\Big|\; -30a^4bc + 24a^3bc^2 - 36a^2bc^3 + 9a^4c^2$$

$$-30a^4bc + 24a^3bc^2 - 36a^2bc^3 + 9a^4c^2$$

* *

The root required is therefore $5a^2b - 4abc + 6bc^2 - 3a^2c$.

Example 3.

Extract the square root of

$$a^{2m}x^{2n+2} + 10ca^{2m-2}x^{2n+3} + 25c^2a^{2m-4}x^{2n+4} - 6a^{m+1}x^{n+1} - 30ca^{m-1}x^{n+2} + 9a^2 \;\Big|\; a^mx^{n+1} + 5ca^{m-2}x^{n+2} - 3a$$

$$a^{2m}x^{2n+2}$$

$$2a^mx^{n+1} + 5ca^{m-2}x^{n+2} \;\Big|\; 10ca^{2m-2}x^{2n+3} + 25c^2a^{2m-4}x^{2n+4} - \ldots$$

$$10ca^{2m-2}x^{2n+3} + 25c^2a^{2m-4}x^{2n+4}$$

$$2a^mx^{n+1} + 10ca^{m-2}x^{n+2} - 3a \;\Big|\; -6a^{m+1}x^{n+1} - 30ca^{m-1}x^{n+2} + 9a^2$$

$$-6a^{m+1}x^{n+1} - 30ca^{m-1}x^{n+2} + 9a^2$$

* *

The root is therefore $a^mx^{n+1} + 5ca^{m-2}x^{n+2} - 3a$.

Example 4.

Extract the square root of

$$\begin{array}{r|l} & 4x^{3}-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}+24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}} \quad \Big| \; 2x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}}+6y^{\frac{5}{6}}b^{\frac{1}{2}} \\ & 4x^{3} \\ \hline 4x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}} & -20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}} \cdots\cdots \\ & -20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}} \\ \hline 4x^{\frac{3}{2}}-10x^{\frac{3}{4}}a^{\frac{2}{3}}+6y^{\frac{5}{6}}b^{\frac{1}{2}} & 24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}} \\ & 24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}} \\ \hline \end{array}$$

The root is therefore $2x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}}+6y^{\frac{5}{6}}b^{\frac{1}{2}}$.

(5) Extract the square root of $4x^{4}+8ax^{3}+4a^{2}x^{2}+16b^{2}x^{2}+16ab^{2}x+16b^{4}$ Ans. $2x^{2}+2ax+4b^{2}$

(6) Extract the square root of $\frac{4}{9}a^{2}x^{4}-\frac{4}{3}abx^{3}z+\frac{8}{3}a^{2}bx^{2}z^{2}+b^{2}x^{2}z^{2}-4ab^{2}xz^{3}+4a^{2}b^{2}z^{4}$

Ans. $\frac{2}{3}ax^{2}-bxz+2abz^{2}$

(7) Extract the square root of $\frac{9a^{2m-2}c^{2}}{4d^{6p}}-\frac{3a^{m+n-1}b^{2n-1}c}{d^{3p-3}}-\frac{2^{8}a^{m-1}b^{x}c}{d^{3p}}+a^{2n}b^{4n-2}d^{6}+\frac{2^{9}}{3}a^{n}b^{x+2n-1}d^{3}+\frac{2^{16}b^{2x}}{9}$

Ans. $\frac{3a^{m-1}c}{2d^{3p}}-a^{n}b^{2n-1}d^{3}-\frac{2^{8}b^{x}}{3}$

(8) Extract the square root of $\frac{a^{2}x^{2}+2ab^{2}x^{3}+b^{4}x^{4}}{a^{2m}+2a^{m}x^{n}+x^{2n}}$ Ans. $\frac{ax+b^{2}x^{2}}{a^{m}+x^{n}}$

(9) Extract the square root of

$$x^3 + 4x^{\frac{3}{2}}y^{\frac{5}{4}} + 6x^{\frac{3}{2}}z^{\frac{7}{3}} + 4y^{\frac{5}{2}} + 12y^{\frac{5}{4}}z^{\frac{7}{3}} + 9z^{\frac{14}{3}}$$

Ans. $x^{\frac{3}{2}} + 2y^{\frac{5}{4}} + 3z^{\frac{7}{3}}$

(10) Extract the square root of

$$4xa + 12x^{\frac{1}{2}}y^{\frac{3}{2}}a^{\frac{1}{2}}b^{\frac{3}{2}} - 16x^{\frac{1}{2}}z^{\frac{5}{2}}a^{\frac{1}{2}}c^{\frac{5}{2}} - 20x^{\frac{1}{2}}a^{\frac{1}{2}}d^{\frac{7}{2}} + 9b^3y^3$$
$$- 24b^{\frac{3}{2}}y^{\frac{3}{2}}c^{\frac{5}{2}}z^{\frac{5}{2}} - 30b^{\frac{3}{2}}y^{\frac{3}{2}}d^{\frac{7}{2}} + 16c^5z^5 + 40c^{\frac{5}{2}}z^{\frac{5}{2}}d^{\frac{7}{2}} + 25d^7$$

Ans. $2x^{\frac{1}{2}}a^{\frac{1}{2}} + 3b^{\frac{3}{2}}y^{\frac{3}{2}} - 4c^{\frac{5}{2}}z^{\frac{5}{2}} - 5d^{\frac{7}{2}}$

80. If the proposed polynomial contain several terms affected with the same power of the principal letter, we must arrange the polynomial in the manner explained in division (Art. 20.); and in applying the above process we shall be obliged to perform several *partial extractions of the square roots of the coefficients* of the different powers of the principal letter, before we can arrive at the root required. Such examples however very rarely occur.

Before quitting this subject we may make the following remarks :—

I. *No binomial can be a perfect square;* for the square of a monomial is a monomial, and the square of the most simple polynomial, that is, a binomial, consists of three distinct terms, which do not admit of being reduced with each other. Thus, such an expression as $a^2 + b^2$ is not a square; it wants the term $\pm 2ab$ to render it the square of $(a \pm b)$.

II. *In order that a trinomial, when arranged according to the powers of some one letter, may be a perfect square, the two extreme terms must be perfect squares, and the middle term must be equal to twice the product of the square roots of the extreme terms.* When these conditions are fulfilled, we may obtain the square root of a trinomial immediately, by the following

Rule.

Extract the square roots of the extreme terms, and connect the two terms thus found by the sign $+$, when the second term of the trinomial is positive, and by the sign $-$, when the second term of the trinomial is negative. Thus the expression

$$9a^6 - 48a^4b^2 + 64a^2b^4$$

is a perfect square; for the two extreme terms are perfect squares, and the middle term is twice the product of the square roots of the extreme terms, hence the square root of the trinomial is

$$\sqrt{9a^6} - \sqrt{64a^2b^4}$$

Or,

$$3a^3 - 8ab^2$$

An expression such as $4a^2 + 12ab - 9b^2$ cannot be a perfect square, although $4a^2$ and $9b^2$, considered independently of their signs, are perfect squares, and $12ab = 2a \times 6b$, for $-9b^2$ is not a square, since no quantity, when multiplied by itself, can have the sign $-$.

III. In performing the operations required by the general rule, if we find that the first term of one of the remainders is not exactly divisible by twice the first term of the root, we may immediately conclude that the polynomial is not a perfect square.

IV. We may apply to the square roots of polynomials which are not perfect squares, the simplifications already employed in the case of monomials (art. 51). Thus, in the expression,

$$\sqrt{a^3 b + 4 a^2 b^2 + 4 a b^3}$$

The quantity under the radical is not a perfect square, but it may be put under the form

$$\sqrt{a b (a^2 + 4 a b + 4 b^2)}$$

The factor within brackets is manifestly the square of $a + 2 b$, hence

$$\begin{aligned} \sqrt{a^3 b + 4 a^2 b^2 + 4 a b^3} &= \sqrt{a b (a^2 + 4 a b + 4 b^2)} \\ &= \sqrt{a b (a + 2 b)^2} \\ &= (a + 2 b) \sqrt{a b} \end{aligned}$$

81. Let us next proceed to form the *Cube* of $x + a$.

$$\begin{aligned} (x + a)^3 &= (x + a) \times (x + a) \times (x + a) \\ &= x^3 + 3 x^2 a + 3 x a^2 + a^3 \quad \text{by rules of multiplication.} \end{aligned}$$

Let it be required to form the cube of a trinomial $(x + a + b)$; represent the two last terms $a + b$ by the single letter s, then

$$\begin{aligned} (x + a + b)^3 &= (x + s)^3 \\ &= x^3 + 3 x^2 s + 3 x s^2 + s^3 \\ &= x^3 + 3 x^2 (a + b) + 3 x (a + b)^2 + (a + b)^3 \\ &= x^3 + 3 x^2 a + 3 x^2 b + 3 x a^2 + 6 x a b + 3 x b^2 + a^3 \\ &\quad + 3 a^2 b + 3 a b^2 + b^3 \end{aligned}$$

This expression is composed of the sum of the cubes of all the terms, together with three times the sum of the squares of each term, multiplied by the simple power of each of the others in succession, together with six times the product of the simple power of all the terms.

By following a process of reasoning analogous to that employed in (Art. 78), we can prove that the above law of formation will hold good for any polynomial of whatever number of terms. We shall thus find

$$\begin{aligned} (a+b+c+d)^3 &= a^3+b^3+c^3+d^3+3a^2 b+3a^2 c+3a^2 d+3b^2 a+3b^2 c+3b^2 d \\ &\quad +3c^2 a+3c^2 b+3c^2 d+3d^2 a+3d^2 b+3d^2 c+6abcd \\ (2a^2-4ab+3b^2)^3 &= 8 a^6 - 64 a^3 b^3 + 27 b^6 - 48 a^5 b + 36 a^4 b^2 + 96 a^4 b^2 \\ &\quad + 144 a^2 b^4 + 54 a^2 b^4 - 108 a b^5 - 144 a^3 b^3 \\ &= 8a^6 - 48 a^5 b + 132 a^4 b^2 - 208 a^3 b^3 + 198 a^2 b^4 - 108 a b^5 \\ &\quad + 27 b^6 \end{aligned}$$

In a similar manner, we can obtain the 4th, 5th, &c. powers of any polynomial.

82. We shall now explain the process by which we can extract the cube root of any polynomial, a method analogous to that employed for the square root, and which may easily be generalized, so as to be applicable to the extraction of roots of any degree.

Let P be the given polynomial, R its cube root. Let these two polynomials be arranged according to the powers of some one letter, a for example. It follows, from the law of formation of the cube of a polynomial, that the cube of R contains two terms, which are not susceptible of reduction with any others; these are, the cube of the first term, and three times the square of the first term multiplied by the second term; for it is manifest that these two terms will involve a affected with an exponent higher than any that is to be found in the succeeding terms. Consequently these two terms must form the two first terms of P Hence, if we extract the cube root of the first term of P, we shall obtain the first term of R, and then, dividing the second term of P by three times the square of the first term of R thus found, the quotient will be the second term of R. Having thus determined the two first terms of R, cube this binomial, and subtract it from P. The remainder P′ will contain three times the product of the square of the first term of R by the third, together with a series of terms involving a, affected with a less exponent than that with which it is affected in this product, which consequently forms the first term of P′. Dividing the first term of P′ by three times the square of the first term of R, the quotient will be the third term of R. Forming the cube of the trinomial root thus determined, and subtracting this cube from the original polynomial P, we obtain a new polynomial P″, which we may treat in the same manner as P′, and continue the operation till the whole root is determined.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

83. We have already given rules in our Arithmetic, by which we are enabled to extract the Square and Cube Roots of any proposed number; we shall now proceed to explain the principles upon which these rules are founded.

The numbers

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 100, 1000,

when squared become

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 10000, 1000000,

and reciprocally, the numbers in the first line are the square roots of the numbers in the second.

Upon inspecting these two lines we perceive, that among numbers expressed by one or two figures, there are only nine which are the squares of other whole numbers; consequently, the square root of all the rest must be a whole number, plus a fraction.

Thus the square root of 53, which lies between 49 and 64, is 7 plus a fraction. So also, the square root of 91 is 9 plus a fraction.

84. It is however very remarkable, that *the square root of a whole number, which is not a perfect square, cannot be expressed by an exact fraction, and is therefore incommensurable with unity.*

To prove this, let $\frac{a}{b}$, a fraction in its lowest terms, be, if possible, the square root of some whole number N, then the square of $\frac{a}{b}$, or $\frac{a^2}{b^2}$, must be equal to N. But since a and b are, by supposition, prime to each other, a^2 and b^2 are also prime to each other; therefore, $\frac{a^2}{b^2}$ is an irreducible fraction, and cannot be equal to a whole number.

85. The difference between the squares of two consecutive whole numbers is greater in proportion as the numbers themselves are greater; the expression for this difference can easily be found.

Let a and $a+1$ be two consecutive whole numbers;
Then

$$(a+1)^2 = a^2 + 2a + 1$$

Hence

$$(a+1)^2 - a^2 = 2a + 1$$

that is to say, *the difference of the squares of two consecutive whole numbers, is equal to twice the less of the two numbers, plus unity.*

Thus the difference between the squares of 348 and 347 is equal to $2 \times 347 + 1$, or 695.

Let us now proceed to investigate a process for the extraction of the square root of any number, beginning with whole numbers.

Extraction of the square root of whole numbers.

86. If the number proposed consist of one or two figures only, its root may be found immediately, by inspecting the squares of the nine first numbers in (Art. 83.). Thus, the square root of 25 is 5, the square root of 42 is 6 plus a fraction, or 6 is the approximate square root of 42, and is within one unit of the true value; for 42 lies between 36, which is the square of 6, and 49, which is the square of 7.

Let us consider, then, a number composed of more than two figures, 6084 for example.

```
      60'84 | 78
      49
148 | 118'4
        0.
```

Since this number is comprised between 100, which is the square of 10, and 10000, which is the square of 100, its root must necessarily consist of two figures, that is to say, of tens and units. Designating the tens in the root sought by a, and the units by b, we have

$$60'84 = (a+b)^2 = a^2 + 2ab + b^2$$

which shows, that *the square of a number consisting of tens and units, is composed of the square of the tens, plus twice the product of the tens by the units, plus the square of the units.*

This being premised, since the square of a certain number of tens can contain nothing lower than hundreds, it follows that the squares of the tens contained in the root must be found in the part 60 (or 60 hundreds), to the left of the two last figures of 6084 (which written at full length is $6000 + 80 + 4$); we therefore separate the two last figures from the others by a point. The part 60 is comprised between the two squares 49 and 64, the roots of which are 7 and 8, hence 7 is the figure which expresses the number of tens in the root

sought; for 6000 is evidently comprised between 4900 and 6400, which are the squares of 70 and 80, and the root of 6084 must, therefore, be comprised between 70 and 80; hence, the root sought is composed of 7 tens and a certain number of units less than ten.

The figure 7 being thus found, we place it on the right of the given number, separating them by a vertical line as in division; we then subtract 49, which is the square of 7, from 60, which leaves as remainder 11, after which we write the remaining figures 84. The number 1184 which results from this first operation contains, as we have seen above, twice the product of the tens multiplied by the units, plus the square of the units. But the product of the tens multiplied by the units cannot be less than tens, and therefore the last figure 4 cannot form any part of the product of the tens by the units; hence this product is contained in the part 118 to the right of the figure 4, which we therefore separate from the others by a point.

If we double the tens, which gives 14, and divide 118 by 14, the quotient 8 is the figure of units in the root sought, or a figure greater than the one required. It may manifestly be greater than the figure sought, for 118 may contain, in addition to twice the product of the tens by the units, other tens arising from the square of the units. In order to determine whether 8 expresses the real number of units in the root, it is sufficient to place it on the right of 14, and then multiply the number 148, thus obtained, by 8. In this manner we form, 1°, the square of the units; 2°, twice the product of the units by the tens. This operation being effected the product is 1184, a number equal to the result of the first operation; subtracting this product the remainder is 0, which shows that 6084 is a perfect square, and 78 the root sought.

It will be seen, in reviewing the above process, that we have successively subtracted from 6084, the square of 7 tens or 70, plus twice the product of 70 by 8, plus the square of 8, that is, the three parts which enter into the composition of the square of $70 + 8$, or 78; and since the result of this subtraction is 0, it follows that 6084 is the square of 78.

Take as a second example the number 841.

	8'41	29
	4	
49	44'1	
	441	
	0.	

This number being comprised between 100 and 10000, its root must consist of two figures, that is to say, of tens and units. We can prove, as in the last example, that the root of the greatest square contained in 8, or in that portion of the number to the left of the two last figures, expresses the number of tens in the root required. But the greatest square contained in 8 is 4, whose root is 2, which is therefore the figure of the tens. Squaring 2, and subtracting the result from 8, the remainder is 4; bringing down the figures of the second period 41, and annexing them on the right of 4, the result is 441, a number which contains twice the product of the tens by the units, plus the square of the units.

We may further prove, as in the last case, that if we point off the last figure 1, and divide the preceding figures 44 by twice the tens, or 4, the quotient will be either the figure which expresses the number of units in the root, or a figure greater than the one sought. In this case the quotient is 11, but it is manifest that we cannot have a number greater than 9 for the units, for otherwise we must suppose that the figure already found for the tens is incorrect. — Let us try 9; place 9 to the right of 4, and then multiply this number 49 by 9; the product is 441, which, when subtracted from the result of the first operation, leaves a remainder 0, showing that 29 is the root required.

Let us take, as a third example, a number which is not a perfect square, such as 1287.

```
    12'87 | 35
    9     |
65 | 38'7
   | 325
   | 62
```

Applying to this number the process described in the preceding examples, we find that the root is 35, with a *remainder 62. This shows that 1287 is not a perfect square, but that it is comprised between the square of 35 and that of 36.* Thus, when the number is not a perfect square, the above process enables us at least to determine the root of the greatest square contained in the number, or the integral part of the root of the number.

87. Let us pass on to consider the extraction of the square root of a number composed of more than four figures.

```
       56'82'14'44 | 7538
       49          |
145   | 78'2
      | 725
1503  | 571'4
      | 4509
15068 | 12054'4
      | 120544
      |   0.
```

Let 56821444 be the number.

Since the number is greater than 10000, its root must be greater than 100; that is to say, it must consist of more than two figures. But, whatever the number may be, we may always consider it as composed of units and of tens, the tens being expressed by one or more figures. (Thus, any number such as 37142 may be resolved into $37140 + 2$, or 3714 tens, plus two units.)

Now, the square of the root sought, that is, the proposed number, contains the square of the tens, plus twice the product of the tens by the units, plus the square of the units. But the square of the tens must give at least hundreds; hence, the two last figures, 44, can form no part of it, and it is in the portion of the number to the left hand that we must look for that square. We further assert, that if we find the root of the greatest square contained in this portion of the root to the left hand, considering its *absolute value* 568214 alone, without reference to the remaining figures 44, we shall determine the whole number of tens in the root sought.

For let a be the root of the greatest square contained in 568214, then this last number must be comprised between a^2 and $(a+1)^2$, hence, 568214×100, or 56821400 is comprised between $a^2 \times 100$ and $(a+1)^2 \times 100$, and since these two last numbers differ from each other by more than a hundred (Art. 85.), it follows, that the proposed number itself, 56821444, is comprised between $a^2 \times 100$ and $(a+1)^2 \times 100$, and the root required must be comprised between the square roots of these two numbers, that is, between $a \times 10$ and $(a+1) \times 10$. Hence, it appears that the root sought consists of a tens, plus a certain number of units less than ten. The question is thus reduced to finding the square root of 568214, considering its *absolute value* alone.

Reasoning with regard to this number in the same manner as for the original number, in order to find the tens contained in its root, we must extract the root of the greatest square contained in the part to the left of the two last figures, 14, (which we therefore separate from the preceding ones by a point,) that is, in 5682.

The question is now farther reduced to extracting the square root of 5682, considering its *absolute value* alone, without reference to the figures which follow it. In order to find the number of tens in this new root, we must again separate the two last figures 82 by a point, and extract the root of the greatest square contained in 56.

Extracting then the root of 56, we find 7 for the root of 49, the greatest

square contained in 56; we place 7 on the right of the proposed number, and squaring it, subtract 49 from 56, which gives a remainder 7, to which we annex the following period 82, because we must determine the second figure of the root of the greatest square contained in 5682. Separating the last figure to the right of 782, and then dividing 78 by 14, which is twice the root already found, we have 5 for a quotient, which we annex to 14, we then multiply the whole number 145 by 5, and subtract the product 725 from 782; 75 represents the number of tens contained in the root of the number 568214.

In order to obtain the units in the root of the above number, we bring down the period 14, annex it to the second remainder 57, and point off the last figure of this number 5714. Dividing 571 by 150, which is twice the root already found, the quotient is 3, which we place to the right of 150, and multiplying the whole number 1503 by 3, we subtract the product 4509 from 5714. Hence, 753 expresses the whole number of tens in the root of the number 56821444.

Finally, in order to find the figure of units, we bring down the last period 44, annex it to the third remainder 1205, and point off the last figure of this number 120544. Dividing 12054 by 1506, which is twice the root already found, the quotient is 8, which we place on the right of 1506, and multiplying the whole number 15068 by 8, we subtract the product 120544 from the last result 120544. The remainder is 0; hence, 7538 is the root sought.

From what has been said above, it is easy to deduce the rule, which we have already given in arithmetic, for the extraction of the square root of a number consisting of any number of figures, and which it is unnecessary here to repeat.

Extraction of the square root by approximation.

88. When a whole number is not the square of another whole number, we have seen (Art. 84.) that its root cannot be expressed by a whole number, and an exact fraction; but although it is impossible to determine the precise value of the fraction which completes the root sought, we can approximate to it as nearly as we please.

Suppose that a is a whole number which is not a perfect square, and that we are required to extract the root to $\frac{1}{n}$, that is, to determine a number which shall differ from the true root of a, by a quantity less than the fraction $\frac{1}{n}$.

To effect this, let us observe that the quantity a may be put under the form $\frac{a n^2}{n^2}$; if we designate the integral portion of the root of $a n^2$ by r, this number $a n^2$ will be comprised between r^2 and $(r+1)^2$, hence, $\frac{a n^2}{n^2}$ is comprised between $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$, and consequently, the root of a is comprised between the roots of $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$, that is, between $\frac{r}{n}$ and $\frac{r+1}{n}$. Thus, it appears, that $\frac{r}{n}$ represents the square root of a within $\frac{1}{n}$ of the true value.

From this we derive the following

RULE.

Multiply the given number a, *by the square of* n, (n *being the denominator of the fraction which determines the required degree of approximation), extract the integral part of the square root of the product, and divide this integral part by the denominator* n.

Let it be required, for example, to find the square root of 59 within $\frac{1}{12}$ of the true value.

Multiply 59 by the square of 12, that is 144, the product is 8496, the integral part of the root of 8496 is 92. Hence, $\frac{92}{12}$ or $7\frac{8}{12}$ is the approximate root of 59, the result differing from the true value by a quantity less than $\frac{1}{12}$.

So also,

$$\sqrt{11} = 3\tfrac{4}{15} \text{ true to } \tfrac{1}{15}$$
$$\sqrt{223} = 14\tfrac{37}{40} \text{ true to } \tfrac{1}{40}$$

89. The method of *approximation in decimals*, which is the process most frequently employed, is an immediate consequence of the preceding rule.

In order to obtain the square root of a whole number within $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ ... of the true value, we must, according to the above rule, multiply the proposed number by $(10)^2$, $(100)^2$, $(1000)^2$, ... or, which comes to the same thing, place to the right of the number, two, four, six, ... cyphers, then extract the integral part of the root of the product, and divide the result by 10, 100, 1000 .....

Hence, in order to obtain any required number of decimals in the root, we must

Place on the right hand of the proposed number twice as many zeros as we wish to have decimal figures; extract the integral part of the root of this new number, and then mark off in the result the required number of decimal places.

This rule has already been sufficiently exemplified in our arithmetic.

Extraction of the square root of fractions.

We have seen (Art. 62) that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, hence, in order to extract the square root of a fraction, it is sufficient to extract the square roots of the numerator and denominator, and then divide the former result by the latter. This method may be employed with advantage when either one or both of the terms of the proposed fraction are perfect squares; but when this is not the case, it will be found inconvenient in practice. If, for example, we take the fraction $\frac{3}{5}$, although $\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}}$, (since each of these expressions, when multiplied by itself, produces the same quantity $\frac{3}{5}$,) we must find an approximate value both for $\sqrt{3}$ and also for $\sqrt{5}$, and after all we shall not be able to determine at once the degree of approximation in the result. Under such circumstances the following process may be employed:—

Let the proposed fraction be $\frac{a}{b}$, this may be put under the form $\frac{ab}{b^2}$, this being premised, let r represent the integral part of the root of the numerator $a\,b$, hence $\frac{ab}{b^2}$, or $\frac{a}{b}$, is comprised between $\frac{r^2}{b^2}$ and $\frac{(r+1)^2}{b^2}$, consequently, the

root of $\frac{a}{b}$ is comprised between $\frac{r}{b}$ and $\frac{r+1}{b}$. Thus it appears, that $\frac{r}{b}$ represents the root of $\frac{a}{b}$ within $\frac{1}{b}$ of the true value. Hence, in order to obtain the square root of a fraction,

Make the denominator of the fraction a perfect square, by multiplying both terms of the fraction by the denominator, extract the integral part of the root of the numerator, and divide the result by the denominator.

Let it be required to extract the square root of $\frac{7}{13}$.

This fraction is the same as $\frac{7 \times 13}{(13)^2}$ or $\frac{91}{(13)^2}$. But the integral part of the square root of 91 is 9, hence, $\frac{9}{13}$ is the root sought; a result within $\frac{1}{13}$ of the true value.

A greater degree of approximation may, perhaps, be required. In this case, returning to the number $\frac{91}{(13)^2}$, extract the root of 91 to any required degree of approximation. Suppose, for example, we wish to find the root of 91 within $\frac{1}{100}$ of the real value, it will become, by (Art. 88) $\sqrt{91} = 9.53\ldots\ldots$ Hence the root of $\frac{7}{13}$, or $\frac{91}{(13)^2}$, will be $\frac{9.53}{13}$, or a result within $\frac{1}{1300}$ of the true value. For it is evident that $\frac{91}{(13)^2}$ is comprised between $\frac{(9.53)^2}{(13)^2}$ and $\frac{(9.54)^2}{13^2}$; hence the square root of $\frac{91}{(13)^2}$, or $\frac{7}{13}$, differs from $\frac{9.53}{13}$ by a quantity less than $\frac{1}{1300}$.

REMARK.—It frequently happens that the denominator of the fraction, although not a perfect square, has a perfect square for one of its factors, in which case the above operation may be simplified

Let the fraction, for example, be $\frac{23}{48}$. 48 is equal to 16×3, or $(4)^2 \times 3$; hence, multiplying both terms of the fraction by 3, it becomes $\frac{23 \times 3}{(4)^2 \times (3)^2}$ or $\frac{69}{(12)^2}$; and the denominator is thus made a perfect square. Extracting the root of 69 to $\frac{1}{10}$, which gives 8.3, we find $\frac{8.3}{12}$, or, $\frac{83}{120}$ for the root required, a result within $\frac{1}{120}$ of the true value.

In general, therefore, *whenever the denominator of the fraction involves a factor which is a perfect square, multiply both terms of the fraction by the factor which is not a perfect square.*

Extraction of the square root of decimal fractions.

90. This process is an immediate consequence of the preceding remark. Required, for example, the square root of 2.36.

This fraction is the same as $\frac{236}{100}$; in this case the denominator is a perfect

square, extracting therefore the integral part of the root of the numerator we have $\frac{15}{10}$, a result within $\frac{1}{10}$ of the true value.

Again, let it be required to extract the square root of 3.425.

This fraction is the same as $\frac{3425}{1000}$. But 1000 is not a perfect square, it is however equal to 100×10, or $(10)^2 \times 10$; thus, in order to render the denominator a perfect square it is sufficient to multiply both terms of the fraction by 10, which gives $\frac{34250}{10000}$, or $\frac{34250}{(100)^2}$. Extracting the integral part of the root 34250 we find 185, hence the root required is $\frac{185}{100}$, or 1.85, a result which is within $\frac{1}{100}$ of the true value.

If we wish to have a greater number of decimal places in the root, we must add on the right of 34250 twice as many zeros as we wish to have additional decimal figures.

From what has just been observed, we readily deduce the general rule for the extraction of the square root of a decimal fraction which has been already given in our Arithmetic.

EXTRACTION OF THE CUBE ROOT OF NUMBERS.

91. The numbers,

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 100, 1000,

when cubed become

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1000000, 1000000000;

and reciprocally, the numbers in the first line are the cube roots of the numbers in the second.

Upon inspecting the two lines we perceive, that, among the numbers expressed by one, two, or three figures, there are only nine which are *perfect cubes*, consequently, the cube root of all the rest must be a whole number plus a fraction.

92. But we can prove, in the same manner as in the case of the square root, that *the cube root of a whole number, which is not the perfect cube of some other whole number, cannot be expressed by an exact fraction, and consequently its cube root is incommensurable with unity.*

For if we suppose $\frac{a}{b}$, an exact fraction in its lowest terms, to be the cube root of some whole number N, it follows that the cube of $\frac{a}{b}$, or $\frac{a^3}{b^3}$, must be equal to N. But since a and b are, by supposition, prime to each other, a^3 and b^3 are also prime to each other; and therefore $\frac{a^3}{b^3}$ cannot be equal to a whole number.

93. The difference between the cubes of two consecutive whole numbers is greater in proportion as the numbers themselves are greater; the expression for this difference can easily be found.

Let

a and $a + 1$ be two consecutive whole numbers;

Then,

$$(a + 1)^3 = a^3 + 3a^2 + 3a + 1;$$

Hence,

$$(a+1)^3-a^3=3a^2+3a+1;$$

that is to say, *the difference of the cubes of two consecutive whole numbers is equal to three times the square of the less of the two numbers, plus three times the simple power of the number, plus unity.*

Thus, the difference between the cube of 90 and the cube of 89 is equal to $3\times(89)^2+3\times 89+1=24031$.

Let us now proceed to investigate a process for the extraction of the cube root of any number.

Extraction of the Cube Root.

94. The cube root of a proposed number, consisting of one, two, or three figures only, will be found immediately by inspecting the cubes of the first nine numbers in Art. 91. Thus the cube root of 125 is 5, and the cube root of 54 is 3 plus a fraction, for $3\times3\times3=27$, and $4\times4\times4=64$; therefore 3 is the approximate cube root of 54 within one unit of the true value.

For the purpose of investigating a new and simple rule for the extraction of the cube root, it will be necessary to attend to the composition of a complete power of the third degree. Now, since we have

$$(a+b)^3=(a+b)\ (a+b)\ (a+b)=a^3+3a^2b+3ab^2+b^3,$$

it is obvious that the cube of a number, consisting of tens and units, will be algebraically indicated by the polynomial

$$a^3+3a^2b+3ab^2+b^3$$

where a designates the number of tens, and b the number of units in the root sought. The number in the tens' place will evidently be found by extracting the cube root of the monomial a^3, for $\sqrt[3]{a^3}=a$, and removing a^3 from the polynomial $a^3+3a^2b+3ab^2+b^3$, we have the remainder

$$3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)\ b;$$

and the difficulty that has been hitherto experienced in the extraction of the cube root entirely consists in the composition of the expression $3a^2+3ab+b^2$, which is obviously the true divisor for the determination of b, the figure of the root in the place of units. The part $3a^2$ of the expression $3a^2+3ab+b^2$, being independent of b, the yet unknown part of the root, is employed as a *trial* divisor for the determination of b; but since the expression $3a^2+3ab+b^2$ involves the unknown part of the root in its composition, it is obvious that the trial divisor $3a^2$, which does not contain b, will at the first step of the operation give no certain indication of the next figure of the root, unless the figure denoted by b be very small in comparison with that denoted by a; for the trial divisor $3a^2$ will be considerably augmented by the addend $3ab+b^2$, when b is a large number, while the augmentation, when b is a small number, will not so materially affect the trial divisor.

When the figure in the tens' place is a small number, as 1 or 2, it is hence obvious that little or no dependence can be placed on the trial divisor; but if a be great and b small, the trial divisor $3a^2$ will generally point out the value

of b. All this will be evident if we consider that the relative values of a and b materially affect the true divisor, $3a^2+3ab+b^2$. In the successive steps, however, of the cube root, this uncertainty diminishes; for conceiving a to designate a number consisting of tens and hundreds, and b the number of units, then the value of b being small in comparison with a, the amount of the effect of b in the addend $3ab+b^2$ will be very inconsiderable; hence the trial divisor, $3a^2$, will generally indicate the next figure in the root.

To remove, in some measure, the difficulty which has hitherto been experienced in the extraction of the cube root, we shall proceed to point out *two* methods of composing the true divisor, $3a^2+3ab+b^2$, and leave the student to select that which he conceives to possess greater facility of operation.

95. *First method of composition of* $3a^2+3ab+b^2$.

$$
\begin{array}{lll}
a \times a = & a^2 & a^3+3a^2b+3ab^2+b^3\ (a+b = \text{root sought} \\
a & a^2 \times a = \ldots\ldots & a^3 \\
a & a^2 & \rule{3cm}{0.4pt} \\
\rule{1cm}{0.4pt} & \rule{1cm}{0.4pt} & 3a^2b+3ab^2+b^3 \\
 & 3a^2 & \\
(3a+b)\times b = & 3ab+b^2 & \\
b & \rule{3cm}{0.4pt} & \\
b & (3a^2+3ab+b^2)\times b = \ldots\ldots & 3a^2b+3ab^2+b^3 \\
 & b^2 & \rule{3cm}{0.4pt} \\
\rule{2cm}{0.4pt} & \rule{3cm}{0.4pt} & \\
3a+3b & 3a^2+6ab+3b^2 &
\end{array}
$$

Distinguishing the three columns from left to right, by *first*, *second*, and *third* columns, we write a in the root, and also three times vertically in the first column; then $a\times a$ produces a^2, which write also three times vertically in the second column; multiply the second a^2 by a, placing the product, a^3, under a^3 in the third column; then subtracting a^3 from the proposed quantity, we have the remainder $3a^2b+3ab^2+b^3$. The sum of the three quantities in the second column gives $3a^2$ for the trial divisor, by which find b, the next figure of the root, and to $3a$, the sum of the three last written quantities in the first column, annex b; then the sum, $3a+b$, is multiplied by b, and the product, $3ab+b^2$, is placed in the second column; then the trial divisor $3a^2$, and the addend $3ab+b^2$ being collected, give the true divisor, $3a^2+3ab+b^2$, which multiply by b, and place the product, $3a^2b+3ab^2+b^3$, under the remainder $3a^2b+3ab^2+b^3$. When there is a remainder after this operation, the process may be continued by writing b twice in the first column, under $3a+b$, and b^2 once in the second column, under the last true divisor; then $3a^2+6ab+3b^2$, the sum of the last written three lines in the second column, will be another trial divisor, with which proceed as above. We have written a^2 in the second column three times in succession, to assimilate the first step in the operation to the other successive steps, but the first trial divisor, $3a^2$, may be written at once, and the symmetry of the disposition of the quantities in the first steps disregarded

96. *Second method of composing $3a^2+3ab+b^2$, the true divisor.*

$$
\begin{array}{lll}
a & & a^3+3a^2b+3ab^2+b^3\ (a+b \\
\underline{a} & a^2 \cdots\cdots\cdots & \underline{a^3} \\
2a \cdots\cdots\cdots & \underline{2a^2} & 3a^2b+3ab^2+b^3 \\
\underline{a} & 3a^2 & \\
3a\ +b \ \cdots & \underline{3ab+\ b^2} & \\
\underline{\quad b} & 3a^2+3ab+\ b^2 \cdots & \underline{3a^2b+3ab^2+b^3} \\
3a+2b \ \cdots & 3ab+2b^2 & \\
\underline{\quad b} & \overline{3a^2+6ab+3b^2} = \text{second trial divisor.} & \\
3a+3b & &
\end{array}
$$

In this method we write a under a in the first column, and the sum $2a$ being multiplied by a, gives $2a^2$ to place under a^2 in the second column, and the sum of $2a^2$ and a^2 is $3a^2$ for the trial divisor. Again, under $2a$ in the first column write a, and the sum of $2a$ and a gives $3a$. Now, having found b by the trial divisor, annex it to $3a$ in the first column, making $3a+b$, which, multiplied by b, and the product placed in the second column, gives, by addition, the true divisor $3a^2+3ab+b^2$, as before. We shall exhibit the operation of extracting the cube root by both these methods.

Examples.

(1.) What is the cube root of $x^6-9x^5+39x^4-99x^3+156x^2-144x+64$?

By the first method.

$$
\begin{array}{lll}
x^2 & x^4 & x^6-9x^5+39x^4-99x^3+156x^2-144x+64\ (x^2-3x+4 \\
x^2 & x^4 \cdots\cdots\cdots & \underline{x^6} \\
\underline{x^2} & \underline{x^4} & -9x^5+39x^4-99x^3 \\
 & 3x^4 & \\
3x^2-3x \cdots & \underline{-9x^3+\ 9x^2} & \\
\quad -3x & 3x^4-\ 9x^3+\ 9x^2 \cdots\cdots & \underline{-9x^5+27x^4-27x^3} \\
 & \qquad\qquad 9x^2 & \\
\underline{\quad -3x} & \overline{3x^4-18x^3+27x^2} & 12x^4-72x^3+156x^2-144x+64 \\
3x^2-9x+4 \cdots & \qquad 12x^2-36x+16 & \\
 & \overline{3x^4-18x^3+39x^2-36x+16} \cdots & \underline{12x^4-72x^3+156x^2-144x+64}
\end{array}
$$

(2.) What is the cube root of $x^6+6x^5-40x^3+96x-64$?

By the second method.

$$
\begin{array}{lll}
x^2 & & x^6+6x^5-40x^3+96x-64\ (x^2+2x-4 \\
x^2 & x^4 \ \cdots\cdots & x^6 \\
\hline
2x^2 \cdots\cdots & 2x^4 & 6x^5-40x^3 \\
x^2 & \underline{\qquad} & \\
\hline
 & 3x^4 & \\
3x^2+2x \cdots & \quad 6x^3+4x^2 & \\
\quad 2x & \underline{\qquad\qquad} & \\
\hline
 & 3x^4+\ 6x^3+\ 4x^2 \cdots\cdots & 6x^5+12x^4+8x^3 \\
3x^2+4x \cdots & \quad 6x^3+\ 8x^2 & \underline{\qquad\qquad\qquad} \\
\quad 2x & \underline{\qquad\qquad} & -12x^4-48x^3+96x-64 \\
\hline
 & 3x^4+12x^3+12x^2 & \\
3x^2+6x-4 \cdots & \quad -12x^2-24x+16 & \\
 & \underline{\qquad\qquad\qquad} & \\
 & 3x^4+12x^3\ -\ 24x+16 & -12x^4-48x^3+96x-64. \\
 & & \underline{\qquad\qquad\qquad}
\end{array}
$$

(3.) What is the cube root of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$? Ans. $a+b+c$.

(4.) Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
Ans. x^2-2x+

97. The same process is employed in the extraction of the cube root of numbers, as in the subsequent examples.

Examples.

(1.) Extract the cube root of 403583419.

$$
\begin{array}{lll}
7 \cdots\cdots & 49 & 403583419\ (739 = \text{root} \\
7 & 49 \cdots\cdots\cdots\cdots & 343 \\
7 & 49 & \underline{\qquad} \\
\underline{\quad} & \underline{\quad} & 60583 \\
 & 147 & \\
213 \cdots\cdots & 639 & \\
3 & \underline{\qquad} & \\
3 & 15339 \cdots\cdots\cdots\cdots & 46017 \\
 & \quad 9 & \underline{\qquad} \\
 & \underline{\qquad} & 14566419 \\
\underline{\quad} & 15987 & \\
2199 \cdots\cdots & 19791 & \\
 & \underline{\qquad} & \\
 & 1618491 \cdots\cdots\cdots\cdots & 14566419
\end{array}
$$

(2.) What is the cube root of 115501303?

```
                                         .     .
                                    115501303 (487 = root
4 . . . . . . . . 16 . . . . . . . . .  64
                                        ----
4                                       51501
---
8 . . . . . . . . 32

4                 48
---
128 . . . . . . . 1024
  8               ----
----              5824 . . . . . . . . 46592
136 . . . . . .   1088                 -------
  8               ----                 4909303
----              6912
1447 . . . . . .  10129
                  ------
                  701329 . . . . . . . 4909303
```

98. The local values of the figures in the root determine the arrangement of the figures in the several columns, as is exemplified by working the last example as below; and by omitting the terminal ciphers, the arrangement is precisely the same as in the preceding example.

```
                                            .  .  .
                                        115501303 (400+80+7
 400 . . . . . . 160000 . . . . . . . .  64000000     =487
 400                                     --------
----                                     51501303
 800 . . . . . . 320000
 400             ------
----             480000
1200
  80
----
1280 . . . . . . 102400
  80             ------
----             582400 . . . . . . . .  46592000
1360 . . . . . . 108800                  --------
  80             ------                   4909303
----             691200
1440
   7
----
1447 . . . . . .  10129
                 ------
                 701329 . . . . . . . .  4909303
                                         -------
```

99. *Extraction of the fourth root of whole numbers.*

The investigation of a method for extracting the fourth root of any number is similar to that employed for the cube root. Thus, since

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

we may conceive a to denote the number of tens, and b the number of units in the root of the number expressed by $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Then $\sqrt[4]{a^4}=a$, figure in the tens' place, and the remainder, when a^4 is removed, is

$$4a^3b+6a^2b^2+4ab^3+b^4=(4a^3+6a^2b+4ab^2+b^3)\,b.$$

The method of composing the divisor $4a^3+6a^2b+4ab^2+b^3$, for the determination of b, the figure in the units' place, may be illustrated as follows:

$$\begin{array}{llllll}
a\times a = a^2 & & & & & a^4+4a^3b+6a^2b^2+4ab^3+b^4\ (a+b \\
\underline{a} & \overline{a^2}\times a & = a^3 & & & \\
2a\times a = 2a^2 & & \overline{a^3}\times a & = a^4 & & \\
\underline{a} & 3a^2\times a & = 3a^3 & & & 4a^3b+6a^2b^2+4ab^3+b^4 \\
3a\times a = 3a^2 & & \overline{4a^3} & & & \\
\underline{a} & \overline{6a^2} & & & & \\
(4a+b)b = 4ab+b^2 & & & & & \\
 & (6a^2+4ab+b^2)b = 6a^2b+4ab^2+b^3 & & & & \\
 & & (4a^3+6a^2b+4ab^2+b^3)b = & \underline{4a^3b+6a^2b^2+4ab^3+b^4.} & &
\end{array}$$

100. From this mode of composing the complete divisor we easily derive the following process for the extraction of the fourth root of any number.

Example.—What is the fourth root of 1185921?

$$\begin{array}{llllll}
3\times 3 = 9 & & & & & 11\dot{8}59\dot{2}1\ (33=\text{root} \\
\underline{3} & \overline{9}\times 3 = 27 & & & & \\
6\times 3 = 18 & & \overline{27}\times 3 & = & & \underline{81} \\
\underline{3} & \overline{27}\times 3 = 81 & & & & 375921 \\
9\times 3 = 27 & & \overline{108\ldots} & & & \\
\underline{3} & \overline{54\ldots} & & & & \\
123\times 3 = 369 & & & & & \\
 & \overline{5769}\times 3 = 17307 & & & & \\
 & & \overline{125307}\times 3 & = & & \underline{375921}
\end{array}$$

In the same manner the student may readily investigate rules for the extraction of the higher roots of numbers, simply observing to use an additional column for each successive root.

101. *To represent a rational quantity as a surd.*

Let it be required to represent a in the form of a surd of the nth order; then, by Art. 63, the form will be $\sqrt[n]{a^n}$, or $(a^n)^{\frac{1}{n}}$; for by raising a to the nth power, and then extracting the nth *root* of the nth *power* of a, we must evidently revert to the proposed quantity, a. Hence we have

$$a = \sqrt{a^2} = \sqrt[3]{a^3} = \sqrt[4]{a^4} = \sqrt[15]{a^{15}} = \sqrt[m]{a^m} = \sqrt[n]{a^n}$$

$$a = (a^2)^{\frac{1}{2}} = (a^{\frac{3}{2}})^{\frac{2}{3}} = (a^{\frac{6}{5}})^{\frac{5}{6}} = (a^{\frac{m}{n}})^{\frac{n}{m}}.$$

102. When the given quantity is the product of a rational quantity and surd, we must represent the rational quantity in the form of the given surd and then express the product by means of the radical sign, or fractional index Thus we have

$$\begin{aligned}
a\sqrt{b} &= \sqrt{a^2}\times\sqrt{b} = \sqrt{a^2b} \\
3a\sqrt{5b} &= \sqrt{3a\times 3a}\times\sqrt{5b} = \sqrt{9a^2\times 5b} = \sqrt{45a^2b} \\
a\sqrt[3]{xy} &= \sqrt[3]{a\times a\times a}\times\sqrt[3]{xy} = \sqrt[3]{a^3}\times\sqrt[3]{xy} = \sqrt[3]{a^3xy} \\
12\sqrt{7} &= \sqrt{144}\times\sqrt{7} = \sqrt{144\times 7} = \sqrt{1008} \\
a(-a^{-2}x^2)^{\frac{1}{2}} &= (a^2)^{\frac{1}{2}}(1-a^{-2}x^2)^{\frac{1}{2}} = (a^2-a^0x^2)^{\frac{1}{2}} = \sqrt{a^2-x^2}.
\end{aligned}$$

EXAMPLES.

(1.) Represent a^2 in the form of a surd, whose index is $\frac{1}{5}$.

(2.) Represent $2-\sqrt{3}$ in the form of a quadratic surd.

(3.) Transform $6\sqrt{11}$ into the form of a quadratic surd.

(4.) Transform $a\sqrt{a-b}$ into the form of a quadratic surd.

(5.) Represent as a surd the mixed quantity $(x+y)\sqrt{\dfrac{x-y}{x+y}}$.

(6.) Represent as a surd the mixed quantity $(x+4)\sqrt{\dfrac{1}{x+4}}$.

ANSWERS.

(1.) $\sqrt[5]{a^{10}}$ or $(a^{10})^{\frac{1}{5}}$.

(2.) $\sqrt{7-4\sqrt{3}}$.

(3.) $\sqrt{396}$.

(4.) $\sqrt{a^3-a^2b}$ or $(a^3-a^2b)^{\frac{1}{2}}$.

(5.) $\sqrt{x^2-y^2}$ or $(x^2-y^2)^{\frac{1}{2}}$.

(6.) $\sqrt{x+4}$ or $(x+4)^{\frac{1}{2}}$.

103. *To find multipliers which will render binomial surds rational.*

The product of two irrational quantities is, in many instances, a rational quantity, and therefore an irrational quantity may frequently be found, which, employed as a factor to multiply some other given irrational quantity, will

produce a rational result; and since the product of the sum and difference of two quantities is equal to the difference of their squares, we have, evidently,

$$\sqrt{a}\times\sqrt{a}=a;\ (\sqrt{a}-\sqrt{b})\,(\sqrt{a}+\sqrt{b})=a-b$$
$$\sqrt[3]{x}\times\sqrt[3]{x^2}=x;\ (x+\sqrt{y})\,(x-\sqrt{y})=x^2-y$$
$$\sqrt[m]{y}\times\sqrt[m]{y^{m-1}}=y;\ (\sqrt{x}-y)\,(\sqrt{x}+y)=x-y^2.$$

Hence it is obvious that, in these and similar equations, if one of the factors be given, the other factor or multiplier is readily known, and the proposed irrational quantity is thus rendered rational. In the same manner, since

$$(x\pm y)\,(x^2\mp xy+y^2)=x^3\pm y^3$$
$$\therefore\ (\sqrt[3]{x}\pm\sqrt[3]{y})\,(\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2})=x\pm y,$$

and the expression $\sqrt[3]{x}\pm\sqrt[3]{y}$ may therefore be rationalized by multiplying it by $\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2}$, and $\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2}$, multiplied by $\sqrt[3]{x}\pm\sqrt[3]{y}$, will produce a rational result.

Again, by division,

$$\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3+\ \ldots\ +y^{n-1}$$
$$\frac{x^n-y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+\ \ldots\ -y^{n-1}$$
$$\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+\ \ldots\ +y^{n-1}.$$

Put $x^n=a$; then $x=\sqrt[n]{a}$; $x^{n-1}=\sqrt[n]{a^{n-1}}$; $x^{n-2}=\sqrt[n]{a^{n-2}}$; &c.

$y^n=b$; then $y=\sqrt[n]{b}$; $y^2=\sqrt[n]{b^2}$; $y^3=\sqrt[n]{b^3}$; &c.;

hence, by substitution in the three preceding equations, we have

$$\frac{a-b}{\sqrt[n]{a}-\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}+\sqrt[n]{a^{n-4}b^3}+\ \ldots\ +\sqrt[n]{b^{n-1}}\ .\ (1)$$
$$\frac{a-b}{\sqrt[n]{a}+\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{a^{n-4}b^3}+\ \ldots\ -\sqrt[n]{b^{n-1}}\ .\ (2)$$
$$\frac{a+b}{\sqrt[n]{a}+\sqrt[n]{b}}=\sqrt[n]{a^{n-1}}-\sqrt[n]{a^{n-2}b}+\sqrt[n]{a^{n-3}b^2}-\sqrt[n]{a^{n-4}b^3}+\ \ldots\ +\sqrt[n]{b^{n-1}}\ .\ (3)$$

Now the dividend being the product of the divisor and quotient, it is obvious that a binomial surd of the form $\sqrt[n]{a}-\sqrt[n]{b}$ will be rendered rational by multiplying it by n terms of the second side of equation (1), and a binomial surd of the form $\sqrt[n]{a}+\sqrt[n]{b}$ will be rationalized by employing n terms of the second side of equation (2) or (3), according as n is even or odd, the product in the former case being $a-b$, and in the latter $a-b$ or $a+b$.

Note.—When n is an even number employ equation (2), and when it is an odd number use equation (3), in order to rationalize $\sqrt[n]{a}+\sqrt[n]{b}$.

Examples.

(1.) Find a multiplier to rationalize $\sqrt[3]{11}-\sqrt[3]{7}$.

Employing equation (1), we have $a=11$, $b=7$, and $n=3$; hence required multiplier $=\sqrt[3]{11^2}+\sqrt[3]{11.7}+\sqrt[3]{7^2}=\sqrt[3]{121}+\sqrt[3]{77}+\sqrt[3]{49}$.

$$\begin{array}{rl}
\text{For} & \sqrt[3]{121}+\sqrt[3]{77}+\sqrt[3]{49} \\
& \sqrt[3]{11}\ -\sqrt[3]{7} \\ \hline
& \sqrt[3]{1331}+\sqrt[3]{847}+\sqrt[3]{539} \\
& \qquad -\sqrt[3]{847}-\sqrt[3]{539}-\sqrt[3]{343} \\ \hline
& 11 \quad * \quad * \quad -\ 7\ =4,\ \text{a rational product.}
\end{array}$$

(2.) Rationalize the binomial surd $\sqrt[3]{5}+\sqrt[3]{4}$.

Here we have $a=5$, $b=4$, $n=3$, an odd number; hence by equation (3) we have multiplier required $=\sqrt[3]{25}-\sqrt[3]{20}+\sqrt[3]{16}$; for by multiplication $(\sqrt[3]{5}+\sqrt[3]{4})(\sqrt[3]{25}-\sqrt[3]{20}+\sqrt[3]{16})=5+4=9=$ a rational number.

(3.) What multiplier will render the denominator of the fraction $\dfrac{1}{\sqrt[5]{7}-\sqrt[5]{2}}$ a rational quantity.

(4.) Change $\dfrac{5}{\sqrt[3]{4}-\sqrt[3]{2}}$ into a fraction that shall have a rational denominator

(5.) Change $\dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}\pm\sqrt[3]{xy}+\sqrt[3]{y^2}}$ into a fraction that shall have a rational denominator.

(6.) Change $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$ into a fraction that shall have a rational denominator.

ANSWERS.

(3.) $\sqrt[5]{7^4}+\sqrt[5]{7^3.2}+\sqrt[5]{7^2.2^2}+\sqrt[5]{7.2^3}+\sqrt[5]{2^4}$

(4.) $\dfrac{5(\sqrt[3]{16}+\sqrt[3]{8}+\sqrt[3]{4})}{2}$

(5.) $\dfrac{\sqrt[3]{x^2}(\sqrt[3]{x}\mp\sqrt[3]{y})}{x\pm y}=\dfrac{x\mp\sqrt[3]{x^2y}}{x+y}$

(6.) $\dfrac{a+\sqrt{a^2-x^2}}{x}$.

104. *To extract the square root of a binomial surd.*

Before commencing the investigation of the formula for the extraction of the square root of a binomial surd, it will be necessary to premise two or three lemmas.

Lemma 1. The square root of a quantity cannot be partly rational and partly irrational.

For if $\sqrt{a}=b+\sqrt{c}$, then by squaring we have

$$a=b^2+c+2b\sqrt{c}\text{: therefore }\sqrt{c}=\frac{a-b^2-c}{2b};$$

that is, an irrational equal to a rational quantity, which is absurd.

Lemma 2. If $a\pm\sqrt{b}=x\pm\sqrt{y}$ be an equation consisting of rational and irrational quantities, then $a=x$, and $\sqrt{b}=\sqrt{y}$.

For if a be not equal to x, let $a-x=d$, then we have

$$\pm\sqrt{y}\mp\sqrt{b}=a-x;\text{ but }a-x=d;\text{ therefore}$$
$$\pm\sqrt{y}\mp\sqrt{b}=d,\text{ which is impossible,}$$
$$\therefore\ a=x,\text{ and }\sqrt{b}=\sqrt{y}.$$

Lemma 3. If $\sqrt{a+\sqrt{b}}=x+y$; then $\sqrt{a-\sqrt{b}}=x-y$; where x and y are supposed to be one or both irrational quantities.

For since $a+\sqrt{b}=x^2+y^2+2xy$; and since x^2 and y^2 are both rational, $2xy$ must be irrational, otherwise $\sqrt{b}=x^2+y^2+2xy-a$, a rational quantity which is impossible by Lemma 1; hence by Lemma 2, we have

$$a=x^2+y^2;\ \sqrt{b}=2xy$$
$$\therefore\ a-\sqrt{b}=x^2-2xy+y^2$$
$$\text{and }\sqrt{a-\sqrt{b}}=x-y.$$

Let it now be required to extract the square root of $a+\sqrt{b}$.

Assume $\sqrt{a+\sqrt{b}}=x+y$; then $\sqrt{a-\sqrt{b}}=x-y$

$$\therefore\ a+\sqrt{b}=x^2+y^2+2xy$$
$$a-\sqrt{b}=x^2+y^2-2xy$$

$\therefore$ By addition $2a \quad =2\,(x^2+y^2)$, or $a=x^2+y^2$.

Again, $\sqrt{a+\sqrt{b}}\times\sqrt{a-\sqrt{b}}=x^2-y^2$, or $\sqrt{a^2-b}=x^2-y^2$.

Hence $x^2+y^2=a$

$x^2-y^2=\sqrt{a^2-b}=c$, suppose.

Therefore, by addition and subtraction we have

$$x^2=\frac{a+c}{2} \text{ and } y^2=\frac{a-c}{2}$$

$$\therefore\ x=\sqrt{\frac{a+c}{2}} \text{ and } y=\sqrt{\frac{a-c}{2}}$$

$$\text{Hence } \sqrt{a+\sqrt{b}}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}\ \ldots\ldots\ldots\ (1)$$

$$\sqrt{a-\sqrt{b}}=\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}\ \ldots\ldots\ldots\ (2)$$

where $c=\sqrt{a^2-b}$; and therefore a^2-b must be a perfect square; and this is the *test* by which we discover the possibility of the operation proposed

EXAMPLES.

(1.) What is the square root of $11+\sqrt{72}$, or $11+6\sqrt{2}$?

Here $a=11$; $b=72$; $c=\sqrt{a^2-b}=\sqrt{121-72}=7$

$$\therefore\ \sqrt{11+6\sqrt{2}}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}=3+\sqrt{2}.$$

(2.) What is the square root of $23-8\sqrt{7}$?

Here $a=23$; $b=8^2\times7=448$; $c=\sqrt{a^2-b}=\sqrt{529-448}=9$

$$\therefore\ \sqrt{23-8\sqrt{7}}=\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}=4-\sqrt{7}.$$

(3.) What is the square root of $14\pm6\sqrt{5}$? Ans. $3\pm\sqrt{5}$.

(4.) What is the square root of $18\pm 2\sqrt{77}$? Ans. $\sqrt{7}\pm\sqrt{11}$

(5.) What is the square root of $94+42\sqrt{5}$? Ans. $7+3\sqrt{5}$.

(6.) What is $\sqrt{np+2m^2-2m\sqrt{np+m^2}}$ equal to? Ans. $\sqrt{np+m^2}-m$.

(7.) Simplify the expression $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}$. Ans. 10.

(8.) What is $\sqrt{28+10\sqrt{3}}$ equal to? Ans. $5+\sqrt{3}$.

(9.) $\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}=\pm2b$.

(10.) $\sqrt{ab+4c^2-d^2+2\sqrt{4abc^2-abd^2}}=\sqrt{ab}+\sqrt{4c^2-d^2}$.

(11.) What is the square root of $-2\sqrt{-1}$? Ans. $1-\sqrt{-1}$.

(12.) What is the square root of $3-4\sqrt{-1}$? Ans. $2-\sqrt{-1}$.

(13.) What is the square root of $\frac{3\sqrt{3}+2\sqrt{6}}{4}$ $\frac{112+20\sqrt{12}}{\sqrt{3}}$?

Ans. $(1+\sqrt{2})\,.\,(5+\sqrt{3})$

BINOMIAL THEOREM.

105. It is manifest, from what has been said above, that algebraic polynomials may be raised to any power merely by applying the rules of multiplication. We can however in all cases obtain the desired result without having recourse to this operation, which would frequently prove exceedingly tedious. When a binomial quantity of the form $x+a$ is raised to any power, the successive terms are found in all cases to bear a certain relation to each other. This law, when expressed generally in algebraic language, constitutes what is called the "Binomial Theorem." It was discovered by Sir Isaac Newton, who seems to have arrived at the general principle by examining the results of actual multiplication in a variety of particular cases, a method which we shall here pursue, and give a rigorous demonstration of the proposition in a subsequent article of this treatise.

Let us form the successive powers of $x+a$ by actual multiplication.

$$
\begin{array}{ll}
x+a & \\
x+a & \\
\hline
x^2+xa & \\
\quad +xa+a^2 & \\
\hline
x^2+2xa+a^2 & \ldots\ldots\text{2d power} \\
x+a & \\
\hline
x^3+2x^2a+xa^2 & \\
\quad +x^2a+2xa^2+a^3 & \\
\hline
x^3+3x^2a+3xa^2+a^3 & \ldots\ldots\text{3d power.} \\
x+a & \\
\hline
x^4+3x^3a+3x^2a^2+xa^3 & \\
\quad +x^3a+3x^2a^2+3xa^3+a^4 & \\
\hline
x^4+4x^3a+6x^2a^2+4xa^3+a^4 & \ldots\ldots\text{4th power.} \\
x+a & \\
\hline
x^5+4x^4a+6x^3a^2+4x^2a^3+xa^4 & \\
\quad +x^4a+4x^3a^2+6x^2a^3+4xa^4+a^5 & \\
\hline
x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5 & \ldots\ldots\text{5th power.} \\
x+a & \\
\hline
x^6+5x^5a+10x^4a^2+10x^3a^3+5x^2a^4+xa^5 & \\
\quad +x^5a+5x^4a^2+10x^3a^3+10x^2a^4+5xa^5+a^6 & \\
\hline
x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6 & \ldots\ldots\text{6th power.} \\
x+a & \\
\hline
x^7+6x^6a+15x^5a^2+20x^4a^3+15x^3a^4+6x^2a^5+xa^6 & \\
\quad +x^6a+6x^5a^2+15x^4a^3+20x^3a^4+15x^2a^5+6xa^6+a^7 & \\
\hline
x^7+7x^6a+21x^5a^2+35x^4a^3+35x^3a^4+21x^2a^5+7xa^6+a^7 & \text{7th power.}
\end{array}
$$

In order that these results may be more clearly exhibited to the eye, we shall arrange them in a table.

TABLE OF THE POWERS OF $x + a$.

$(x + a)$	$x + a$
$(x+a)^2$	$x^2 + 2xa + x^2$
$(x+a)^3$	$x^3 + 3x^2a + 3xa^2 + a^3$
$(x+a)^4$	$x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^3$
$(x+a)^5$	$x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$
$(x+a)^6$	$x^6 + 6x^5a + 15x^4a^2 + 20x^3a^3 + 15x^2a^4 + 6xa^5 + a^6$
$(x+a)^7$	$x^7 + 7x^6a + 21x^5a^2 + 35x^4a^3 + 35x^3a^4 + 21x^2a^5 + 7xa^6 + a^7$
$(x+a)^8$	$x^8 + 8x^7a + 28x^6a^2 + 56x^5a^3 + 70x^4a^4 + 56x^3a^5 + 28x^2a^6 + 8xa^7 + a^8$

In the above table, the quantities in the left hand column are called the *expressions* for a binomial raised to the *first*, *second*, *third*, &c. *power*; the corresponding quantities in the right hand column are called the *expansions*, or, *developements* of the others.

106. The developements of the successive powers of $x - a$ are precisely the same with those of $x + a$, with this difference, that the signs of the terms are alternately $+$ and $-$; thus,

$$(x - a)^5 = x^5 - 5x^4a + 10x^3a^2 - 10x^2a^3 + 5xa^4 - a^5$$

and so for all the others.

107. On considering the above table we shall perceive, that

I. In each case the first term of the expansion is the first term of the binomial raised to the given power, and the last term of the expansion is the second term of the binomial raised to the given power. Thus, in the expansion of $(x + a)^4$ the first term is x^4 and the last term is a^4, and so for all the rest.

II. The quantity a does not enter into the first term of the expansion, but appears in the second term with the exponent unity. The powers of x decrease by unity, and the powers of a increase by unity in each successive term. Thus, in the expansion of $(x + a)^6$ we have, x^6, x^5a, x^4a^2, x^3a^3, x^2a^4, xa^5, a^6.

III. The coefficient of the first term is unity, and the coefficient of the second term is in every case the exponent of the power to which the binomial is to be

raised. Thus the coefficient of the second term of $(x+a)^2$ is 2, of $(x+a)^6$ is 6, of $(x+a)^7$ is 7.

IV. If the coefficient in any term be multiplied by the index of x in that term and divided by the number of terms up to the given place, the resulting quotient will be the coefficient of the succeeding term. Thus in the expansion of $(x+a)^4$ the coefficient of the second term is 4; this multiplied by 3, the index of x in that term, gives 12, which when divided by 2 the number of terms up to the given place gives 6, the coefficient of the third term. Again, 6 the coefficient of the third term multiplied by 2, the exponent of x in that term, gives 12, which, when divided by 3, the number of terms up to the given place, gives 4, the coefficient of the 4th term. So also 35, the coefficient of the 5th term in the expansion of $(x+a)^7$, when multiplied by 3, the index of x in that term, gives 105, which, when divided by 5, the number of terms up to the given place, gives 21, the coefficient of the succeeding term.

By attending to the above observations, we can always raise a binomial of the form $(x+a)$ to any required power, without the process of actual multiplication.

Example I.

Raise $x+a$ to the power of 9.

The first term is		$x^9 a^0$
— second —		$9x^8 a$
— third —	$\frac{9 \times 8}{2} x^7 a^2 =$	$36 x^7 a^2$
— fourth —	$\frac{36 \times 7}{3} x^6 a^3 =$	$84 x^6 a^3$
— fifth —	$\frac{84 \times 6}{4} x^5 a^4 =$	$126 x^5 a^4$
— sixth —	$\frac{126 \times 5}{5} x^4 a^5 =$	$126 x^4 a^5$
— seventh —	$\frac{126 \times 4}{6} x^3 a^6 =$	$84 x^3 a^6$
— eighth —	$\frac{84 \times 3}{7} x^2 a^7 =$	$36 x^2 a^7$
— ninth —	$\frac{36 \times 2}{8} x^1 a^8 =$	$9 x^1 a^8$
— tenth —	$\frac{9 \times 1}{9} x^0 a^9 =$	$x^0 a^9$.

Hence,

$$(x+a)^9 = x^9 + 9x^8 a + 36 x^7 a^2 + 84 x^6 a^3 + 126 x^5 a^4 + 126 x^4 a^5 + 84 x^3 a^6 + 36 x^2 a^7 + 9 x a^8 + a^9.$$

Example II.

In like manner,

$$(x-a)^{10} = x^{10} - 10 x^9 a + 45 x^8 a^2 - 120 x^7 a^3 + 210 x^6 a^4 - 252 x^5 a^5 + 210 x^4 a^6 - 120 x^3 a^7 + 45 x^2 a^8 - 10 x a^9 + a^{10}.$$

108. The labour of determining the coefficients may be much abridged by attending to the following additional considerations:

V. The number of terms in the expanded binomial is always greater by unity than the index of the binomial. Thus the number of terms in $(x + a)^4$ is $4 + 1$, or 5, in $(x + a)^{10}$ is $10 + 1$, or 11.

VI. Hence, when the exponent is an even number, the number of terms in the expansion will be odd, and it will be observed, on examining the examples already given, that after we pass the middle term the coefficients are repeated in a reverse order; thus,

The coefficients of $(x + a)^4$ are 1, 4, $\underline{6}$, 4, 1.
—— $(x + a)^6$ — 1, 6, 15, $\underline{20}$, 15, 6, 1.
—— $(x + a)^8$ — 1, 8, 28, 56, $\underline{70}$, 56, 28, 8, 1.

VII. When the exponent is an odd number, the number of terms in the expansion will be even, and there will be two middle terms, or two contiguous terms, each of which is equally distant from the corresponding extremities of the series; in this case the coefficient of the two middle terms is the same, and then the coefficients of the preceding terms are reproduced in a reverse order; thus,

The coefficients of $(x + a)^3$ are 1, 3, 3, 1.
—— $(x + a)^5$ — 1, 5, $\underline{10}$, $\underline{10}$, 5, 1
—— $(x + a)^7$ — 1, 7, 21, $\underline{35}$, $\underline{35}$, 21, 7, 1.
—— $(x + a)^9$ — 1, 9, 36, 84, $\underline{126}$, $\underline{126}$, 84, 36, 9, 1.

109. If the terms of the given binomial be affected with coefficients or exponents, they must be raised to the required powers, according to the principles already established for the involution of monomials; thus:

Example III.

Raise $(2x^3 + 5a^2)$ to the power of 4.

The first term will be .... $(2x^3)^4 = 16x^{12}$

— second —— .... $4(2x^3)^3 \times (5a^2) = 4 \times 8 \times 5x^9a^2$

— third —— .... $\frac{4 \times 3}{2} \times (2x^3)^2 \times (5a^2)^2 = 6 \times 4 \times 25x^6a^4$

— fourth —— .... $\frac{6 \times 2}{3}(2x^3)^1 \times (5a^2)^3 = 4 \times 2 \times 125x^3a^6$

— fifth —— .... $\frac{4}{4}(2x^3)^0 \times (5a^2)^4 = 625a^8$

$$\therefore\ (2x^3+5a^2)^4 = 16x^{12} + 160x^9a^2 + 600x^6a^4 + 1000x^3a^6 + 625a^8$$

Example IV.

In like manner,

$$\begin{aligned}(a^3+3ab)^9 = {} & (a^3)^9 + 9(a^3)^8 \times (3ab) + 36(a^3)^7 \times (3ab)^2 + 84(a^3)^6 \times (3ab)^3 \\ & + 126(a^3)^5 \times (3ab)^4 + 126(a^3)^4 \times (3ab)^5 + 84(a^3)^3 \times (3ab)^6 \\ & + 36(a^3)^2 \times (3ab)^7 + 9a^3 \times (3ab)^8 + (3ab)^9 \\ = {} & a^{27} + 27a^{25}b + 324a^{23}b^2 + 2268a^{21}b^3 + 10206a^{19}b^4 + 30618a^{17}b^5 \\ & + 61236a^{15}b^6 + 78732a^{13}b^7 + 59049a^{11}b^8 + 19683a^9b^9\end{aligned}$$

110. We shall now proceed to exhibit the binomial theorem in a general form. Let it be required to raise any binomial $(x + a)$ to the power represented by

the general algebraic symbol n. Then by the preceding principles we shall have,

The first term	....	x^n
— second —	....	$nx^{n-1}a$
— third —	....	$\frac{n(n-1)}{1.2}x^{n-2}a^2$
— fourth —	....	$\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3$
— fifth —	....	$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^{n-4}a^4$
&c.	....	&c.
— last —	....	a^n

The whole number of terms will be $n+1$, and the coefficients be repeated in a reverse order after the $(\frac{n+1}{2})^{\text{th}}$, or $(\frac{n}{2}+1)^{\text{th}}$ term, according as n is odd or even; moreover, the terms will all have the sign $+$, if the quantity to be expanded be of the form of $x+a$, and they will have the sign $+$ and $-$ alternately, if the quantity be of the form $x-a$. Hence generally,

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{1\,.\,2}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}a^3 + \ldots\ldots$$
$$+\ldots\ldots\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3a^{n-3} + \frac{n(n-1)}{1\,.\,2}x^2a^{n-2} + nxa^{n-1} + a^n$$

$$(x-a)^n = x^n - nx^{n-1}a + \frac{n(n-1)}{1\,.\,2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \pm a^n$$

In this last case, if n be an even number, the last term, being one of the odd terms, will have the sign $+$; and if n be an odd number, the last term, being one of the even terms, will have the sign $-$.

Both forms may be included in one, by employing the double sign; thus,

$$(x\pm a)^n = x^n \pm nx^{n-1}a + \frac{n(n-1)}{1\,.\,2}x^{n-2}a^2 \pm \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}a^3 + \ldots\ldots \text{ \&c.}$$

Example V.

To exemplify the application of the theorem in this form, let it be required to raise $x+a$ to the power of 5.

Here we have $n = 5$, $n-1 = 4$, $n-2 = 3$, &c.

Hence,

x^n	is	x^5	$= x^5$
$nx^{n-1}a$	—	$5x^4a$	$= 5x^4a$
$\frac{n(n-1)}{1\,.\,2}x^{n-2}a^2$	—	$\frac{5\,.\,4}{1\,.\,2}x^3a^2$	$= 10x^3a^2$
$\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}a^3$	—	$\frac{5\,.\,4\,.\,3}{1\,.\,2\,.\,3}x^2a^3$	$= 10x^2a^3$
$\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}x^{n-4}a^4$	—	$\frac{5\,.\,4\,.\,3\,.\,2}{1\,.\,2\,.\,3\,.\,4}xa^4$	$= 5xa^4$
$\frac{n(n-1)(n-2)(n-3)(n-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5}x^0a^5$	—	$\frac{5\,.\,4\,.\,3\,.\,2\,.\,1}{1\,.\,2\,.\,3\,.\,4\,.\,5}x^0a^5$	$= x^0a^5$

$$(x+a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$$

Example VI.

Raise $5c^2 - 2yz$ to the power of 4.

Hence,

$$\left.\begin{array}{l} x = 5c^2 \\ a = 2yz \\ n = 4 \end{array}\right\} \begin{array}{lll} x^n \ldots\ldots\ldots\ldots \text{ becomes} & (5c^2)^4 & = 625\,c^8 \\ nx^{n-1}a \ldots\ldots\ldots\ldots & 4\,(5c^2)^3 \times (2yz) & = 1000\,c^6yz \\ \frac{n(n-1)}{1\,.\,2}\,x^{n-2}a^2 \ldots\ldots & \frac{4\,.\,3}{1\,.\,2}\,(5c^2)^2 \times (2yz)^2 & = 600\,c^4y^2z^2 \\ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,x^{n-3}a^3 & \frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}(5c^2)^1 \times (2yz)^3 & = 160\,c^2y^3z^3 \\ \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\,x^{n-4}a^4 & \frac{4.3.2.1}{1.2.3.4}(5c^2)^0 \times (2yz)^4 & = 16\,y^4z^4 \end{array}$$

$$\therefore\ (5c^2 - 2yz)^4 = 625\,c^8 - 1000c^6yz + 600\,c^4y^2z^2 - 160\,c^2y^3z^3 + 16\,y^4z^4$$

111. We have sometimes occasion to employ a particular term in the expansion of a binomial, while the remainder of the series does not enter into our calculations. Our labour will, in a case like this, be much abridged, if we can at once determine the term sought, without reference either to those which precede, or to those which follow it. This object will be attained, by finding what is called the *general term* of the series.

If we examine the general formula, we shall soon perceive that a certain relation subsists between the coefficients and exponents of each term in the expanded binomial, and the place of the term in the series; thus,

The *first* term is x^n which may be put under the form x^{n-1+1}

... *second* ...		......	$nx^{n-2+1}\,a^{2-1}$
... *third* ...	$\frac{n(n-1)}{1\,.\,2}\,x^{n-2}\,a^2$	......	$\frac{n(n-3+2)}{1\,.\,(3-1)}\,x^{n-3+1}\,a^{3-1}$
... *fourth* ...	$\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,x^{n-3}\,a^3$	......	$\frac{n(n-1)(n-4+2)}{1\,.\,2\,.\,(4-1)}\,x^{n-4+1}\,a^{4-1}$
... *fifth* ...	$\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\,x^{n-4}\,a^4$	......	$\frac{n(n-1)(n-2)(n-5+2)}{1\,.\,2\,.\,3\,.\,(5-1)}\,x^{n-5+1}\,a^{5-1}$
... *sixth* ...	$\frac{n(n-1)(n-2)(n-3)(n-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\,x^{n-5}\,a^5$	......	$\frac{n(n-1)(n-2)(n-3)(n-6+2)}{1\,.\,2\,.\,3\,.\,4\,.\,(6-1)}\,x^{n-6+1}\,a^{6-1}$

Observing the connection between the numerical quantities, it is manifest, that if we designate the place of any term by the general symbol p, the p^{th} term is,

$$\frac{n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots(n-p+2)}{1\,.\,2\,.\,3\,.\,4\quad\ldots\ldots\ldots\ldots\ (p-1)}\,x^{n-p+1}a^{p-1}$$

This is called the *general term*, because by giving to p the values 1, 2, 3, 4, ... we can obtain in succession the different terms of the series for $(x+a)^n$.

Example VII.

Required the 7^{th} term of the expansion of $(x+a)^{12}$.

Here $\left.\begin{array}{l} n = 12 \\ p = 7 \end{array}\right\}\ \therefore\ \begin{array}{l} n - p + 2 = 7, \\ p - 1 = 6, \end{array}\quad n - p + 1 = 6$

Substituting these values in the general expression, we find that the term sought is,

$$\frac{12 \,.\, 11 \,.\, 10 \,.\, 9 \,.\, 8 \,.\, 7}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6} x^6 a^6, \quad \text{or} \quad 924 x^6 a^6.$$

Example VIII.

Required the 5^{th} term of $(2c^4 - 4h^5)^9$.

Here $n = 9, \quad p = 5, \quad x = 2c^4, \quad a = 4h^5$

$\therefore \; n - p + 2 = 6, \quad n - p + 1 = 5, \quad p - 1 = 4$

$\therefore$ the 5^{th} term is $\frac{9 \,.\, 8 \,.\, 7 \,.\, 6}{1 \,.\, 2 \,.\, 3 \,.\, 4}(2c^4)^5 \times (4h^5)^4, \quad$ or $126 \times 32 \times 256\, c^{20} h^{20}$.

Since the second term of the proposed binomial has the sign —, all the even terms of the expansion will have the sign —, and all the odd terms the sign +; therefore the 5^{th} term is,

$$+ 1032192\, c^{20} h^{20}$$

Example IX.

Required the middle term of the expansion of $(x - a)^{18}$.

Since the exponent is 18, the whole number of terms will be 19, and hence the middle term will be the 10^{th}; and since it is an even term, it will have the sign —; hence it will be,

$$-\frac{18 \,.\, 17 \,.\, 16 \,.\, 15 \,.\, 14 \,.\, 13 \,.\, 12 \,.\, 11 \,.\, 10}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7 \,.\, 8 \,.\, 9} x^9 a^9 \quad \text{or} \quad -48620\, x^9 a^9$$

112. By employing the Binomial Theorem, we can raise any polynomial to any power, without the process of actual multiplication.

For example, let it be required to raise $x+a+b$ to the power of 5.

Put

$a+b \quad = y$

Then,

$$\begin{aligned}(x+a+b)^4 &= (x+y)^4, \\ &= x^4 + 4x^3y + 6x^2y^2 + xy^3 + y^4, \text{ putting for } y \text{ its value,} \\ &= x^4 + 4x^3(a+b) + 6x^2(a+b)^2 + 4x(a+b)^3 + (a+b)^4\end{aligned}$$

Expanding $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, by the Binomial Theorem, and performing the multiplications indicated, we shall arrive at the expansion of $(x+a+b)^4$.

It is manifest, that we may apply a similar process to any polynomial.

113. In the observations made upon the expansion of $(x+a)^n$, we have supposed n to be a positive integer. The binomial theorem, however, is applicable, whatever may be the nature of the quantity n, whether it be positive or negative, integral or fractional. * When n is a positive integer, the series consists of $n + 1$ terms; in every other case the series never terminates, and the developement of $(x + a)^n$ constitutes what is called *an infinite series.*

Before proceeding to consider this extension of the theorem, we may remark that in all our reasonings with regard to a quantity, such as $(x + a)^n$, we may

* No algebraist has succeeded in proving this in a manner altogether satisfactory. The least exceptionable of the demonstrations which have been proposed, will be given in a subsequent chapter.

confine our attention to the more simple form $(1+a)^n$ to which the former may always be reduced. For,

$$(x+a)=x\left(1+\frac{a}{x}\right)$$

$$\therefore (x+a)^n=\left\{x\left(1+\frac{a}{x}\right)\right\}^n$$

$$=x^n\left(1+\frac{a}{x}\right)^n, \text{ or } x^n(1+u)^n \quad \text{if we put } u=\frac{a}{x}$$

$$=x^n\left\{1+n.\frac{a}{x}+\frac{n(n-1)}{1.\ 2}\cdot\frac{a^2}{x^2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{a^3}{x^3}+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cdot\frac{a^4}{x^4}+\&c.\right\}$$

Suppose $n=\frac{r}{s}$, where r and s are any whole numbers whatever,

Then $(x+a)^n$ becomes $(x+a)^{\frac{r}{s}}$, and substituting $\frac{r}{s}$ for n in the series.

$$(x+a)^{\frac{r}{s}}=x^{\frac{r}{s}}\left(1+\frac{a}{x}\right)^{\frac{r}{s}}$$

$$=x^{\frac{r}{s}}\left(1+\frac{r}{s}\cdot\frac{a}{x}+\frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1.2}\cdot\frac{a^2}{x^2}+\frac{\frac{r}{s}\left(\frac{r}{s}-1\right)\left(\frac{r}{s}-2\right)}{1.\ 2.\ 3}\cdot\frac{a^3}{x^3}+\frac{\frac{r}{s}\left(\frac{r}{s}-1\right)\left(\frac{r}{s}-2\right)\left(\frac{r}{s}-3\right)}{1.2.3.4}\cdot\frac{a^4}{x^4}+\&c.\right\}$$

Or reduced,

$$=x^{\frac{r}{s}}\left(1+\frac{r}{s}\cdot\frac{a}{x}+\frac{r(r-s)}{1.2.s^2}\cdot\frac{a^2}{x^2}+\frac{r(r-s)(r-2s)}{1.2.3.s^3}\cdot\frac{a^3}{x^3}+\frac{r(r-s)(r-2s)(r-3s)}{1.2.3.4\ s^4}\cdot\frac{a^4}{x^4}+\&c.\right\}$$

114. The binomial theorem, under this form, is extensively employed in analysis for developing algebraic expressions in series.

Example I.

Expand $\sqrt{x+a}$ in a series.

$$\sqrt{x+a}=(x+a)^{\frac{1}{2}}$$

$$=x^{\frac{1}{2}}\left(1+\frac{a}{x}\right)^{\frac{1}{2}} \quad \text{Here } r=1,\ s=2.$$

$$=x^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{a}{x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}\cdot\frac{a^2}{x^2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}\cdot\frac{a^3}{x^3}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}\cdot\frac{a^4}{x^4}+\ldots\ldots\ldots\ldots\right\}$$

$$=x^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{a}{x}+\frac{\frac{1}{2}\times-\frac{1}{2}}{1.2}\cdot\frac{a^2}{x^2}+\frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{1.2.3.}\cdot\frac{a^3}{x^3}\right.$$

$$\left.+\frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}\times-\frac{5}{2}}{1.2.3.4}\cdot\frac{a^4}{x^4}+\ldots\ldots\ldots\ldots\right\}$$

$$=x^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{a}{x}-\frac{1}{1.2.4}\cdot\frac{a^2}{x^2}+\frac{1.3}{1.2.3.8}\cdot\frac{a^3}{x^3}-\frac{1.3.5}{1.2.3.4.16}\right.$$

$$\left.\frac{a^4}{x^4}+\ldots\ldots\ldots\ldots\ldots\ldots\right\}$$

which may be put under the form

$$=x^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{a}{x}-\frac{1}{2.4}\cdot\frac{a^2}{x^2}+\frac{1.3}{2.4.6}\cdot\frac{a^3}{x^3}-\frac{1.3.5}{2.4.6.8}\cdot\frac{a^4}{x^4}\right.$$

$$\left.+\frac{1.3.5.7}{2.4.6.8.10}\cdot\frac{a^5}{x^5}-\&c.\ \ldots\ldots\ldots\ldots\ldots\right\}$$

where the law of the series is evident.

Example II.

Expand $\sqrt{a^2-a^2e^2}$ in a series

$$\sqrt{a^2-a^2e^2}=(a^2-a^2e^2)^{\frac{1}{2}}$$

$$=a(1-e^2)^{\frac{1}{2}}\quad\text{Here } r=1,\quad s=2,\quad\frac{a}{x}=-e^2$$

$$=a\left\{1-\frac{1}{2}\cdot e^2+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}\cdot e^4-\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2\ 3}\cdot e^6\right.$$

$$\left.+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{1\ .\ 2\ .\ 3\ .\ 4}\ e^8-\&c.\ \ldots\ldots\ldots\right\}$$

$$=a\left\{1-\frac{1}{2}e^2-\frac{1}{2.4}e^4-\frac{1.3}{2.4.6}e^6-\frac{1.3.5}{2.4.6.8}e^8-\&c.\right\}$$

Example III.

Expand $\frac{m}{\sqrt{b^2+c^4}}$ in a series.

$$\frac{m}{\sqrt{b^2+c^4}}\ =\ m\,(b^2+c^4)^{-\frac{1}{2}}$$

$$=\ m\,b^{-1}\left(1+\frac{c^4}{b^2}\right)^{-\frac{1}{2}}.\ \text{Here } r=-1, s=2, \frac{a}{x}=\frac{c^4}{b^2}$$

$$=\ \frac{m}{b}\ \left\{1-\frac{1}{2}\ .\ \frac{c^4}{b^2}+\frac{-\frac{1}{2}\ \left(-\frac{1}{2}-1\right)}{1\ .\ 2}\cdot\frac{c^8}{b^4}\right.$$

$$+\frac{-\frac{1}{2}\ \left(-\frac{1}{2}-1\right)\ \left(-\frac{1}{2}-2\right)}{1\ .\ 2\ .\ 3}\ .\ \frac{c^{12}}{b^6}$$

$$+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)}{1 . 2 . 3 . 4}$$

$$\times\left(\frac{c^4}{b^2}\right)^4 \&c. \ldots\ldots \Big\}$$

$$= \frac{m}{b}\Big\{1-\frac{1}{2}\cdot\frac{c^4}{b^2}+\frac{-\frac{1}{2}\times-\frac{3}{2}}{1 . 2}\cdot\frac{c^8}{b^4}+\frac{-\frac{1}{2}\times-\frac{3}{2}\times-\frac{5}{2}}{1 . 2 . 3}$$

$$\cdot\frac{c^{12}}{b^6}+\frac{-\frac{1}{2}\times-\frac{3}{2}\times-\frac{5}{2}\times-\frac{7}{2}}{1 . 2 . 3 . 4}\frac{c^{16}}{b^8}+ \&c.\ldots\ldots\Big\}$$

$$= \frac{m}{b}\Big\{1-\frac{1}{2}\cdot\frac{c^4}{b^2}+\frac{1 . 3}{2 . 4}\cdot\frac{c^8}{b^4}-\frac{1 . 3 . 5}{2 . 4 . 6}\cdot\frac{c^{12}}{b^6}+\frac{1 . 3 . 5 . 7}{2 . 4 . 6 . 8}$$

$$\cdot\frac{c^{16}}{b^8}- \&c. \ldots\ldots\Big\}$$

Example IV.

Expand $\dfrac{n}{\sqrt{b^2-c^2e^2}}$ in a series.

$$\frac{n}{\sqrt{b^2-c^2e^2}} = n\,(b^2-c^2e^2)^{-\frac{1}{2}}$$

$$= n\,b^{-1}\left(1-\frac{c^2e^2}{b^2}\right)^{-\frac{1}{2}}.\ \text{Here, } r=-1,\ s=2,\ \frac{a}{x}=-\frac{c^2e^2}{b^2}$$

$$= \frac{n}{b}\Big\{1+\frac{1}{2}\cdot\frac{c^2e^2}{b^2}+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{1 . 2}\cdot\left(\frac{c^2e^2}{b^2}\right)^2$$

$$-\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1 . 2 . 3}\cdot\left(\frac{c^2e^2}{b^2}\right)^3$$

$$+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}-3\right)}{1 . 2 . 3 . 4}\left(\frac{c^2e^2}{b^2}\right)^4-\&c.\Big\}$$

$$= \frac{n}{b}\Big\{1+\frac{1}{2}\cdot\frac{c^2e^2}{b^2}+\frac{1 . 3}{2 . 4}\cdot\frac{c^4e^4}{b^4}+\frac{1 . 3 . 5}{2 . 4 . 6}\cdot\frac{c^6e^6}{b^6}$$

$$+\frac{1 . 3 . 5 . 7}{2 . 4 . 6 . 8}\cdot\frac{c^8e^8}{b^8}+\&c. \ldots\ldots\Big\}$$

Example V.

Expand $\dfrac{p+q}{\sqrt[4]{(m^3+n^5)^3}}$ in a series.

$$\frac{p+q}{\sqrt[4]{(m^3+n^5)^3}} = (p+q)(m^3+n^5)^{-\frac{3}{4}}$$

$$= m^{-\frac{9}{4}}(p+q)\left(1+\frac{n^5}{m^3}\right)^{-\frac{3}{4}}.\ \text{Here, } r=-3,\ s=4,$$

$$\frac{a}{x} = \frac{n^5}{m^3}$$

$$= \frac{(p+q)}{m^{\frac{9}{4}}}\left\{1-\frac{3}{4}\cdot\frac{n^5}{m^3}+\frac{-\frac{3}{4}\left(-\frac{3}{4}-1\right)}{1.2}\cdot\left(\frac{n^5}{m^3}\right)^2\right.$$

$$\left.+\frac{-\frac{3}{4}\left(-\frac{3}{4}-1\right)\left(-\frac{3}{4}-2\right)}{1.2.3}\cdot\left(\frac{n^5}{m^3}\right)^3 \ldots\ldots\ldots\ldots\right\}$$

$$= \frac{(p+q)}{m^{\frac{9}{4}}}\left\{1-\frac{3}{4}\cdot\frac{n^5}{m^3}+\frac{3.7}{1.2.4^2}\frac{n^{10}}{m^6}-\frac{3.7.11}{1.2.3.4^3}\right.$$

$$\left.\cdot\frac{n^{15}}{m^9}+\frac{3.7.11.15}{1.2.3.4.4^4}\frac{n^{20}}{m^{12}}-\&c.\ldots\ldots\ldots\ldots\right\}$$

Ex. 6. $\frac{1}{(c+x)^2} = (c+x)^{-2} = \frac{1}{c^2}\left\{1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+\&c.\right\}$

Ex. 7. $(c^2-x^2)^{\frac{3}{2}} = c^{\frac{3}{2}}\left\{1-\frac{3}{2^2}\cdot\frac{x^2}{c^2}-\frac{3}{2^5}\cdot\frac{x^4}{c^4}-\frac{5}{2^7}\cdot\frac{x^6}{c^6}-\frac{7}{2^9}\cdot\frac{x^8}{c^8}-\&c.\right\}$

Ex. 8. $(a^2-ax)^{-\frac{3}{10}} = \frac{1}{a^{\frac{3}{5}}}\left\{1+\frac{3}{10}\cdot\frac{x}{a}+\frac{3.13}{10^2}\cdot\frac{x^2}{1.2.a^2}+\frac{3.13.23}{10^3}\right.$

$$\left.\cdot\frac{x^3}{1.2.3.a^3}+\frac{3.13.23.33}{10^4}\,\frac{x^4}{1.2.3.4.a^4}+\&c.\right\}$$

Ex. 9. $\frac{1}{(1+x)^{\frac{1}{5}}} = 1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11.x^3}{5.10.15}+\frac{6.11.16.x^4}{5.10.15.20}-\&c.\}$

Ex. 10. The 11th term of the series for $(a^3-x^3)^{\frac{7}{3}}$ is $= -\frac{2618}{4782969}\cdot\frac{x^{30}}{a^{23}}$.

115. The binomial theorem is also employed to determine approximate values of the roots of numbers.

In the formula,

$$(x+a)^n = x^n\left(1+n.\frac{a}{x}+\frac{n(n-1)}{1.2}.\frac{a^2}{x^2}+\frac{n(n-1)(n-2)}{1.2.3}.\frac{a^3}{x^3}+\cdots\right)$$

Let us put $n = \frac{1}{r}$, the expression becomes $(x+a)^{\frac{1}{r}}$ or $\sqrt[r]{x+a}$, and we have

$$\sqrt[r]{x+a} = \sqrt[r]{x}\left(1+\frac{1}{r}\cdot\frac{a}{x}+\frac{\frac{1}{r}\left(\frac{1}{r}-1\right)}{1.2}\cdot\frac{a^2}{x^2}+\frac{\frac{1}{r}\left(\frac{1}{r}-1\right)\left(\frac{1}{r}-2\right)}{1.2.3}\right.$$

$$\left.\cdot\frac{a^3}{x^3}+\ldots\ldots\ldots\ldots\ldots\ldots\right)$$

$$= \sqrt[r]{x}\left(1+\frac{1}{r}\cdot\frac{a}{x}-\frac{1}{r}\cdot\frac{r-1}{2r}\cdot\frac{a^2}{x^2}+\frac{1}{r}\cdot\frac{r-1}{2r}\cdot\frac{2r-1}{3r}\cdot\frac{a^3}{x^3}-\&c.\right)$$

If we wished to form a new term, it would manifestly be obtained by multiplying the fourth by $\frac{3r-1}{4r}$ and $\frac{a}{x}$, then changing the sign, and so on for the rest, the terms after the first being alternately positive and negative.

This being premised, let it be required to extract the cube root of 31. The greatest cube contained in 31 is 27; in the above formula let us make $r = 3$, $x = 27$, $a = 4$, and we shall then have

$$\begin{aligned}
\sqrt[3]{31} &= \sqrt[3]{27+4} \\
&= \sqrt[3]{27}\left(1+\frac{4}{27}\right)^{\frac{1}{3}} \\
&= 3\left(1+\frac{1}{3}\cdot\frac{4}{27}-\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{16}{729}+\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{5}{9}\cdot\frac{64}{19683}\right.-\text{ \&c.} \\
&= 3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}-
\end{aligned}$$

The succeeding term will be found by multiplying $\frac{320}{531441}$ by $\frac{3r-1}{4r}\cdot\frac{a}{x}$ or $\frac{2}{3}\cdot\frac{4}{27}$, and then changing the sign, which will give us $-\frac{2560}{43046721}$.

In like manner, we shall find the next term by multiplying $\frac{2560}{43046721}$ by $\frac{4r-1}{5r}\cdot\frac{a}{x}$, it will therefore be $\frac{2560}{43046721}\times\frac{11}{15}\times\frac{4}{27}=\frac{112640}{17433922005}$ and so on for any number of terms.

Let us however confine our attention to the first five terms of the series, and reduce them to decimals, we shall have, for the sum of the additive terms,

$$\left\{\begin{aligned} 3 &= 3.00000 \\ \frac{4}{27} &= 0.14815 \\ \frac{320}{531441} &= 0.00060 \end{aligned}\right\} = 3.14875$$

And for the sum of the subtractive terms,

$$\left\{\begin{aligned} -\frac{16}{2187} &= -0.00731 \\ -\frac{2560}{43046721} &= -0.00006 \end{aligned}\right\} = -0.00737$$

Hence,

$$\sqrt[3]{31} = 3.14138 \ldots\ldots\ldots\ldots$$

a result which we shall proceed to show is within 0.00001 of the truth.

116. When the expression for a number is expanded in a series of terms, the numerical value of which go on decreasing continually, we easily perceive that the greater the number of terms which we take, the more nearly shall we approach to the real value of the proposed expression. But if, in addition to this, we suppose that the terms of the series are alternately positive and negative, we can, upon stopping at any particular term, determine precisely the degree of approximation at which we have arrived.

Let there be a series $a-b+c-d+e-f+g-h+k-l+m- - - -$ composed of an indefinite number of terms, in which we suppose that the quantities $a, b, c, d,$ go on diminishing in succession, and let us designate by N the number repre-

sented by this series, we shall prove that, *the numerical value of* N *lies between any two consecutive sums of any number of the terms of the above series.*

For let us take any two consecutive sums,

$$a - b + c - d + e - f, \text{ and, } a - b + c - d + e - f + g$$

Upon considering the first of these, we perceive that the terms which follow $-f$ are, $+ (g - h) + (k - l) + \text{-----}$; but since the series is a decreasing one, the positive differences $g - h$, $k - l$, &c. are all positive numbers, hence it follows, that, in order to obtain the complete value of N, we must add to the sum $a - b + c - d + e - f$ some positive number. Hence,

$$a - b + c - d + e - f < \text{N}$$

With regard to the second sum, the terms which follow $+ g$ are, $-(h - k)$, $-(l - m)$, ----; but the partial differences, $h - k$, $l - m$, &c. are positive, hence, $-(h - k)$, $-(l - m)$, ---- are all negative, and therefore, in order to obtain the complete value of N, we must subtract some positive number from the sum $a - b + c - d + e - f + g$. Hence,

$$a - b + c - d + e - f + g > \text{N}$$

and it has been shown that

$$a - b + c - d + e - f \quad < \text{N}$$

therefore N lies between these two sums.

From this it follows, that, since g is the numerical value of the difference of these two sums, *the error committed when we assume a certain number of terms* $a - b + c - d + e - f$ *for the value of* N, *is numerically less than the term which immediately follows that at which we stopped.*

In the preceding example, all the terms after the first being alternately positive and negative, we may conclude that the numerical value of the first five terms

$$3 + \frac{4}{27} - \frac{16}{2187} + \frac{320}{531441} - \frac{2560}{43046721}$$

differs from the true value of $\sqrt[3]{31}$, by a quantity less than the value of the sixth term, which was found to be equal to $\frac{112640}{17433922005}$, but this fraction is by mere inspection less than $\frac{1}{100000}$, therefore when we assume that $\sqrt[3]{31} = 3.14138$, the result is within 0.00001 of the truth.

117. From what has been said above it will be seen, that, in order to obtain an approximate value of the n^{th} root of any number N by the method of series, we may make use of the following

RULE.

Resolve the given number N *into two parts of the form* $p^n + q$, *where* p^n *is the highest* n^{th} *power contained in* N, *and in the developement of* $(x + a)^{\frac{1}{n}}$ *make* $x = p^n$, $a = q$. *The number of terms to be taken in the resulting series will*

depend upon the degree of accuracy required, and can be determined by the principle just explained. Convert all the terms of which account is taken into decimals, and then effect the reduction between the additive and subtractive terms.

This method cannot be employed with advantage except when $\frac{q}{p^n}$ is a small fraction; for unless this be the case the terms of the series will not diminish with sufficient rapidity, and it will be necessary to take account of a great number of terms in order to arrive at a near approximation.

It may happen that p^n is $\angle q$, we must then modify the above process, for then $\frac{p^n}{q}$ or $\frac{a}{x}$ is greater than unity, and therefore all the powers of $\frac{a}{x}$ will increase in numerical value as the degree of the power increases.

Suppose, for example, that the cube root of 56 is sought, 27 being the greatest cube contained in 56, we shall have

$$x = 27, \quad a = 29 \quad \text{and} \therefore \frac{a}{x} = \frac{29}{27},$$

and the terms of the series will go on increasing instead of diminishing, (we do not speak of the coefficients, which are fractions differing but little from unity). But we may resolve 56 into $64 - 8$, or, $4^3 - 8$; but $\frac{8}{64}$, or, $\frac{1}{8}$ is a small fraction. On the other hand, if we substitute $-a$ for a in the expression for $\sqrt[n]{x+a}$ we have

$$\sqrt[n]{x-a} = x^{\frac{1}{n}}\left(1 - \frac{1}{n}\cdot\frac{a}{x} - \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{a^2}{x^2} - \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a}{x^3} - - -\right.$$

If we put $x = 64$, $a = 8$, we shall obtain a series of terms which will decrease with great rapidity.

Here all the terms, with the exception of the first, are negative, and we cannot apply to this series the criterion established in Art. (116.) for fixing the degree of approximation. But we shall approach very nearly to the required degree of approximation if we take into account such a number of terms that the first which we neglect shall be less, by one tenth, for example, than the decimal place to which we wish to limit the approximation.

The student may take the following examples as exercises:

Ex. 1.	$\sqrt[5]{39}$	$=$	$\sqrt[5]{32+7}$	$=$	2.0807.... true to	0.0001.
2.	$\sqrt[3]{65}$	$=$	$\sqrt[3]{64+1}$	$=$	4.02073... ——	0.00001.
3.	$\sqrt[4]{260}$	$=$	$\sqrt[4]{256+4}$	$=$	4.01553... ——	0.00001.
4.	$\sqrt[7]{108}$	$=$	$\sqrt[7]{128-20}$	$=$	1.95204... ——	0.00001.

RATIOS AND PROPORTION.

118. Numbers may be compared in two ways.

When it is required to determine by how much one number is greater or less than another, the answer to this question consists in stating the *difference* between these two numbers. This difference is called the *Arithmetical Ratio* of the two numbers. Thus, the arithmetical ratio of 9 to 7 is $9 - 7$ or 2, and if a, b designate two numbers, their arithmetical ratio is represented by $a - b$.

When it is required to determine how many times one number contains, or, is contained in, another, the answer to this question consists in stating the *quotient* which arises from dividing one of these numbers by the other. This quotient is called the *Geometrical Ratio* of the two numbers. The term *Ratio*, when used without any qualification, is always understood to signify a geometrical ratio, and we shall, at present, confine our attention to ratios of this description.

119. By the *ratio* of two numbers, then, we mean the *quotient* which arises from dividing one of these numbers by the other. Thus the ratio of 12 to 4 is represented by $\frac{12}{4}$ or 3, the ratio of 5 to 2 is $\frac{5}{2}$ or 2.5, the ratio of 1 to 3 is $\frac{1}{3}$ or .333... We here perceive that the value of a ratio cannot always be expressed exactly, but that, by taking a sufficient number of terms of the decimal, we can approach as nearly as we please to the true value. It may happen that one or both terms of the ratio can only be expressed in decimal fractions which do not terminate; thus, in the ratio of 1 to $\sqrt{2}$, and in the ratio of $\sqrt{3}$ to $\sqrt[3]{7}$, the quantities $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{7}$ can only be expressed in decimals which do not terminate, and therefore the values of the above ratios cannot be exactly expressed, although we can approach to them as nearly as we please.

120. If a, b, designate two numbers, the ratio of a to b is the quotient arising from dividing a by b, and will be represented by writing them $a : b$, or, $\frac{a}{b}$.

121. A ratio being thus expressed, the first term, or a, is called the *antecedent* of the ratio, the last term, or b, is called the *consequent* of the ratio.

122. It appears, therefore, that, in arithmetic and algebra, the theory of ratios becomes identified with the theory of fractions, and a ratio may be defined as a fraction whose numerator is the antecedent, and whose denominator is the consequent of the ratio.

123. When the antecedent of a ratio is greater than the consequent, the ratio is called *a ratio of greater inequality;* when the antecedent is less than the consequent it is called *a ratio of less inequality;* and when the antecedent and consequent are equal it is called *a ratio of equality.* Thus, $\frac{12}{4}$ is a ratio of greater inequality, $\frac{12}{144}$ is a ratio of less inequality, $\frac{3}{3}$ or 1 is a ratio of equality. It is manifest that a ratio of equality may always be represented by unity.

124. When the antecedents of two or more ratios are multiplied together to form a new antecedent, and their consequents multiplied together to form a new consequent, the several ratios are said to be *compounded,* and the resulting ratio is called the *sum* of the compounding ratios. Thus, the ratio $\frac{a}{b}$ is compounded with the ratio $\frac{c}{d}$, by multiplying the antecedents a, c, for a new antecedent, and the consequents b, d, for a new consequent, and the resulting ratio $\frac{a\,c}{b\,d}$ is called the sum of the ratios $\frac{a}{b}$ and $\frac{c}{d}$.

In like manner, the ratios of $\frac{m}{n}$, $\frac{p}{q}$, $\frac{r}{s}$, $\frac{t}{w}$, are compounded by multiplying all the antecedents together for a new antecedent, and all the consequents for

a new consequent, and the resulting ratio $\frac{m\,p\,r\,t}{n\,q\,s\,w}$ is called the sum of the ratios $\frac{m}{n}, \frac{p}{q}, \frac{r}{s}, \frac{t}{w}$.

125. When a ratio is compounded with itself the resulting ratio is called the *duplicate ratio*, or, *double ratio* of the primitive. Thus, if we compound the ratio $\frac{a}{b}$ with $\frac{a}{b}$, the resulting ratio $\frac{a^2}{b^2}$ is called the duplicate ratio of $\frac{a}{b}$.

Similarly, $\frac{a^3}{b^3}$ is called the *triplicate ratio* or *triple ratio* of $\frac{a}{b}$.

And generally, $\frac{a^n}{b^n}$ is called the sum of the ratio $\frac{a}{b}$ added n times together.

According to the same principle, the ratio $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ is called the *subduplicate ratio*, or, *half ratio* of $\frac{a}{b}$; for the duplicate ratio of $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ is $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} \times \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \frac{a}{b}$.

So also the ratio $\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$ is called the *subtriplicate ratio*, or, *one third of the ratio*, of $\frac{a}{b}$. For the triple of $\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$ is $\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}} \times \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}} \times \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}} = \frac{a}{b}$.

And in general, $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$ is called *one n^{th} of the ratio* $\frac{a}{b}$; for n times the ratio $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$ is $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \times \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \times \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \times \ldots$ to n terms $= \frac{a}{b}$.

NOTE. The ratio $\frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}$ is called the sesquiplicate ratio of $\frac{a}{b}$ for it is compounded of the simple and subduplicate ratio; thus, $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} \times \frac{a}{b} = \frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}$.

126. *If the terms of a ratio be both multiplied, or both divided, by the same quantity, the value of the ratio remains unchanged.*

The ratio of a to b is represented by the fraction $\frac{a}{b}$, and since the value of a fraction is not changed, if we multiply, or divide, both numerator and denominator by the same quantity, the truth of the proposition is evident. Thus,

$$\frac{a}{b} = \frac{m\,a}{m\,b} = \frac{\frac{a}{n}}{\frac{b}{n}}, \text{ or, } a : b = m\,a : m\,b = \frac{a}{n} : \frac{b}{n}.$$

127. *Ratios are compared with each other by reducing the fractions, by which they are represented, to a common denominator.*

If we wish to ascertain whether the ratio of 2 to 7 is greater or less than that of 3 to 8, since these ratios are represented by the fractions $\frac{2}{7}$ and $\frac{3}{8}$, which are equivalent to $\frac{16}{56}$ and $\frac{21}{56}$; and since the latter of these is greater than the former, it appears that the ratio of 2 to 7 is less than the ratio of 3 to 8.

128. *A ratio of greater inequality is diminished, and a ratio of a less quality is increased, by adding the same quantity to both terms.*

Let $\frac{a}{b}$ represent any ratio, and let x be added to each of its terms. The two ratios will then be

$$\frac{a}{b},\ \frac{a+x}{b+x}$$

which, reduced to a common denominator, become

$$\frac{ab+ax}{b(b+x)},\ \frac{ab+bx}{b(b+x)}.$$

If $a > b$, i. e. $\frac{a}{b}$, a ratio of greater inequality, then

$$\frac{ab+ax}{b(b+x)} > \frac{ab+bx}{b(b+x)}$$

and $\therefore \frac{a}{b}$ is diminished by the addition of the same quantity to each of its terms.

Again if $a < b$, i. e. $\frac{a}{b}$, a ratio of less inequality, then

$$\frac{ab+ax}{b(b+x)} < \frac{ab+bx}{b(b+x)},$$

and $\therefore \frac{a}{b}$ is increased by the addition of the same quantity to each of its terms.

129. *If there be any number of ratios in which the consequent of the first ratio is the antecedent of the second, and the consequent of the second the antecedent of the third, and so on, the sum of any number of said ratios is the ratio of the first antecedent to the last consequent.*

Let the proposed ratios be

$$\frac{a}{b},\ \frac{b}{c},\ \frac{c}{d},\ \frac{d}{e},\ \frac{e}{f},\ ----$$

Then by (Art. 124.) their sum is

$$\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e}\times\frac{e}{f}\ ----$$

Or,

$$\frac{a\,b\,c\,d\,e\,----}{b\,c\,d\,e\,f\,----}$$

$$\text{i. e. } \frac{a}{f}$$

130. Proportion consists in the equality of ratios.

Thus, if a, b, c, d be four quantities, such that a when divided by b gives the

same quotient as c when divided by d, then a, b, c, d, are said to be *proportionals;* the numbers 20, 5, 36, 9 are proportionals, for $\frac{20}{5} = 4$, and $\frac{36}{9} = 4$.

131. When four quantities are proportionals, it is usually enunciated by saying that *the first is to the second as the third is to the fourth.* Thus, if a, b, c, d are proportionals, we say that a is to b as c is to d, and this is expressed by writing them

$$a : b :: c : d$$

Or as fractions,

$$\frac{a}{b} = \frac{c}{d}$$

The former notation is usually employed in geometry, the latter in analytical investigations.

132. The expression $a:b::c:d$ or $\frac{a}{b} = \frac{c}{d}$ is called a proportion, and a, b, c, d, are severally called the *terms* of the proportion. The first and last are called the *extreme terms*, the second and third the *mean terms.* The first term is called the *first antecedent*, the second term the *first consequent*, the third term the *second antecedent*, and the fourth term the *second consequent.*

133. When the second and third terms of a proportion are identical, the quantity which forms these terms is called a *mean proportional* to the other two; thus, if we have three quantities a, b, c, such, that

$$a : b :: b : c \quad \text{or} \quad \frac{a}{b} = \frac{b}{c}$$

then b is said to be a mean proportional to a and c, and c is called a *third proportional* to a and b.

If, in a series of proportional magnitudes, each consequent be identical with the next antecedent, these quantities are said to be in *continued proportion;* thus, if we have a series of quantities, a, b, c, d, e, f, g, h, such that

$$a : b :: b : c :: c : d :: d : e :: e : f :: f : g :: g : h$$

Or,

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \frac{e}{f} = \frac{f}{g} = \frac{g}{h}$$

then the quantities a, b, c, d, e, f, g, h, are in continued proportion.

The following are the most important propositions connected with the subject of proportion.

I. *If four quantities be proportionals, the product of the extreme terms will be equal to the product of the mean terms.*

Let,

$$a : b :: c : d$$

Or,

$$\frac{a}{b} = \frac{c}{d}$$

Multiplying these equals by $b\,d$ the expression becomes

$$a\,d = b\,c$$

II. Conversely, *If the product of any two quantities be equal to the product of any other two, these four quantities will constitute a proportion, the terms of one of the products being the means, and the terms of the other the extremes.*

Let,

$$a\,d = b\,c$$

viding these equals by $b\,d$ the expression becomes

$$\frac{a}{b} = \frac{c}{d} \text{ or, } \frac{c}{d} = \frac{a}{b}$$

i. e. $a : b :: c : d$, or, $c : d :: a : b$

In the first, a and d are the extremes, and b and c the means, in the second b and c are the extremes, and a and d the means.

III. *If three quantities be in continued proportion, the product of the extreme terms is equal to the square of the mean.*

This follows immediately from the last proposition, for let a, b, c, be three quantities in continued proportion, then

$$a : b :: b : c \text{ or, } \frac{a}{b} = \frac{b}{c}$$

$$\therefore a\,c = b \times b \text{ by last prop.}$$

$$= b^2$$

IV. Conversely, *If the product of any two quantities be equal to the square of a third, the last quantity will be a mean proportional to the other two.*

Thus, if $a\,c = b^2$, b is a mean proportional to a and c, for since

$$a\,c = b^2$$

dividing these equals by $b\,c$

$$\frac{a}{b} = \frac{b}{c} \text{ or } a : b :: b : c$$

V. *Quantities which have the same ratio to the same quantity are equal to one another, and those to which the same quantity has the same ratio are equal to one another.*

First, let a and b have the same ratio to the same quantity c, then $a = b$,

Since,

$$a : c :: b : c$$

Or,

$$\frac{a}{c} = \frac{b}{c}$$

Multiply these equals by c $\therefore$ $a = b$

Again, let c have the same ratio to each of the quantities a and b, then $a = b$.

Since,

$$c : a :: c : b$$

Or,

$$\frac{c}{a} = \frac{c}{b}$$

Dividing these equals by c

$$\frac{1}{a} = \frac{1}{b}$$

$$\therefore a = b$$

VI. *Ratios that are the same to the same ratio are the same to one another.*

Let,

$$\left.\begin{array}{l} a : b :: x : y \\ c : d :: x : y \end{array}\right\} \text{Then, } a : b :: c : d$$

And,

Since,

$$a : b :: x : y, \text{ or } \frac{a}{b} = \frac{x}{y}$$

And,

$$c : d :: x : y, \text{ or } \frac{c}{d} = \frac{x}{y}$$

We have,

$$\frac{x}{y} = \frac{a}{b} \text{ and also } = \frac{c}{d}$$

$$\therefore \frac{a}{b} = \frac{c}{d} \text{ or, } a : b :: c : d$$

VII. *If four quantities be proportionals, they will be proportionals also* alternando, *that is, the first will have the same ratio to the third that the second has to the fourth.*

Let $a : b :: c : d$, then also, $a : c :: b \ d$

Since $\frac{a}{b} = \frac{c}{d}$, divide each of these equals by c and multiply each by b.

$$\text{Then } \frac{a}{c} = \frac{b}{d} \text{ i. e. } a : c :: b : d$$

VIII. *If four quantities be proportionals, they will be proportionals also* invertendo, *that is, the second will have to the first the same ratio that the fourth has to the third.*

Let $a : b :: c : d$, then also $b : a :: d : c$

Since $\frac{a}{b} = \frac{c}{d}$, divide unity by each of these equals. We have

$$\overline{\left(\frac{a}{b}\right)} = \overline{\left(\frac{c}{d}\right)}$$

Or,

$$\frac{b}{a} = \frac{d}{c} \text{ i. e. } b : a :: d : c$$

IX. *If four quantities be proportionals, they will be proportionals also* componendo, *that is, the first together with the second, will have to the second the same ratio that the third together with the fourth has to the fourth.*

Let $a : b :: c : d$, then also, $a + b : b :: c + d : d$

Since $\frac{a}{b} = \frac{c}{d}$, add 1 to each of these equals, then

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

Or,

$$\frac{a+b}{b} = \frac{c+d}{d} \text{ i. e. } a + b : b :: c + d : d$$

X. *If four quantities be proportionals, they will be proportionals also* dividendo, *that is, the difference of the first and second will have to the second the same ratio that the difference of the third and fourth has to the fourth.*

Let $a : b :: c : d$, then also, $a - b : b :: c - d : d$

Since $\frac{a}{b} = \frac{c}{d}$, subtract unity from each of these equals, then

$$\frac{a}{b} - 1 = \frac{c}{d} - 1$$

Or,

$$\frac{a-b}{b} = \frac{c-d}{d} \text{ i. e. } a - b : b :: c - d : d$$

XI. *If four quantities be proportionals, they will be proportionals also* convertendo, *that is, the first will have to the difference of the first and second the same ratio that the third has to the difference of the third and fourth.*

Let $a : b :: c : d$, then also, $a : a - b :: c : c - d$

Since $\frac{a}{b} = \frac{c}{d}$ then by prop. VIII. $\frac{b}{a} = \frac{d}{c}$ and hence subtracting these equal quantities from unity,

$$1 - \frac{b}{a} = 1 - \frac{d}{c}$$

Or,

$$\frac{a-b}{a} = \frac{c-d}{c}$$

Or,

$$\frac{a}{a-b} = \frac{c}{c-d} \text{ i. e. } a : a - b :: c : c - d$$

XII. *If four quantities be proportionals, the sum of the first and second will have to their difference the same ratio that the sum of the third and fourth has to their difference.*

Let $a : b :: c : d$, then also, $a + b : a - b :: c + d : c - d$

Since $\frac{a}{b} = \frac{c}{d}$ we have

By Prop. IX. $\frac{a+b}{b} = \frac{c+d}{d}$

And,

$$\text{By Prop. X. } \frac{a-b}{h} = \frac{c-d}{d}$$

Dividing these equals by each other,

$$\frac{\dfrac{a+b}{b}}{\dfrac{a-b}{b}} = \frac{\dfrac{c+d}{d}}{\dfrac{c-d}{d}}.$$

Or,

$$\frac{a+b}{a-b} = \frac{c+d}{c-d} \quad \text{i. e. } a+b : a-b :: c+d : c-d$$

XIII. *If there be any number of quantities more than two, and as many others, which, taken two and two in order, are proportionals,* (ex æquali,) *the first will have to the last of the first rank the same ratio that the first of the second rank has to the last.*

Let

$a, b, c, d \ldots$ be any numbers of quantities

And

$e, f, g, h \ldots$ as many others

Let

$$\left.\begin{array}{l} a : b :: e : f \\ b : c :: f : g \\ c : d :: g : h \end{array}\right\} \text{ Then also, } a : d :: e : h$$

For since

$$\frac{a}{b} = \frac{e}{f}$$
$$\frac{b}{c} = \frac{f}{g}$$
$$\frac{c}{d} = \frac{g}{h}$$

Multiplying the first column together, and also the second,

$$\frac{abc}{bcd} = \frac{efg}{fgh}$$

or,

$$\frac{a}{d} = \frac{e}{h} \quad \text{i. e. } a : d :: e : h$$

XIV. *If there be any number of quantities more than two, and as many others, which, taken two and two in a cross order, are proportionals,* (ex æquali perturbatâ,) *the first will have to the last of the first rank the same ratio that the first of the second rank has to the last.*

Let

$a, b, c, d \ldots$ be any number of quantities,

And,

$e, f, g, h \ldots$ as many others.

Let

$$\left.\begin{array}{l} a : b :: g : h \\ b : c :: f : g \\ c : d :: e : f \end{array}\right\} \text{Then also, } a : d :: e : h$$

For since

$$\frac{a}{b} = \frac{g}{h}$$
$$\frac{b}{c} = \frac{f}{g}$$
$$\frac{c}{d} = \frac{e}{f}$$
$$\frac{a\,b\,c}{b\,c\,d} = \frac{g\,f\,e}{h\,g\,f}$$

or

$$\frac{a}{d} = \frac{e}{h} \text{ i. e. } a : d :: e : h$$

XV. *If four quantities be proportionals, any powers or roots of these quantities will also be proportionals.*

Let

$$a : b :: c : d, \text{ then also, } a^n : b^n :: c^n : d^n$$

Since

$$\frac{a}{b} = \frac{c}{d} \text{ raising each of these equals to the power of } n, \left(\frac{a}{b}\right)^n = \left(\frac{c}{d}\right)^n$$

or,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n} \text{ i. e. } a^n : b^n :: c^n : d^n$$

Where n may be either integral or fractional.

XVI. *If there be any number of proportional quantities, the first will have to the second the same ratio that the sum of all the antecedents has to the sum of all the consequents.*

Let

$a, b, c, d, e, f, g, h,$ be any number of proportional quantities, such that

$$a : b :: c : d :: e : f :: g : h$$

Then,

$$a : b :: a + c + e + g : b + d + f + h$$

Since,

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$$

We have,

$$a\,b = b\,a$$
$$a\,d = b\,c$$
$$a\,f = b\,e$$
$$a\,h = b\,g$$

$$\text{and } \therefore \quad a(b+d+f+h) = b(a+c+e+g)$$

$$\therefore \quad \frac{a}{b} = \frac{a+c+e+g}{b+d+f+h}$$

$$a : b :: a+c+e+g : b+d+f+h$$

XVII. *If three quantities be in continued proportion, the first will have to the third the duplicate ratio of that which it has to the second.*

Let $a : b :: b : c$, then, $a : c :: a^2 : b^2$

Since

$\frac{a}{b} = \frac{b}{c}$, multiply each of these equals by $\frac{a}{b}$, then,

$\frac{a}{b} \times \frac{a}{b} = \frac{b}{c} \times \frac{a}{b}$ or $\frac{a^2}{b^2} = \frac{a}{c}$ i. e. $a : c :: a^2 : b^2$

XVIII. *If four quantities be in continued proportion, the first will have to the fourth the triplicate ratio of that which it has to the second.*

Let a, b, c, d, be four quantities in continued proportion, so that,

$a : b :: b : c :: c : d$, then also, $a : d :: a^3 : b^3$

Since,

$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, we have,

$$\frac{a}{b} = \frac{b}{c}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a}{b} = \frac{a}{b}$$

Multiplying these equals together,

$$\frac{a^3}{b^3} = \frac{bca}{cdb}$$

or,

$\frac{a^3}{b^3} = \frac{a}{d}$ i. e. $a : d :: a^3 : b^3$

ON EQUATIONS.

PRELIMINARY REMARKS.

134. An *equation*, in the most general acceptation of the term, signifies two algebraic expressions which are equal to each other, and are connected by the sign $=$.

Thus, $a x = b$, $c x^2 + d x = e$, $c x^3 + g x^2 = h x + k$, $m x^4 + n x^3 + p x^2 + q x + r = o$, are equations.

The two quantities separated by the sign $=$ are called the *members* of the equation, the quantity to the left of the sign $=$ is called the *first member*, the quantity to the right the *second member*. The quantities separated by the signs $+$ and $-$ are called the *terms* of the equation.

135. Equations are usually composed of certain quantities which are known and given, and others which are unknown. The known quantities are in general represented either by numbers, or by the first letters in the alphabet, a, b, c, &c.; the unknown quantities by the last letters, s, t, x, y, z, &c.

136. Equations are of different kinds.

1°. An equation may be such, that one of the members is a repetition of the other, as, $2x - 5 = 2x - 5$.

2°. One member may be merely the result of certain operations *indicated* in the other member, as, $5x + 16 = 10x - 5 - (5x - 21)$, $(x + y)(x - y) = x^2 - y^2$, $\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2$.

3°. All the quantities in each member may be known and given, as, $25 = 10 + 15$, $a + b = c - d$, in which, if we substitute for a, b, c, d, the known quantities which they represent, the equality subsisting between the two members will be self-evident.

In each of the above cases the equation is called *an identical equation.*

4°. Finally, the equation may contain both known and unknown quantities, and be such, that the equality subsisting between the two members cannot be made manifest, until we substitute for the unknown quantity or quantities certain other numbers, the value of which depends upon the known numbers which enter into the equation. The discovery of these unknown numbers constitutes what is called the *solution of the equation.*

The word *equation,* when used without any qualification, is always understood to signify an equation of this last species; and these alone are the objects of our present investigations.

$x + 4 = 7$ is an equation properly so called, for it contains an unknown quantity x, combined with other quantities which are known and given, and the equality subsisting between the two members of the equation cannot be made manifest, until we find a value for x, such, that when added to 4, the result will be equal to 7. This condition will be satisfied, if we make $x = 3$, and this value of x being determined, the equation is solved.

The value of the unknown quantity thus discovered is called the *root* of the equation

137. Equations are divided into *degrees* according to the highest power of the unknown quantity which they contain. Those which involve the simple power only of the unknown quantity, are called *simple equations*, or *equations of the first degree;* those into which the square of the unknown quantity enters, are called *quadratic equations*, or, *equations of the second degree;* so we have *cubic equations*, or, *equations of the third degree; biquadratic equations*, or, *equations of the fourth degree; equations of the fifth, sixth, ... n^{th} degree.* Thus,

$a x + b = c x + d$ is a simple equation.
$4 x^2 - 2 x = 5 - x^2$ is a quadratic equation.
$x^3 + p x^2 = 2 q$ is a cubic equation.
$x^n + p x^{n-1} + q x^{n-2} +$, &c. $= r$, is an equation of the n^{th} degree.

138. *Numerical equations* are those which contain particular numbers only, in addition to the unknown quantity. Thus, $x^3 + 5 x^2 = 3 x + 17$, is a numerical equation.

139. *Literal equations* are those in which the known quantities are represented by letters only, or by both letters and numbers. Thus, $x^3 + p x^2 + q x = r$, $x^4 - 3 p x^3 + 5 q x^2 + r x = 5$ are literal equations.

140. Let us now pass on to consider the solution of equations, it being understood, that, *to solve an equation, is to find the value of the unknown quantity, or to find a number which, when substituted for the unknown quantity in the equation, renders the first member identical with the second.*

The difficulty of solving equations depends upon the degree of the equations, and the number of unknown quantities. We first consider the most simple

ON THE SOLUTION OF SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

141. The various operations which we perform upon equations, in order to arrive at the value of the unknown quantities, are founded upon the following principles:—

If to two equal quantities, the same quantity be added, the sums will be equal.

If from two equal quantities, the same quantity be subtracted, the remainders will be equal.

If two equal quantities be multiplied by the same quantity, the products will be equal.

If two equal quantities be divided by the same quantity, the quotients will be equal.

These principles, when applied to the two equal quantities which constitute the two members of every equation, will enable us to deduce from them new equations, which are all satisfied by the same value of the unknown quantity, and which will lead us to discover the value of that unknown quantity.

142. The unknown quantity may be combined with the known quantities in the given equation, by the operations of addition, subtraction, multiplication, and division. We shall consider these different cases in succession.

I. Let it be required to solve the equation,

$$x + a = b$$

If, from the two equal quantities $x + a$ and b, we subtract the same quantity a, the remainders will be equal, and we shall have,

$$x + a - a = b - a$$

or,

$$x = b - a, \text{ the value of } x \text{ required.}$$

So also, in the equation,

$$x + 6 = 24$$

Subtracting 6 from each of the equal quantities $x + 6$ and 24, the result is,

$$x = 24 - 6$$
$$= 18, \quad \text{the value of } x \text{ required.}$$

II. Let the equation be,

$$x - a = b$$

If, to the two equal quantities $x - a$ and b, the same quantity a be added, the sums will be equal, then we have,

$$x - a + a = b + a$$

or,

$$x = b + a \text{ the value of } x \text{ required.}$$

So also in the equation,

$$x - 6 = 24$$

Adding 6 to each of these equal quantities, the result is,

$$x = 24 + 6$$
$$= 30, \quad \text{the value of } x \text{ required.}$$

It follows from (I.) and (II.) that,

We may transpose any term of an equation from one member to the other, by changing the sign of that term.

We may change the signs of every term in each member of the equation, without altering the value of the expression. This is, in fact, the same thing as transposing every term in each member of the equation.

If the same quantity appear in each member of the equation affected with the same sign, it may be suppressed.

III. Let the equation be,

$$a x = b$$

Dividing each of these equals by a, the result is,

$$x = \frac{b}{a} \text{ the value of } x \text{ required.}$$

So also in the equation,

$$6x = 24$$

Dividing each of these equals by 6, the result is,

$$x = 4, \quad \text{the value of } x \text{ required.}$$

From this it follows, that,

When one member of an equation contains the unknown quantity alone, affected with a coefficient, and the other member contains known quantities only, the value of the unknown quantity is found by dividing each member of the equation by the coefficient of the unknown quantity.

IV. Let the equation be,

$$\frac{x}{a} = b$$

Multiplying each of these equals by a, the result is,

$$x = a b, \quad \text{the value of } x \text{ required.}$$

So also in the equation,

$$\frac{x}{6} = 24$$

Multiplying each of these equals by 6, the result is,

$$x = 144$$

From this it follows, that,

When one member of the equation contains the unknown quantity alone, divided by a known quantity, and the other member contains known quantities only, the value of the unknown quantity is found by multiplying each member of the equation by the quantity which is the divisor of the unknown quantity.

V. Let the equation be,

$$\frac{a\,x}{b} - c = \frac{d\,x}{e} \quad \frac{m}{n}$$

In order to solve this equation, we must clear it of fractions; to effect this, reduce the fractions to equivalent ones, having a common denominator (Art. 52), the equation becomes,

$$\frac{a\,e\,n\,x}{b\,e\,n} - c = \frac{b\,d\,n\,x}{b\,e\,n} - \frac{b\,e\,m}{b\,e\,n}$$

Multiply these equal quantities by the same quantity $b\,e\,n$, or, which is evidently the same thing, suppress the denominator $b\,e\,n$ in each of the fractions, and multiply the integral term by $b\,e\,n$, the result is,

$a\,e\,n\,x - b\,c\,e\,n = b\,d\,n\,x - b\,e\,m$, an equation clear of fractions.

So also in the equation,

$$\frac{2\,x}{3} - \frac{3}{4} = 11 + \frac{x}{5}$$

Reducing the fractions to a common denominator,

$$\frac{40\,x}{60} - \frac{45}{60} = 11 + \frac{12\,x}{60}$$

Multiplying both members of the equation by 60, the result is,

$40\,x - 45 = 660 + 12\,x$, an equation clear of fractions.

If the denominators have common factors, we can simplify the above operation by reducing them to their least common denominator, which is done (see arithmetic) by finding the least common multiple of the denominators. Thus, in the equation,

$$\frac{5\,x}{12} - \frac{4\,x}{3} - 13 = \frac{7}{8} - \frac{13\,x}{6}$$

The least common multiple of the numbers 12, 3, 8, 6, is 24, which is therefore the least common denominator of the above fractions, and the equation will become,

$$\frac{10\,x}{24} - \frac{32\,x}{24} - 13 = \frac{21}{24} - \frac{52\,x}{24}$$

Multiplying both members of the equation by 24, the result is,

$10\,x - 32\,x - 312 = 21 - 52\,x$, an equation clear of fractions.

Hence, it appears that,

In order to clear an equation of fractions, reduce the fractions to a common denominator, multiply each integral term by this common denominator, and then suppress it.

143. From what has been said above, we deduce the following general

RULE FOR THE SOLUTION OF A SIMPLE EQUATION CONTAINING ONE UNKNOWN QUANTITY.

1°. *Clear the equation of fractions, and perform in both members all the algebraic operations indicated.*

2°. *Transpose all the terms containing the unknown quantity to one member of the equation, and all the terms containing known quantities only to the other member of the equation, and reduce each member to its most simple form.*

3°. *We thus obtain an equation, one member of which contains the unknown quantity alone, affected with a coefficient, and the other member contains known quantities only; the value of the unknown quantity will be found by dividing the member composed of the known quantities by the coefficient of the unknown quantity.*

The terms containing the unknown quantity are usually collected in the *first member* of the equation.

Example 1.

Given, $19x + 13 = 59 - 4x$

Transposing, $19x + 4x = 59 - 13$

Reducing, $23x = 46$

Dividing by 23, $x = 2$

Example 2.

Given, $\frac{x}{6} - \frac{x}{4} + 10 = \frac{x}{3} - \frac{x}{2} + 11$

Reducing to least common denominator 12,

$$\frac{2x}{12} - \frac{3x}{12} + 10 = \frac{4x}{12} - \frac{6x}{12} + 11$$

Multiplying both members by 12,

$$2x - 3x + 120 = 4x - 6x + 132$$

Transposing, $2x - 3x - 4x + 6x = 132 - 120$

Reducing, $x = 12$

Example 3.

Given, $\frac{5x+3}{4} + 7 = \frac{4x-10}{10} + 10$

Reducing to least common denominator 20,

$$\frac{25x+15}{20} + 7 = \frac{8x-20}{20} + 10$$

Multiplying both members by 20,

$$25x + 15 + 140 = 8x - 20 + 200$$

Transposing, $25x - 8x = 200 - 20 - 15 - 140$

Reducing, $17x = 25$

Dividing by 17, $x = \frac{25}{17}$

Example 4.

Given, $\dfrac{2x-5}{4}-\dfrac{7x+10}{3} = 16-\dfrac{12x-10}{5}$

Reducing to common denominator,

$$\frac{30x-75}{60}-\frac{140x+200}{60} = 16-\frac{144x-120}{60}$$

Multiplying both members by 60,

$$30x-75-140x-200 = 960-144x+120$$

Transposing, $30x-140x+144x = 960+75+200+120$

Reducing, $34x = 1355$

Dividing by 34, $x = \dfrac{1355}{34}$

Example 5.

Given, $\dfrac{12-4x}{10}-\dfrac{2x+5}{5} = 3+\dfrac{7x+60}{2}-50$

Reducing to least common denominator,

$$\frac{12-4x}{10}-\frac{4x+10}{10} = 3+\frac{35x+300}{10}-50$$

Multiplying by 10,

$$12-4x-4x-10 = 30+35x+300-500$$

Transposing, $-4x-4x-35x = 30+300-12+10-500$

Reducing, $-43x = -172$

Changing the signs of both members,

$$43x = 172$$

Dividing by 43, $x = 4$

Example 6.

Given, $ax+b = cx+d$

Transposing, $ax-cx = d-b$

Simplifying, $(a-c)x = d-b$

Dividing by $(a-c)$, $x = \dfrac{d-b}{a-c}$

Example 7.

$$\frac{ax}{b}+\frac{cx}{d}+e = fx+\frac{gx}{h}+m$$

Reducing to a common denominator,

$$\frac{adhx}{bdh}+\frac{bchx}{bdh}+e = fx+\frac{bdgx}{bdh}+m$$

Multiplying by bdh,

$$adhx+bchx+bdeh = bdfhx+bdgx+bdhm$$

Transposing,

$$adhx+bchx-bdfhx-bdgx = bdhm-bdeh$$

Simplifying,

$$(adh+bch-bdfh-bdg)x = bdhm-bdeh$$

Dividing by coefficient of x, $$x = \frac{bdhm - bdeh}{adh + bch - bdfh - bdg}$$
$$= \frac{bdh\,(m - e)}{adh + bch - bdfh - bdg}$$

Example 8.

Given, $$\frac{x}{a} - 1 - \frac{dx}{c} + 3ab = 0$$

Reducing to common denominator,
$$\frac{cx}{ac} - 1 - \frac{adx}{ac} + 3ab = 0$$

Multiplying by ac,
$$cx - ac - adx + 3a^2bc = 0$$

Transposing and simplifying,
$$(c - ad)\,x = ac - 3a^2bc$$

Dividing by coefficient of x, $$x = \frac{ac\,(1 - 3ab)}{c - ad}$$

144. In addition to the principles detailed in (Art. 150.) we may subjoin the following:

If two equal quantities be raised to the same power, the results will be equal.

If the same root of two equal quantities be extracted, the results will be equal.

Hence, any equation may be cleared of a single radical quantity, by transposing all the other terms to the opposite side, and then raising each member to the power denoted by the index of the radical. If there be more than one radical, the operation must be repeated. Thus

Example 9.

Given,
$$\sqrt{3x + 7} = 10$$

Squaring each member of the equation,
$$3x + 7 = 100$$

Transposing, $$3x = 100 - 7$$

Reducing and dividing by 3, $$x = 31$$

Example 10.

Given,
$$\sqrt{4x + 2} = \sqrt{4x} + 5$$

Squaring both sides of the equation,
$$4x + 2 = 4x + 10\sqrt{4x} + 25$$

Reducing, $$-10\sqrt{4x} = 23$$

Squaring both sides, $$400x = 529$$
$$x = \frac{529}{400}$$

Example 11.

Given, $$\frac{\sqrt{x} + 28}{\sqrt{x} + 4} = \frac{\sqrt{x} + 38}{\sqrt{x} + 6}$$

Clearing the equation of fractions,

$$x + 28\sqrt{x} + 6\sqrt{x} + 168 \quad = \quad x + 38\sqrt{x} + 4\sqrt{x} + 152$$

Transposing and reducing, $\quad 16 \quad = \quad 8\sqrt{x}$

Dividing both members by 8, $\quad 2 \quad = \quad \sqrt{x}$.

Squaring both members, $\quad 4 \quad = \quad x$

Example 12.

ven,

$$\sqrt[m]{a + x} \quad = \quad \sqrt[2m]{x^2 + 5ax + b^2}$$

Raising both members to the power of m,

$$a + x \quad = \quad \sqrt{x^2 + 5ax + b^2}$$

Squaring both members,

$$a^2 + 2ax + x^2 \quad = \quad x^2 + 5ax + b^2$$

Transposing and reducing, $\quad -3ax \quad = \quad b^2 - a^2$

Changing the signs, $\quad 3ax \quad = \quad a^2 - b^2$

Dividing by $3a$, $\quad x \quad = \quad \dfrac{a^2 - b^2}{3a}$

Ex. 13. Given $4x + 36 \quad = \quad 5x + 34$

Ans. $x = 2$

Ex. 14. Given $4x - 12 + 3x + 1 \quad = \quad 2x + 4$

Ans. $x = 3$

Ex. 15. Given $3a + x - 6b + 2 \quad = \quad 7b - a + c + 6$

Ans. $x = 12b - 4a + c + 4$

Ex. 16. Given $13\frac{3}{4} - \dfrac{x}{2} \quad = \quad 2x - 8\frac{3}{4}$

Ans. $x = 9$

Ex. 17. Given $12\frac{1}{4} + 3x - 6 - \dfrac{7x}{3} \quad = \quad \dfrac{3x}{4} - 5\frac{3}{8}$

Ans. $x = 139\frac{1}{2}$

Ex. 18. Given $\dfrac{x}{2} + \dfrac{x}{3} \quad = \quad \dfrac{x}{4} + 7$

Ans. $x = 12$

Ex. 19. Given $21 + \dfrac{3x - 11}{16} \quad = \quad \dfrac{5x - 5}{8} + \dfrac{97 - 7x}{2}$

Ans. $x = 9$

Ex. 20. Given $\dfrac{bx}{a} - \dfrac{d}{c} \quad = \quad \dfrac{a}{b} - \dfrac{cx}{d}$

Ans. $x = \dfrac{ad}{bc}$

Ex. 21. Given $23 + \dfrac{5x - 1}{11} + \dfrac{3x - 2}{5} - \dfrac{11x - 3}{12} \quad = \quad \dfrac{13x - 15}{3} - \dfrac{8x - 2}{7}$

Ans. $x = 9$

Ex. 22. Given $4x+\frac{1}{10}-\frac{3x-13}{16}-\frac{12+7x}{9} = 7x-33-\frac{9+5x}{10}-\frac{11x-17}{8}$

Ans. $x = 15$

Ex. 23. Given $\frac{ace}{d}-\frac{(a+b)^2x}{a}-bx = ae-3bx$

Ans. $x=\frac{a^2e(c-d)}{(a^2+b^2)d}$

Ex. 24. Given $\frac{a+3x}{4a}-\frac{7a-5x}{6b}+3-\frac{9x}{4} = \frac{x}{ab}+\frac{5x}{6b}$

Ans. $x=\frac{39ab-14a^2}{27ab-9b+12}$

Ex. 25. Given $\frac{bx}{2b-a}-\frac{(3bc+ad)x}{2ab(a+b)}-\frac{5ab}{3c-d}=\frac{(3bc-ad)x}{2ab(a-b)}-\frac{5a(2b-a)}{a^2-b^2}$

Ans. $x=\frac{5a(2b-a)}{3c-d}$

Ex. 26. Given $\sqrt{10x+3} = 7$

Ans. $x=\frac{46}{10}$

Ex. 27. Given $\sqrt{x-32} = 16-\sqrt{x}$

Ans. $x = 81$

Ex. 28. Given $\frac{5x-9}{\sqrt{5x}+3}-1=\frac{\sqrt{5x}-3}{2}$

Ans. $x = 5$

Ex. 29. Given $h\sqrt[3]{ax-b}=k\sqrt[3]{cx+dx-f}$

Ans. $x=\frac{bh^3-fk^3}{ah^3-(c+d)k^3}$

Ex. 30. Given $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\sqrt{m}$

Ans. $x=\frac{2a\sqrt{m}}{1+m}$

Ex. 31. Given $\sqrt[3]{a^2+c}=\sqrt[4]{\frac{a^2+c}{d(x+9)}}$

Ans. $x=\frac{1}{d\sqrt[3]{a^2+c}}-9$

Ex. 32. Given $\frac{mx}{n}\sqrt{p^2x^2+q^2}+\frac{mpx^2}{n}=rx$

Ans. $x=\frac{(nr+mq)(nr-mq)}{2mnpr}$

ON THE SOLUTION OF SIMPLE EQUATIONS, CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

145. A single equation, containing two unknown quantities, admits of an infinite number of solutions; for if we assign any arbitrary value to one of the unknown quantities, the equation will determine the corresponding value of the other unknown quantity. Thus, in the equation $y = x + 10$, each value which we may assign to x will, when augmented by 10, furnish a corresponding value of y. An equation of this nature is called an *indeterminate equation*, and since the value of y depends upon that of x, y is said to be a *function* of x.

In general, every quantity, whose value depends upon one or more quantities, is said to be a FUNCTION *of these quantities.*

Thus, in the equation $y = a\,x + b$, we say that y is a function of x, and that y is *expressed in terms of* x, and the known quantities a, b.

If, however, we have *two* equations between two unknown quantities, and if these equations hold good together, then it will be seen that we can combine them in such a manner as to obtain determinate values for each of the unknown quantities.

In general, in order that questions of this nature may admit of determinate solutions, we must have *as many separate equations as there are unknown quantities;* a groupe of equations of **this nature** is called *a system of simultaneous equations.*

146. In order to solve a system of two simple equations containing two unknown quantities, we must endeavour to deduce from them a single equation, containing only one unknown quantity; we must therefore make one of the unknown quantities disappear, or, as it is termed, we must *eliminate* it. The equation thus obtained, containing one unknown quantity only, will give the value of the unknown quantity which it involves, and substituting the value of this unknown quantity in either of the equations containing the two unknown quantities, we shall arrive at the value of the other unknown quantity.

The process which most naturally suggests itself for the *elimination* of one of the unknown quantities, is to derive from one of the two equations an expression for that unknown quantity *in terms* of the other unknown quantity, and then substitute this expression in the other equation. We shall see that the *elimination* may be effected by different methods, which are more or less simple according to the nature of the question proposed.

Example 1.

Let it be proposed to solve the system of equations,

$$\left.\begin{array}{lll} y - x & = & 6 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1) \\ y + x & = & 12 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \end{array}\right\}$$

147. First Method.—From equation (1) we find the value of y in terms of x, which gives $y = x + 6$; substituting the expression $x + 6$ for y in equation (2) it becomes $x + 6 + x = 12$, from which we find the determinate value $x = 3$; since we have already seen that $y = x + 6$, we find also the determinate value $y = 3 + 6$ or 9.

Thus it appears, that although each of the above equations, considered separately, admits of an infinite number of solutions, yet the *system* of equations admit only one *common solution*, $x = 3$, $y = 9$.

148. Second Method.—Derive from each equation an expression for y in terms of x, we shall then have

$$y = x + 6$$
$$y = 12 - x$$

These two values of y must be equal to one another, and, by *comparing* them, we shall obtain an equation involving only one unknown quantity, viz.

$$x + 6 = 12 - x$$

Whence,

$$x = 3$$

Substituting the value of x in the expression $y = x + 6$, we find $y = 9$.

The substitution of 3, the value of x, in the second expression, $y = 12 - x$, leads necessarily to the same value of y, for we derived the value of x from the equation $x + 6 = 12 - x$.

149. Third Method.—Since the coefficients of y are equal in the two equations, it is manifest that we may eliminate y by *subtracting the two equations from each other*, which gives

$$(y + x) - (y - x) = 12 - 6$$

Whence,

$$x = 3$$

Having thus obtained the value of x, we may deduce that of y by making $x = 3$ in either of the proposed equations; we can however determine the value of y directly, by observing, that, since the coefficients of x in the proposed equations are equal and have opposite signs, we may eliminate x by *adding the two equations together*, which give

$$(y - x) + (y + x) = 12 + 6$$

Whence

$$y = 9$$

If we examine the three above methods, we shall perceive that they consist in expressing that *the unknown quantities have the same values in both equations.*

These methods have derived their names from the processes employed to effect the elimination of the unknown quantities.

The first is called the *method of elimination by substitution.*
The second .. .. .. *comparison.*
The third .. . .. *addition and subtraction.*

Example 2.

Take the equations

$$\left.\begin{array}{l} 2x + 3y = 13 \text{ - - - - - - - - - } (1) \\ 5x + 4y = 22 \text{ - - - - - - - - - } (2) \end{array}\right\}$$

1°. *Eliminating by substitution.*

From equation (1) we find

$$y = \frac{13 - 2x}{3}$$

Substituting the value of y in terms of x in equation (2) it becomes

$$5x + 4 \times \frac{13 - 2x}{3} = 22$$

an equation containing x alone, which, when solved, gives

$$x = 2$$

This value of x, substituted in either of the equations (1) or (2), gives

$$y = 3$$

2°. *Eliminating by comparison.*

From equation (1) $y = \dfrac{13 - 2x}{3}$

From equation (2) $y = \dfrac{22 - 5x}{4}$

Equating these values of y, $\dfrac{13-2x}{3} = \dfrac{22-5x}{4}$ an equation containing x only

Whence,

$$x = 2$$

Substituting this value for x in either of the preceding expressions for y we find

$$y = 3$$

3°. *Eliminating by subtraction.*

In order to eliminate y, we perceive that if we could deduce from the proposed equations two other equations in x and y, in which the coefficients of y should be equal, the elimination of y would be effected by *subtracting* one of these new equations from the other.

It is easily seen that we shall obtain two equations of the form required, if we multiply all the terms of each equation by the coefficient of y in the other. Multiplying, therefore, all the terms of equation (1) by 4, and all the terms of equation (2) by 3, they become

$$\begin{array}{rcl} 8x + 12y & = & 52 \\ 15x + 12y & = & 66. \end{array}$$

Subtracting the former of these equations from the latter, we find

$$7x = 14$$

Whence,

$$x = 2.$$

In like manner, in order to eliminate x, multiply the first of the proposed equations by 5, and the second by 2, they will then become

$$10x + 15y = 65$$
$$10x + 8y = 44.$$

Subtracting the latter of these two equations from the former,

$$7y = 21$$

Whence,

$$y = 3.$$

In order to solve a system of three simple equations between three unknown quantities, we must first eliminate one of the unknown quantities by one of the methods explained above; this will lead to a system of two equations, containing only two unknown quantities; the value of these two unknown quantities may be found by any of the methods described in the last article, and substituting the value of these two unknown quantities in any one of the original equations, we shall arrive at an equation which will determine the value of the third unknown quantity.

Example 3.

Take the system of equations,

$$\left.\begin{array}{lcl} 3x + 2y + z & = & 16 \ldots\ldots\ldots\ldots(1) \\ 2x + 2y + 2z & = & 18 \ldots\ldots\ldots\ldots(2) \\ 2x + 2y + z & = & 14 \ldots\ldots\ldots\ldots(3) \end{array}\right\}$$

1°. *Eliminating by substitution.*

From equation (1) we find

$$z = 16 - 3x - 2y \ldots\ldots\ldots\ldots(4)$$

Substituting this value of z in equation (2) and (3), they become

$$\left.\begin{array}{lcl} 2x + 2y + 2(16 - 3x - 2y) & = & 18 \ldots(5) \\ 2x + 2y + (16 - 3x - 2y) & = & 14 \ldots(6) \end{array}\right\}$$

these two last equations contain x and y only, and if treated according to any of the above methods, will give us

$$x = 2, \qquad y = 3.$$

Substituting these values of x and y in any one of the equations (1), (2), (3), (4), we find

$$z = 4.$$

2°. *Eliminating by comparison.*

In order to eliminate z derive from each of the three proposed equations a value of z in terms of x and y, we then have

$$z = 16 - 3x - 2y$$
$$z = 9 - x - y$$
$$z = 14 - 2x - 2y;$$

equating the first of these values of z with the second and with the third in succession, we arrive at a system of two equations:

$$\left.\begin{array}{lcl} 16 - 3x - 2y & = & 9 - x - y \\ 16 - 3x - 2y & = & 14 - 2x - 2y \end{array}\right\}$$

containing x and y only; these equations give

$$x = 2; \quad y = 3;$$

these values of x and y, when substituted in any of the three expressions for z, give

$$z = 4$$

3°. *Eliminating by subtraction.*

In order to eliminate z between equations (1) and (2),

$$3x + 2y + z = 16$$
$$2x + 2y + 2z = 18;$$

we perceive that in order to reduce these equations to two others in .. the coefficients of z shall be the same, it will be sufficient to divide the two members of the second equation by (2), for we thus have

$$x + y + z = 9.$$

Subtracting this from the first equation,

$$3x + 2y + z = 16,$$

we find an equation between two unknown quantities,

$$2x + y = 7 \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

In order to eliminate z between equations (1) and (3),

$$3x + 2y + z = 16$$
$$2x + 2y + z = 14.$$

Subtract the latter from the former, which gives

$$x = 2;$$

the substitution of this value of x in equation (α) gives

$$y = 3,$$

and the substitution of these values of x and y in any of the proposed equations gives

$$z = 4.$$

The particular form of the proposed equations enables us to simplify the above calculation, for if we subtract equation (3) from equations (1) and (2) in succession, we have

$$(3x + 2y + z) - (2x + 2y + z) = 16 - 14, \text{ whence } x = 2$$
$$(2x + 2y + 2z) - (2x + 2y + z) = 18 - 14, \text{ whence } z = 4;$$

and substituting these values of x and z in any of the proposed equations we find

$$y = 3.$$

In order to solve a system of four equations between four unknown quantities, we reduce this case to the last by eliminating one of the unknown quantities. We thus arrive at a system of three equations between three unknown quantities, from which the value of these three unknown quantities may be found. Substituting these values in any one of the equations which involve the other unknown quantity, we deduce from it the value of that unknown quantity.

Example 4.

Take the system of equations,

$$\left.\begin{array}{l} x + y + z + t = 14 \ldots\ldots\ldots\ldots\ldots\ldots (1) \\ x + y + z - t = 4 \ldots\ldots\ldots\ldots\ldots\ldots (2) \\ x + y - z + 2t = 11 \ldots\ldots\ldots\ldots\ldots\ldots (3) \\ x - y + z + 3t = 18 \ldots\ldots\ldots\ldots\ldots\ldots (4) \end{array}\right\}$$

The first equation gives

$$t = 14 - x - y - z \ldots\ldots\ldots\ldots(5)$$

Substituting this expression for t in the three other equations we find

$$x + y + z = 9 \ldots\ldots\ldots\ldots(6)$$
$$x + y + 3z = 17 \ldots\ldots\ldots\ldots(7)$$
$$x + 2y + z = 12 \ldots\ldots\ldots\ldots(8).$$

In order to solve these three equations between x, y, z, we find from the first

$$z = 9 - x - y \ldots\ldots\ldots\ldots(9),$$

and substituting this value of z in the two other equations, they become

$$x + y = 5 \ldots\ldots\ldots\ldots(10)$$
$$y = 3 \ldots\ldots\ldots\ldots(11)$$

Whence $$x = 2 \ldots\ldots\ldots\ldots(12).$$

Substituting the values of x and y in equation (8), we find

$$z = 4 \ldots\ldots\ldots\ldots(13).$$

Substituting these values of x, y, z, in any of the first five equations, we find

$$t = 5.$$

We can arrive at the same result, more simply, by subtracting equation (1) from the three following in succession; we shall thus find

$$2t = 14 - 4, \quad 2z - t = 14 - 11, \quad 2y - 2t = 14 - 18;$$

the first of these three new equations gives $t = 5$; this value of t substituted in the two other equations gives $z = 4$, $y = 3$, and substituting these values of y, z, t, in any one of the original equations, we find $x = 2$.

By following a process of reasoning analogous to the above, we shall be able to resolve a system of any number of equations of the first degree, provided there be as many equations as unknown quantities.

It frequently happens that each of the proposed equations do not involve all the unknown quantities. In this case, a little dexterity will enable us to effect the elimination very quickly.

Example 5.

Take the system of equations,

$$\left.\begin{aligned} 2x - 3y + 2z &= 13 \ldots\ldots\ldots\ldots(1) \\ 4t - 2x &= 30 \ldots\ldots\ldots\ldots(2) \\ 4y + 2z &= 14 \ldots\ldots\ldots\ldots(3) \\ 5y + 3t &= 32 \ldots\ldots\ldots\ldots(4) \end{aligned}\right\}$$

Upon examining these equations, we perceive, that the elimination of z between equations (1) and (3) will give an equation in x and y, and that the elimination of t between equations (2) and (4) will give a second equation in x and y. These two unknown quantities may thus be easily determined:—

The elimination of z between (1) and (3) gives,…………… $7y - 2x = 1$

— — t — (2) and (4) gives,…………… $20y + 6x = 38$

Multiply the first of these equations by 3, and then add them, we have, …………… $41y = 41$

Whence, …………… $y = 1$

Substituting the value of y in $7y - 2x = 1$, we have,…… $x = 3$

Substitute this value of x in (2), we have, …………… $4t - 6 = 30$

Whence, …………… $t = 9$

Finally, the substitution of the value of y in (3), gives,…… $z = 5$

We have seen in the method of elimination by subtraction, that, in order to render the coefficients of the unknown quantity the same in both equations, we must multiply each of the equations by the coefficient of the unknown quantity which it is required to eliminate, in the other. If the coefficients of the unknown quantity have a common factor, this operation may be simplified; thus,

Example 6.

Take the system of equations,

$$\left.\begin{array}{l} 12x + 32y = 340 \ldots\ldots\ldots\ldots(1) \\ 8x + 24y = 254 \ldots\ldots\ldots\ldots(2) \end{array}\right\}$$

In order to render the coefficients of y equal, observe, that 32 and 24 have a common factor 8; it will suffice then to multiply equation (1) by 3, and equation (2) by 4, they then become,

$$36x + 96y = 1020$$
$$32x + 96y = 1016$$

Subtracting the latter from the former,

$$4x = 4$$
$$x = 1$$

Again, in order to eliminate x, since 12 and 8 have a common factor 4, it will suffice to multiply equation (1) by 2, and equation (2) by 3; we then have,

$$24x + 64y = 680$$
$$24x + 72y = 762$$

Subtracting the former of these two equations from the latter, we have,

$$8y = 82$$
$$y = 10\tfrac{1}{4}$$

Ex. 7. Given, $\left.\begin{array}{l} 5x + 7y = 43 \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ 11x + 9y = 69 \ldots\ldots\ldots\ldots\ldots\ldots(2) \end{array}\right\}$

Ans. $x = 3$, $y = 4$

Ex. 8. Given, $\left.\begin{array}{l} 8x - 21y = 33 \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ 6x + 35y = 177 \ldots\ldots\ldots\ldots\ldots\ldots(2) \end{array}\right\}$

Ans. $x = 12$, $y = 3$

Ex. 9. Given, $\left.\begin{array}{l} \frac{2x}{3} - 4 + \frac{y}{2} + x = 8 - \frac{3y}{4} + \frac{1}{12} \ldots\ldots\ldots\ldots(1) \\ \frac{y}{6} - \frac{x}{2} + 2 = \frac{1}{6} - 2x + 6 \ldots\ldots\ldots\ldots(2) \end{array}\right\}$

Ans. $x = 2$, $y = 7$

Ex. 10. Given, $\left.\begin{array}{l} x - \frac{3x + 5y}{17} + 17 = 5y + \frac{4x + 7}{3} \ldots\ldots\ldots\ldots(1) \\ \frac{22 - 6y}{3} - \frac{5x - 7}{11} = \frac{x + 1}{6} - \frac{8y + 5}{18} \ldots\ldots\ldots(2) \end{array}\right\}$

Ans. $x = 8$, $y = 2$

Ex. 11. Given, $\left.\begin{array}{l} ax + by = c \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ fx + gy = h \ldots\ldots\ldots\ldots\ldots\ldots(2) \end{array}\right\}$

Ans. $x = \frac{cg - bh}{ag - bf}$, $y = \frac{ah - cf}{ag - bf}$

Ex. 12. Given, $\left.\begin{array}{l} bcx = cy - 2b \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ b^2 y + \frac{a(c^3 - b^3)}{bc} = \frac{2b^3}{c} + c^3 x \ldots\ldots(2) \end{array}\right\}$

Ans. $x = \frac{a}{bc}$, $y = \frac{a + 2b}{c}$

Ex. 13 Given, $\left.\begin{array}{l} 5x - 6y + 4z = 15 \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ 7x + 4y - 3z = 19 \ldots\ldots\ldots\ldots\ldots\ldots(2) \\ 2x + y + 6z = 46 \ldots\ldots\ldots\ldots\ldots\ldots(3) \end{array}\right\}$

Ans. $x = 3$, $y = 4$, $z = 6$

Ex. 14. Given, $\left.\begin{array}{l} \frac{1}{x} + \frac{1}{y} = a \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ \frac{1}{x} + \frac{1}{z} = b \ldots\ldots\ldots\ldots\ldots\ldots(2) \\ \frac{1}{y} + \frac{1}{z} = c \ldots\ldots\ldots\ldots\ldots\ldots(3) \end{array}\right\}$

Ans. $x = \frac{2}{a+b-c}$, $y = \frac{2}{a-b+c}$, $z = \frac{2}{b+c-a}$

Ex. 15. Given, $\left.\begin{array}{l} \frac{x}{3} + \frac{y}{5} + \frac{2z}{7} = 58 \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ \frac{5x}{4} + \frac{y}{6} + \frac{z}{3} = 76 \ldots\ldots\ldots\ldots\ldots\ldots(2) \\ \frac{x}{2} + \frac{3z}{8} + \frac{u}{5} = 79 \ldots\ldots\ldots\ldots\ldots\ldots(3) \\ y + z + u = 248 \ldots\ldots\ldots\ldots\ldots\ldots(4) \end{array}\right\}$

Ans. $x = 12$, $y = 30$, $z = 168$, $u = 50$

Ex. 16. Given, $\left.\begin{array}{l} 7x - 2z + 3u = 17 \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ 4y - 2z + t = 11 \ldots\ldots\ldots\ldots\ldots\ldots(2) \\ 5y - 3x - 2u = 8 \ldots\ldots\ldots\ldots\ldots\ldots(3) \\ 4y - 3u + 2t = 9 \ldots\ldots\ldots\ldots\ldots\ldots(4) \\ 3z + 8u = 33 \ldots\ldots\ldots\ldots\ldots\ldots(5) \end{array}\right\}$

Ans. $x = 2$, $y = 4$, $z = 3$, $u = 3$, $t = 1$

ON THE SOLUTION OF PROBLEMS WHICH PRODUCE SIMPLE EQUATIONS.

150. Every problem which can be solved by Algebra, includes in its enunciation a certain number of conditions, by which we are enabled to detect the relations which the unknown quantities bear to the known quantities upon which they depend. These relations can always be expressed by equations, in which the known and unknown quantities are combined with each other, in a manner more or less complicated, according to the degree of difficulty in the question proposed.

It is impossible to give a general rule, which will enable us to translate every problem into algebraic language, this is a faculty which can be acquired by reflection and practice alone; we shall give a few examples, which will serve to initiate the student, and the rest must be left to his own ingenuity.

Problem 1.

To find two numbers, such, that their sum shall be 40, and their difference 16.

Let x denote the least of the two numbers required,
Then will $x + 16 =$ the greater,
And $x + x + 16 = 40$ by the question,
That is, $2x = 40 - 16 = 24$
Or $x = \frac{24}{2} = 12 =$ less number,
And $x + 16 = 12 + 16 = 28 =$ greater number required.

Problem 2.

What number is that, whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 16?

Let $x =$ number required,
Then will its $\frac{1}{3}$ part be $\frac{1}{3}x$, and its $\frac{1}{4}$ part $\frac{1}{4}x$;
And therefore $\frac{1}{3}x - \frac{1}{4}x = 16$ by the question,
That is, $x - \frac{3}{4}x = 48$, or $4x - 3x = 192$;
Hence $x = 192$, the number required.

Problem 3.

Divide £1000 among A, B, and C, so that A shall have £72 more than B, and C £100 more than A.

Let $x =$ B's share of the given sum,
Then will $x + 72 =$ A's share,
And $x + 172 =$ C's share,
And the sum of all their shares $x + x + 72 + x + 172$,
Or $3x + 244 = 1000$ by the question,
That is, $3x = 1000 - 244 = 756$,
Or $x = \frac{756}{3} =$ £252 $=$ B's share;
Hence $x + 72 = 252 + 72 =$ £324 $=$ A's share,
And $x + 172 = 252 + 172 =$ £424 $=$ C's share;

B's share,	£252
A's share,	324
C's share,	424
Sum of all,	£1000 the proof.

Problem 4.

Out of a cask of wine, which had leaked away $\frac{1}{3}$, 21 gallons were drawn, and then, being gauged, it appeared to be half full: how much did it hold?

Let it be supposed to have held x gallons,
Then it would have leaked $\frac{1}{3}x$ gallons,
Conseq. there had been taken away $21 + \frac{1}{3}x$ gallons.
But $21 + \frac{1}{3}x = \frac{1}{2}x$ by the question,
That is, $63 + x = \frac{3}{2}x$
Or $126 + 2x = 3x$
Hence $3x - 2x = 126$
Or $x = 126 =$ number of gallons required.

Problem 5.

A hare pursued by a greyhound is 60 of her own leaps in advance of the dog. She makes 9 leaps during the time that the greyhound makes only 6; but 3 leaps of the greyhound are equivalent to 7 leaps of the hare. How many leaps must the greyhound make before he overtakes the hare?

It is manifest from the enunciation of the problem, that the space which must be traversed by the greyhound, is composed of the 60 leaps which the hare is in advance, together with the space which the hare passes over from the time that the greyhound starts in pursuit until he overtakes her.

Let $x =$ the whole number of leaps made by the greyhound. Since the hare makes 9 leaps during the time that the greyhound makes 6, it follows that the hare will make $\frac{9}{6}$ or $\frac{3}{2}$ leaps during the time that the greyhound makes 1, and she will consequently make $\frac{3x}{2}$ leaps during the time that the greyhound makes x leaps.

We might here suppose, that in order to obtain the equation required, it would be sufficient to put x equal to $60 + \frac{3x}{2}$; in doing this, however, we should commit a manifest mistake, for the leaps of the greyhound are greater than the leaps of the hare, and we should thus be equating two heterogeneous numbers; that is to say, numbers related to a different unit. In order to remove this difficulty, we must express the leaps of the hare in terms of the leaps of the greyhound, or the contrary.

According to the conditions of the problem, 3 leaps of the greyhound are equal to 7 leaps of the hare; hence, 1 leap of the greyhound is equal to $\frac{7}{3}$ leaps of the hare, and consequently x leaps of the greyhound are equal to $\frac{7x}{3}$ leaps of the hare; hence we have at length the equation,

$$\frac{7x}{3} = 60 + \frac{3x}{2}$$

Clearing of fractions, $$14x = 360 + 9x$$

$$x = 72$$

Hence the greyhound will make 72 leaps before he reaches the hare, and in that time the hare will make $72 \times \frac{3}{2}$, or 108 leaps,

Problem 6.

Find a number such, that when it is divided by 3 and by 4, and the quotients afterwards added, the sum is 63.

Let x be the number, then, by the conditions of the problem, we have

$$\frac{x}{3} + \frac{x}{4} = 63$$

Clearing of fractions,

$$7x = 63 \times 12$$
$$x = 108$$

If we wished to find a number such, that when divided by 5 and by 6, the sum of the quotients is 22, we must again translate the problem into algebraic language, and then solve the equation, in this case we have

$$\frac{x}{5} + \frac{x}{6} = 22$$

Clearing of fractions,

$$11x = 22 \times 30$$
$$x = 60$$

If, however, we desire to solve both these problems at once, and all others of the same class, which differ from the above in the numerical values only, we must substitute for these particular numbers, the symbols $a, b, c,$ ----, which may represent any numbers whatever, and then solve the following question.

Find a number such, that when it is divided by a and by b, and the quotients afterwards added, the sum is p. We have

$$\frac{x}{a} + \frac{x}{b} = p$$
$$(a+b)x = abp$$
$$x = \frac{abp}{a+b}$$

151. This expression is not, strictly speaking, the value of the unknown quantity in our problems, but it presents to our view the calculations which are requisite for the solution of them all. An expression of this nature is called a *formula.* This formula points out to us that the unknown quantity is obtained by multiplying together the three numbers involved in the question, and then dividing their product abp by $a+b$, the sum of the two divisors; or we should rather say, that our formula is a concise method of enunciating the above rule. Algebra, then, may be considered as a language whose object is to express various processes of reasoning, a language which we must be able to write and to read.

Such is the advantage of the above formula, that, by aid of it, the most ignorant Arithmetician could solve either of the proposed problems as readily as the most expert algebraist. The former, however, could only arrive at the result by a blind reliance on his rule; different kinds of problems, moreover, require

different formulæ, and the algebraist alone possesses the secret by which they can be discovered.

Problem 7.

A labourer engaged to serve 40 days, upon these conditions; that for every day he worked, he was to receive 20d., but for every day he was idle he was to forfeit 8d. Now at the end of the time, he was entitled to receive £1 11s. 8d. It is required to find how many days he worked, and how many he was idle?

Let x be the numbers of days he worked,
Then will $40 - x$ be the number of days he was idle,
Also $x \times 20 = 20x =$ the sum earned,
And $(40 - x) \times 8 = 320 - 8x =$ sum forfeited,
Hence $20x - (320 - 8x) = 380\text{d.} = £1\ 11\text{s.}\ 8\text{d.}$ by the question;
That is, $20x - 320 + 8x = 380$,
Or $28x = 380 + 320 = 700$,
Hence $x = \frac{700}{28} = 25 =$ numbers of days he worked,
And $40 - x = 40 - 25 = 15 =$ number of days he was idle.

We may generalize the above problem in the following manner:

Let $n =$ the whole number of days for which he is hired,
$a =$ the wages for each day of work,
$b =$ the forfeit for each day of idleness,
$c =$ the sum which he receives at the end of n days,
$x =$ the number of days of work,
Then $n - x =$ the number of days of idleness,
$ax =$ the sum due to him for the days of work,
$b(n - x) =$ the sum he forfeits for the days of idleness.

We thus find for the equation of the problem,

$$ax - b(n - x) = c$$

Whence

$$ax - bn + bx = c$$
$$(a + b)x = c + bn$$
$$x = \frac{c + bn}{a + b} \text{ the number of days of work.}$$

And $\therefore$

$$n - x = n - \frac{c + bn}{a + b}$$
$$= \frac{an + bn - c - bn}{a + b}$$
$$= \frac{an - c}{a + b} \text{ the number of days of idleness.}$$

Problem 8.

A can perform a piece of work in 6 days, B can perform the same work in 8 days: in what time will they finish it if both work together?

Let $x =$ the time required.

Since A can perform the whole work in 6 days, $\frac{1}{6}$ will denote the quantity he can perform in 1 day, and therefore $\frac{x}{6}$ the quantity he can perform in x days; for the same reason, $\frac{x}{8}$ will be the quantity which B can perform in x days and we shall thus have

$$\begin{aligned} \frac{x}{6} + \frac{x}{8} &= 1 \\ 14x &= 48 \\ x &= 3\tfrac{3}{7} \text{ days.} \end{aligned}$$

Let us generalize the above problem.

A can perform a piece of work in a days, B in b days, C in c days, D in d days: in what time will they perform it if they all work together?

Let $x =$ the time;

Then, since A can perform the whole work in a days, $\frac{1}{a}$ will denote the quantity he can perform in 1 day, and consequently $\frac{x}{a}$ will be the quantity he can perform in x days; for the same reason, $\frac{x}{b}$, $\frac{x}{c}$, $\frac{x}{d}$, will be the quantities which B, C, D, can perform respectively in x days; we thus have

$$\begin{aligned} \frac{x}{a} + \frac{x}{b} + \frac{x}{c} + \frac{x}{d} &= \text{(whole work,)} \\ &= 1 \\ \therefore \qquad x &= \frac{abcd}{abc + abd + acd + bcd} \end{aligned}$$

Problem 9.

A courier, who travelled at the rate of $31\frac{1}{2}$ miles in 5 hours, was despatched from a certain city; 8 hours after his departure, another courier was sent to overtake him. The second courier travelled at the rate of $22\frac{1}{2}$ miles in 3 hours. In what time did he overtake the first, and at what distance from the place of departure?

Let $x =$ number of hours that the second courier travels.

Then, since the first courier travels at the rate of $31\frac{1}{2}$ miles in 5 hours, that is, $\frac{63}{10}$ miles in 1 hour, he will travel $\frac{63}{10}x$ miles in x hours, and since he started 8 hours before the second courier, the whole distance travelled by him will be $(8 + x)\frac{63}{10}$.

Again since the second courier travels at the rate of $22\frac{1}{2}$ miles in 3 hours, that is, $\frac{45}{6}$ miles in 1 hour, he will hence travel $\frac{45}{6}x$ miles in x hours.

The couriers are supposed to be together at the end of the time x, and therefore the distance travelled by each must be the same; hence,

$$\begin{aligned}
\frac{45}{6}x &= (8+x)\frac{63}{10} \\
450\,x &= (8+x)\,378 \\
\therefore\ 72\,x &= 3024 \\
x &= 42
\end{aligned}$$

Hence, the second courier will overtake the first in 42 hours, and the whole distance travelled by each is $\frac{45}{6} \times 42 = 315$ miles

To generalize the above,

A B C

Let a courier, who travels at the rate of m miles in t hours, be despatched from B in the direction C; and n hours after his departure, let a second courier, who travels at the rate of m' miles in t' hours, be sent from A, which is distant d miles from B, in order to overtake the first. In what time will he come up with him, and what will be the whole distance travelled by each?

Let $x =$ number of hours that the second courier travels.

Then, since the first courier travels at the rate of m miles in t hours, that is, $\frac{m}{t}$ miles in 1 hour, he will travel $\frac{m}{t}x$ miles in x hours, and since he started n hours before the second courier, the whole distance travelled by him will be $(n+x)\frac{m}{t}$.

Again, since the second courier travels at the rate of m' miles in t' hours, that is, $\frac{m'}{t'}$ miles in 1 hour, he will travel $\frac{m'}{t'}x$ miles in x hours; but since he started from A, which is distant d miles from B, the whole distance travelled by the second courier, or $\frac{m'}{t'}x$ will be greater than the whole distance travelled by the first courier, by this quantity d; hence,

$$\begin{aligned}
\frac{m'}{t'}x - d &= (n+x)\frac{m}{t} \\
\left(\frac{m'}{t'} - \frac{m}{t}\right)x &= \frac{m\,n}{t} + d \\
\therefore\ x &= \frac{(m\,n + t\,d)\,t'}{m'\,t - m\,t'}
\end{aligned}$$

The whole distance travelled by first courier $= \frac{m}{t}\cdot\left\{\frac{(m\,n+t\,d)\,t'}{m'\,t - m\,t'} + n\right\}$

The whole distance travelled by second courier $= \frac{m'}{t'}\cdot\frac{(m\,n+t\,d)\,t'}{m'\,t - m\,t'} - d$

Problem 10.

A father, who has three children, bequeaths his property by will in the following manner: To the eldest son he leaves a sum a, together with the n^{th} part of what remains; to the second he leaves a sum $2\,a$, together with the n^{th} part of what remains after the portion of the eldest and $2\,a$ have been subtracted; to the third he leaves a sum $3\,a$, together with the n^{th} part of what remains after

the portions of the two other sons and 3 a have been subtracted. The property is found to be entirely disposed of by this arrangement. Required the amount of the property.

Let $x =$ the property of the father.

If we can, by means of this quantity, find algebraic expressions for the portions of the three sons, we must subtract their sums from the whole property x, and putting this remainder $= 0$ we shall determine the equation of the problem.

Let us endeavour to discover these three portions.

Since x represents the whole property of the father, $x - a$ is the remainder after subtracting a; hence,

$$\text{Portion of eldest son,} = a + \frac{x-a}{n}$$

$$= \frac{an + x - a}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{Portion of second son,} = 2a + \frac{x - 2a - \dfrac{an + x - a}{n}}{n}$$

$$= 2a + \frac{nx - 3an - x + a}{n^2}$$

$$= \frac{2an^2 + nx - 3an - x + a}{n^2} \quad \ldots\ldots\ldots\ldots (2)$$

$$\text{Portion of third son,} = 3a + \frac{x - 3a - \dfrac{an+x-a}{n} - \dfrac{2an^2+nx-3an-x+a}{n^2}}{n}$$

$$= 3a + \frac{n^2x - 6an^2 - 2nx + 4an + x - a}{n^3}$$

$$= \frac{3an^3 + n^2x - 6an^2 - 2nx + 4an + x - a}{n^3} \quad (3)$$

According to the conditions of the problem, the property is entirely disposed of. Hence, when the sum of the three portions is subtracted from x, the difference must be equal to zero; this gives us the equation

$$x - \frac{an+x-a}{n} - \frac{2an^2+nx-3an-x+a}{n^2} - \frac{3an^3+n^2x-6an^2-2nx+4an+x-a}{n^3} = 0;$$

clearing the equation of fractions, and reducing,

$$n^3x - 6an^3 - 3n^2x + 10an^2 + 3nx - 5an - x + a = 0$$

$$\therefore (n^3 - 3n^2 + 3n - 1)x = 6an^3 - 10an^2 + 5an - a =$$

$$x = \frac{6an^3 - 10an^2 + 5an - a}{n^3 - 3n^2 + 3n - 1} = \frac{(6n^3 - 10n^2 + 5n - 1)a}{(n-1)^3}$$

By reflecting upon the conditions of the problem, we may obtain an equation much more simple than the preceding. It is stated that the portion of the third son is 3 a, together with the n^{th} of what remains, and that the property is thus

entirely disposed of; in other words, the portion of the third son is $3a$, and the remainder just mentioned is nothing.

We found the expression for that remainder to be

$$\frac{n^2x - 6an^2 - 2nx + 4an + x - a}{}$$

Equating this quantity to zero, we have

$$\frac{n^2x - 6an^2 - 2nx + 4an + x - a}{n^2} = 0$$

$$\therefore\ n^2x - 6an^2 - 2nx + 4an + x - a = 0$$

$$(n^2 - 2n + 1)x = 6an^2 - 4an + a$$

$$x = \frac{6an^2 - 4an + a}{n^2 - 2n + 1}$$

$$= \frac{(6n^2 - 4n + 1)a}{(n-1)^2}$$

This result is, moreover, more simple than the former. We can easily prove that the two expressions are *numerically identical*, for applying to the two polynomials $(6n^3 - 10n^2 + 5n - 1)a$, and $(n^3 - 3n^2 + 3n + 1)$, the process for finding the greatest common measure, we shall find that these two expressions have a common factor $n - 1$; dividing, therefore, both terms of the first result by this common factor, we arrive at the second.

The above problem will point out to the student the importance of examining with great attention the enunciation of any proposed question, in order to discover those circumstances which may tend to facilitate the solution; he will otherwise run the risk of arriving at results more complicated than the nature of the case demands.

The above problem admits of a solution less direct, but more simple and elegant than those given above. It is founded on the observation, that after having subtracted $3a$ from the former portions, nothing ought to remain.

Let us represent by r_1, r_2, r_3, the three remainders mentioned in the enunciation; the algebraic expressions for the three portions must be,

$$a + \frac{r_1}{n}, \quad 2a + \frac{r_2}{n}, \quad 3a + \frac{r_3}{n}$$

1°. By the conditions of the problem we have $r_3 = 0$.

Hence the third portion is $3a$.

2°. The remainder, after the second son has received $2a + \frac{r_2}{n}$ may be represented by $r_2 - \frac{r_2}{n}$, or $\frac{(n-1)r_2}{n}$

But this is the portion of the third son, hence we have

$$\frac{(n-1)r_2}{n} = 3a$$

$$\therefore r_2 = \frac{3an}{n-1}$$

Hence, the portion of the second son is $2a + \frac{3an}{n-1} \div n = 2a + \frac{3a}{n-1}$, or reducing,

$$\frac{2an + a}{n-1}.$$

3°. The remainder, after the eldest son has received $a + \frac{r_1}{n}$, may be represented by $r_1 - \frac{r_1}{n}$, or $\frac{(n-1)r_1}{n}$.

But this remainder forms the portion of the two other sons, hence we have

$$\frac{(n-1)r_1}{n} = \frac{2an+a}{n-1} + 3a$$
$$\therefore r_1 = \frac{5an^2 - 2an}{(n-1)^2}.$$

Hence, the portion of the eldest son is $a + \frac{5an^2 - 2an}{(n-1)^2} \div n = a + \frac{5an - 2a}{(n-1)^2}$, or reducing

$$\frac{an^2 + 3an - a}{n^2 - 2n + 1}.$$

Hence, the whole property is

$$3a + \frac{2an + a}{n-1} + \frac{an^2 + 3an - a}{n^2 - 2n + 1}$$

reducing the whole to a common denominator,

$$\frac{3a(n^2 - 2n + 1) + (2an + a)(n-1) + an^2 + 3an - a}{n^2 - 2n + 1},$$

performing the operations indicated, and reducing

$$\frac{(6n^2 - 4n + 1)a}{n^2 - 2n + 1}.$$

the result obtained above.

This solution is more complete than the former, for we obtain at the sam time the property of the father and the expressions for the portions of his thr sons.

We shall now solve one or two problems, in which it is either necessary or convenient to employ more than one unknown quantity.

Problem 11.

Required two numbers, whose sum is 70 and whose difference is 16.

Let x and y be the two numbers; then, by the conditions of the problem,

$$x + y = 70 \ldots\ldots\ldots\ldots (1)$$
$$x - y = 16 \ldots\ldots\ldots\ldots (2)$$

which are the two equations required for its solution.

Adding the two equations,

$$2x = 86$$
$$x = 43$$

Subtracting the second from the first,

$$2y = 54$$
$$y = 27$$

Hence 43 and 27 are the two numbers.

Problem 12.

A person has two kinds of gold coin, 7 of the larger together with 12 of the smaller make 288 shillings; and 12 of the larger together with 7 of the smaller make 358 shillings. Required the value of each kind of coin.

Let x be the value of the larger coin expressed in shillings, y that of the smaller.

Then, by the conditions of the problem,

$$7x + 12y = 288 \text{..............................} (1)$$

And,

$$12x + 7y = 358 \text{..............................} (2)$$

Multiplying equation (1) by 7, and equation (2) by 12, and subtracting the former product from the latter,........................ $95x = 2280$

$$\therefore x = 24$$

Substituting this value of x in equation (1), it becomes,... $168 + 12y = 288$

$$\therefore y = 10$$

The larger of the two coins is worth 24 shillings, the smaller 10 shillings.

Problem 13.

An individual possesses a capital of £30,000, for which he receives interest at a certain rate; he owes, however, £20,000, for which he pays interest at a certain rate. The interest he receives exceeds that which he pays by £800. Another individual possesses a capital of £35,000, for which he receives interest at the second of the above rates; he owes, however, £24,000, for which he pays interest at the first of the above rates. The interest which he receives exceeds that which he pays by £310. Required the two rates of interest?

Let x and y denote the two rates of interest for £100.

In order to find the interest of £30,000 at the rate x, we have the proportion,

$$100 : 30{,}000 :: x : \frac{30{,}000\,x}{100} = 300\,x$$

In like manner to find the interest of £20,000 at the rate of y

$$100 : 20{,}000 :: y : \frac{20{,}000\,y}{100} = 200\,y$$

But, by the enunciation of the problem, the difference of these two sums is £800, hence we shall have, for the first equation,

$$300\,x - 200\,y = 800 \ldots\ldots (1)$$

Translating, in like manner, the second condition of the problem into algebraic language, we arrive at the second equation

$$350\,y - 240\,x = 310 \ldots\ldots (2)$$

The two members of the first equation are divisible by 100, and those of the second by 10; they may therefore be replaced by the following

$$3\,x - 2\,y = 8 \ldots\ldots (3)$$
$$35\,y - 24\,x = 31 \ldots\ldots (4)$$

In order to eliminate x, multiply equation (3) by 8, and then add equation (4), hence,

$$19\,y = 95$$
$$\therefore \; y = 5$$

Substituting this value of y in equation (3), we have

$$3\,x - 10 = 8$$
$$\therefore \; x = 6$$

Then the first rate of interest is 6 per cent., and the second 5 per cent.

Problem 14.

An artizan has three ingots composed of different metals melted together. A pound of the first contains 7 oz. of silver, 3 oz. of copper, and 6 oz. of tin. A pound of the second contains 12 oz. of silver, 3 oz. of copper, and 1 oz. of tin. A pound of the third contains 4 oz. of silver, 7 oz. of copper, and 5 oz. of tin. How much of each of these three ingots must he take in order to form a fourth, each pound of which shall contain 8 oz. of silver, $3\frac{3}{4}$ oz. of copper, and $4\frac{1}{4}$ oz. of tin?

Let x, y, and z be the number of ounces which he must take in each of the ingots respectively, in order to form a pound of the ingot required?

Since, in the first ingot, there are 7 oz. of silver in a pound of 16 oz. it follows, that in 1 oz. of the ingot there are $\frac{7}{16}$ oz. of silver, and consequently in x oz. of the ingot there must be $\frac{7\,x}{16}$ oz. of silver. In like manner, we shall find that $\frac{12\,y}{16}$, $\frac{4\,z}{16}$ represent the number of ounces of silver taken in the third and fourth ingots in order to form the fourth; but, by the conditions of the problem, the fourth ingot is to contain 8 oz. of silver, we shall thus have

$$\frac{7\,x}{16} + \frac{12\,y}{16} + \frac{4\,z}{16} = 8 \ldots\ldots (1)$$

And reasoning precisely in the same manner for the copper and tin, we find

$$\frac{3\,x}{16} + \frac{3\,y}{16} + \frac{7\,z}{16} = \frac{15}{4} \ldots\ldots (2)$$
$$\frac{6\,x}{16} + \frac{y}{16} + \frac{5\,z}{16} = \frac{17}{4} \ldots\ldots (3)$$

which are the three equations required for the solution of the problem.

Clearing them of fractions they become

$$7\,x + 12\,y + 4\,z = 128 \text{ .... } (4)$$
$$3\,x + 3\,y + 7\,z = 60 \text{ .... } (5)$$
$$6\,x + y + 5\,z = 68 \text{ .... } (6)$$

In these three equations the coefficients of y are most simple; it will, therefore, be convenient to eliminate the unknown quantity first.

Multiply equation (5) by 4, and subtract equation (4), from the product, we have......... $5\,x + 24\,z = 112$......(7)

Multiply equation (6) by 3, and subtract equation (5) from the product, we have......... $15\,x + 8\,z = 144$......(8)

Multiply equation (8) by 3, and subtract equation (7) from the product, we have .... $40\,x = 320$

$\therefore\ x = 8$

Substitute this value of x in equation (8), it becomes $120 + 8\,z = 144$

$\therefore\ z = 3$

Substitute these values of x and z in equation (6), it becomes .... $48 + y + 15 = 68$

$\therefore\ y = 5$

Hence, in order to form a pound of the fourth ingot, we must take 8 ounces of the first, 5 ounces of the second, and 3 ounces of the third.

Problem 15.

There are three workmen, A, B, C. A and B together can perform a certain piece of labour in a days. A and C together in b days, and B and C together in c days. In what time could each, singly, execute it, and in what time could they finish it if all worked together?

Let x = time in which A alone could complete it.
y = time in which B alone could complete it.
z = time in which C alone could complete it.

Since A and B together can execute the whole in a days, the quantity which they perform in one day is $\frac{1}{a}$, and since A alone could do the whole in x days, the quantity he could perform in one day is $\frac{1}{x}$; for the same reason, the quantity which B could perform in one day is $\frac{1}{y}$; the sum of what they could do singly must be equal to the quantity they can do together, hence,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{a} \text{ .... } (1)$$

In like manner we shall have

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{b} \text{ .... } (2)$$
$$\frac{1}{y} + \frac{1}{z} = \frac{1}{c} \text{ .... } (3)$$

Subtract equation (3) from (1)

$$\frac{1}{x} - \frac{1}{z} = \frac{1}{a} - \frac{1}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Add equations (2) and (4)

$$\frac{2}{x} = \frac{1}{a} + \frac{1}{b} - \frac{1}{c}$$

$$\therefore x = \frac{2abc}{ac + bc - ab}$$

In like manner,

$$y = \frac{2abc}{ab + bc - ac}$$

$$z = \frac{2abc}{ab + ac - bc}$$

Let t be the time in which they could finish it if all worked together, then by Prob. 8.

$$t\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 1$$

$$\therefore t\left(\frac{1}{a} + \frac{1}{z}\right) = 1$$

$$t\left(\frac{1}{a} + \frac{ab + ac - bc}{2abc}\right) =$$

$$\therefore t = \frac{2abc}{ab + ac + bc}$$

Prob. 16. What two numbers are those whose difference is 7, and sum 33?
Ans. 13 and 20

Prob. 17. To divide the number 75 into two such parts, that three times the greater may exceed 7 times the less by 15. Ans. 54 and 21.

Prob. 18. In a mixture of wine and cyder, $\frac{1}{2}$ of the whole *plus* 25 gallons was wine, and $\frac{1}{3}$ part *minus* 5 gallons was cyder; how many gallons were there of each? Ans. 85 of wine, and 35 of cyder.

Prob. 19. A bill of 120*l.* was paid in guineas and moidores, and the number of pieces of both sorts that were used was just 100; how many were there of each? Ans. 50 of each.

Prob. 20. Two travellers set out at the same time from London and York, whose distance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet? Ans. In 10 days.

Prob. 21. At a certain election 375 persons voted, and the candidate chosen had a majority of 91; how many voted for each?
Ans. 233 for one, and 142 for the other.

Prob. 22. What number is that from which, if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? Ans. 65.

Prob. 23. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is its whole length? Ans. 24 feet.

Prob. 24. There is a fish whose tail weighs 9lb. his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fish? Ans. 72lb.

Prob. 25. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse; what was in it at first? Ans. 120 guineas.

Prob. 26. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each?

Ans. A's = 84, B's = 42, and C's = 14.

Prob. 27. Two persons, A and B, lay out equal sums of money in trade; A gains £126, and B loses £87, and A's money is now double of B's; what did each lay out? Ans. £300

Prob. 28. A person bought a chaise, horse, and harness, for £60, the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each?

Ans. £13 6s. 8d. for the horse, £6 13s. 4d. for the harness, and £40 for the chaise.

Prob. 29. Two persons, A and B, have both the same income: A saves $\frac{1}{5}$ of his yearly, but B, by spending £50 per annum more than A, at the end of 4 years finds himself £100 in debt; what is their income? Ans. £125.

Prob. 30. A person has two horses, and a saddle worth £50: now, if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first; what is the value of each horse.

Ans. One £30, and the other £40.

Prob. 31. To divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other?

Ans. The parts are 8, 12, and 16.

Prob. 32. A footman agreed to serve his master for £8 a year, and a livery; but was turned away at the end of 7 months, and received only £2 13s. 4d. and his livery; what was its value? Ans. £4 16s.

Prob. 33. A person was desirous of giving 3d. a-piece to some beggars, but found that he had not money enough in his pocket by 8d.; he therefore gave them each 2d., and had then 3d. remaining; required the number of beggars?

Ans. 11.

Prob. 34. A person in play lost $\frac{1}{4}$ of his money, and then won 3s.; after which, he lost $\frac{1}{3}$ of what he then had, and then won 2s.; lastly, he lost $\frac{1}{7}$ of what he then had: and, this done, found he had but 12s. remaining; what had he at first? Ans. 20s.

Prob. 35. To divide the number 90 into 4 such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2; the sum, difference, product, and quotient, shall be all equal to each other? Ans. The parts are 18, 22, 10, and 40, respectively.

Prob. 36. The hour and minute hand of a clock are exactly together at 12 o'clock; when are they next together? Ans. 1 hour $5\frac{5}{11}$ minutes.

Prob. 37. There is an island 73 miles in circumference, and three footmen all start together to travel the same way about it: A goes 5 miles a day, B 8, and C 10; when will they all come together again? Ans. 73 days.

Prob. 38. How much foreign brandy at 8s. per gallon, and British spirits at 3s. per gallon, must be mixed together, so that in selling the compound at 9s. per gallon, the distiller may clear 30 per cent.?

Ans. 51 gallons of brandy, and 14 of spirits.

Prob. 39. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days; how many days would the man alone be in drinking it? Ans. 20 days.

Prob. 40. If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days: how many days will it take each person to perform the same work alone?

Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

Prob. 41. A book is printed in such a manner, that each page contains a certain number of lines, and each line a certain number of letters. If each page were required to contain 3 lines more, and each line 4 letters more, the number of letters in a page would be greater by 224 than before; but if each page were required to contain 2 lines less, and each line 3 letters less, the number of letters in a page would be less by 145 than before. Required the number of lines in each page, and the number of letters in each line?

Ans. 29 lines, 32 letters.

Prob. 42. Five gamblers, A, B, C, D, E, throw dice, upon the condition that he who has the lowest throw shall give all the rest the sum they have already. Each gamester loses in turn, commencing with A, and at the end of the fifth game, all have the same sum, viz. £32. How much had each at first?

Ans. A £81, B £41, C £21, D £11, E £6.

Prob. 43. To divide a number a into two parts, which shall have to each other the ratio of m to n.

Ans. $\frac{ma}{m+n}, \frac{na}{m+n}$

Prob. 44. To divide a number a into three parts, which shall be to each other as $m:n:p$.

Ans. $\frac{ma}{m+n+p}, \frac{na}{m+n+p}, \frac{pa}{m+n+p}$.

Prob. 45. A banker has two kinds of change; there must be a pieces of the first to make a crown, and b pieces of the second to make the same: now, a person wishes to have c pieces for a crown. How many pieces of each kind must the banker give him?

Ans. $\frac{a(b-c)}{b-a}$ of the first kind, $\frac{b(c-a)}{b-a}$ of the second.

Prob. 46. A sportsman promises to pay a friend a shillings for each shot he misses, upon condition that he is to receive b shillings for each shot he hits. After n shots, it may happen that the two friends are quits, or that the first owes the second c shillings, or the contrary. Required a formula which shall comprehend all the three cases, and which shall give x the number of shots missed.

Ans. $x = \frac{bn \pm c}{a+b}$.

In the first case, $c = o$, in the second case we must take the positive sign, in the third case the negative sign.

Prob. 47 If one of two numbers be multiplied by m, and the other by n, the sum of the products is p; but if the first be multiplied by m', and the second by n', the sum of the products is p'. Required the two numbers.

Ans. $\frac{n'p - np'}{mn' - m'n}$, $\frac{mp' - m'p}{mn' - m'n}$.

Prob. 48. An ingot of metal which weighs n pounds, loses p pounds when weighed in water. This ingot is itself composed of two other metals, which we may call M and M'; now, n pounds of M loses q pounds when weighed in water, and n pounds of M' loses r pounds when weighed in water. How much of each metal does the original ingot contain?

Ans. $\frac{n(r-p)}{r-q}$ pounds of M, $\frac{n(p-q)}{r-q}$ pounds of M'.

REMARKS UPON EQUATIONS OF THE FIRST DEGREE.

152. Algebraic formulæ can offer no distinct ideas to the mind, unless they represent a succession of numerical operations which can be actually performed. Thus, the quantity $b - a$, when considered by itself alone, can only signify an absurdity when $a > b$. It will be proper for us, therefore, to review the preceding calculations, since they sometimes present this difficulty.

Every equation of the first degree may be reduced to one which has all its signs positive, such as,

$$ax + b = cx + d \quad \ldots\ldots\ldots (1)*$$

Subtracting $cx + b$ from each member, we then have,

$$ax - cx = d - b$$

Whence,

$$x = \frac{d-b}{a-c} \quad \ldots\ldots\ldots (2)$$

* We can always change the negative terms of an equation into others which are positive, since we can always add any quantity to both members.

This being premised, three different cases present themselves,

1°. $d > b$, and, $a > c$.
2°. One of these conditions only may hold good.
3°. $b > d$, and, $c > a$.

In the first case, the value of x in equation (2), resolves the problem without giving rise to any embarrassment; in the second and third cases, it does not at first appear what signification we ought to attach to the value of x, and it is this that we propose to examine.

In the second case, one of the subtractions, $d - b$, $a - c$, is impossible: for example, let $b > d$ and $a > c$, it is manifest that the proposed equation (1) is absurd, since the two terms $a x$ and b of the first member are respectively greater than the two terms $c x$ and d of the second. Hence, when we encounter a difficulty of this nature, we may be assured that the proposed problem is absurd, since the equation is merely a faithful expression of its conditions in algebraic language.

In the third case, we suppose $b > d$, and $c > a$; here both subtractions ar impossible: but let us observe, that in order to solve equation (1), we subtracted from each member the quantity $c x + b$, an operation manifestly impossible, since each member $< c x + b$. This calculation being erroneous, let us subtract $a x + d$ from each member, we then have,

$$b - d = cx - ax$$

Whence,

$$x = \frac{b - d}{c - a} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

This value of x, when compared with equation (2), differs from it in this only, that the signs of both terms of the fraction have been changed, and the solution is no longer obscure. We perceive, that when we meet with this third case, it points out to us, that instead of transposing all the terms involving the unknown quantity, to the first member of the equation, we ought to place them in the second; and that it is unnecessary, in order to correct this error, to recommence the calculation,—it is sufficient to change the signs of both numerator and denominator.

One of the principal advantages which algebra holds out, is to enable us to obtain formulæ which shall include every variety of the same problem, whatever the numbers may be which it involves. We shall here attain this object by establishing a convention, *to perform upon isolated negative quantities the same operations as if they were accompanied by other magnitudes.* For example, if we had an expression, $m + d - b$, and $b > d$, we might express it, $m - (b - d)$; if m does not exist, by our *convention*, we still write, $d - b = -(b - d)$, when $b > d$.

The value of x, in the second case, becomes $x = -\frac{b - d}{a - c}$, and we conclude, from what has been said above, that *every negative solution denotes an absurdity in the condition of the question proposed.*

In order to divide the polynomial $-a^4 + 3a^2 b^2 + \&c.$ by $-a^2 + b^2 + \&c.$ we divide the first term $-a^4$ by $-a^2$, and we know (Art. 18.) that the quotient a^2 has the sign $+$. We may say the same of the isolated negative quantities,

a^4, $-a^2$; so that, in the third case, the value of x will be of the form $\frac{-(b-d)}{-(c-a)}$, which reduces itself to $\frac{b-d}{c-a}$, the true result in equation (3.)

This convention, which cannot occasion any inconvenience, enables us to include all the cases in the single formula (2). But, we must remember, that the isolated negative quantities $-n$, $\frac{-p}{-q}$, &c. derive their origin from a convention, that they are *symbols*, which have no existence in themselves, and that we treat them as real quantities only because we are thus enabled to accomplish an important object, without the fear of any embarrassment in consequence. In fact, one of two things must happen, either the result will have the sign $-$, from which we conclude that the problem proposed is absurd, the sign $-$ being a symbol which marks the absurdity; or the result will have the sign $+$, and we have proved that this is what it ought to be, although arising from the division of two negative quantities.*

When the equation is absurd, we may nevertheless make use of the negative solution obtained in the second case; for if we substitute $-x$ for $+x$, the proposed equation becomes

$$-ax+b = -cx+d$$

Whence $$x = \frac{b-d}{c-a},$$

a value equal to that in (2), but positive. If, then, we modify the question, in such a manner as to agree with this new equation, this second problem, which will bear a marked resemblance to the first, will no longer be absurd, and, with the exception of the sign, will have the same solution.

Let us take, for example, the following problem.

* The following remarks upon the same subject are from "The Principles of Analytical Calculation," by the late Professor Woodhouse.:

"Since, independently of arbitrary appointment, the symbols of real quantities alone, can be made the object of demonstration, if any rule is laid down for the multiplication of $-a$ by $\pm b$, or of $x-y$ by $\pm b$ (where $x < y$) such rule must be subsequent to certain conventions: the object to be obtained by extending a rule, or, what is the same thing, by making it in part arbitrary, is commodiousness of calculation.

"When $\pm a$ and $-b$ are separately proposed, no reasoning can establish any thing concerning the signs of their product, but since it has appeared that $(x-y)\times a = xa-ya$, where $x > y$, and that $(v-w)\times b = vb-wb$ where $v > w$, suppose $x-y = v-w$ and $a = b$, then by the rule for transposition, these equations may appear under the form $y-x = w-v$ and $-a = -b$, and if $y-x$ be multiplied by $-a$, and $w-v$ by $-b$, making the product of like signs $+$ and the product of unlike signs $-$, the result is $-ya+xa = -wb+vb$, or $xa-ya = vb-wb$, which is known to be a true equation.

"Hence it appears, that the rule for the multiplication of signs may be made general, since it can be proved true in all cases where actual multiplication is to be performed, and since, employed in mere algebraic calculation it cannot lead into error, because the result must always be considered with reference to the premises; thus, if $a\times a$, and $-a\times -a$ be expressed by a^2, the square root of a^2 or that quantity which algebraically multiplied into itself produces a^2, must be put either $+a$ or $-a$, or abridgedly expressed, $\sqrt{a^2} = \pm a$.

"If the rule be not made general, then during the process of calculation, were it necessary to multiply such a quantity as $x-y$ by $a-b$, or c, nothing could be affirmed concerning such multiplication, until it had been ascertained whether a was greater than b, and x greater than y: what tedious impediments would thus be made to clog calculation it is easy to conceive; as this rule for the multiplication of signs embarrasses beginners, and has been frequently made the subject of discussion. I have dwelt rather long upon it, desirous to distinguish in the rule, what may be said to be proved from evident principles and strict reasoning, from what is arbitrary or results from convention, and to show why it is desirable on the grounds of commodiousness to make such a rule general, and why on the score of accuracy and precision it might safely be made general."

A father aged 42 years, has a son aged 12: *in how many years will the age of the son be one-fourth of that of the father?*

Let $x =$ the number of years required;

$$\text{Then} \quad \frac{42 + x}{4} = 12 + x$$

$$\therefore \quad x = -2$$

Thus the problem is absurd. But if we substitute $-x$ for $+x$, the equation becomes

$$\frac{42 - x}{4} = 12 - x$$

and the conditions corresponding to this equation change the problem to the following:

A father aged 42 years, has a son aged 12: *how many years have elapsed since the age of the son was one-fourth of that of the father?*

$$\text{Here} \quad x = 2$$

What number is that, the sum of the third and fifth parts of which, diminished by 7, is equal to the original number? Here,

$$\frac{x}{3} + \frac{x}{5} - 7 = x$$

$$\text{Whence} \quad x = -15$$

The problem is absurd; but substituting $-x$ for $+x$, (or rather $+7$ for -7,) we perceive that 15 is the number, the third and fifth parts of which, *when added to* 7, give the original number 15.

153. With regard to the interpretation of negative results, in the solution of problems, we may establish the following general principle:

When we find a negative value for the unknown quantity in problems of the first degree, it points out an absurdity in the conditions of the problem proposed; provided the equation be a faithful representation of the problem, and of the true meaning of all the conditions.

The value so obtained, neglecting its sign, may be considered as the answer to a problem which differs from the one proposed in this only, that certain quantities which were additive in the first, have become subtractive in the second, and reciprocally.

154. The equation (2) presents still two varieties. If $a = c$, we have

$$x = \frac{d - b}{o}$$

in this case the original equation becomes

$$ax + b = ax + d$$

whence $b = d$; if, therefore, b be not equal to d, the problem is absurd, and cannot be modified as above.

The expression $\frac{d-b}{o}$, or in general $\frac{m}{o}$, when m may be any quantity, represents a number *infinitely great.* For, if we take a fraction $\frac{m}{n}$, the smaller we make n, the greater will the number represented by $\frac{m}{n}$ become; thus for

$=\frac{1}{2}, \frac{1}{100}, \frac{1}{1000}$, the results are 2, 100, 1000 times as great as m. The limit is *infinity*, which corresponds to $n = 0$: we perceive then, that *a problem is absurd when the solution is a number infinitely great*; this is represented by the symbol

$$x = \infty \text{ *}$$

155. If, however, $a = c$, and $b = d$, we have

$$x = \frac{0}{0}$$

in this case the original equation becomes

$$ax + b = ax + b$$

here the two members of the equation are equal, whatever may be the value of x, which is altogether arbitrary. We perceive then, that *a problem is indeterminate, and is susceptible of an infinite number of solutions, when the value of the unknown quantity appears under the form* $\frac{0}{0}$

It is, however, highly important to observe, that the expression $\frac{0}{0}$ does not always indicate that the problem is indeterminate, but merely the existence of a factor common to both terms of the fraction, which factor becomes 0 under a particular hypothesis.

Suppose, for example, that the solution of a problem is exhibited under the form $x = \frac{a^3 - b^3}{a^2 - b^2}$.

If, in this formula, we make $a = b$, then, $x = \frac{0}{0}$.

But we must remark, that $a^3 - b^3$ may be put under the form $(a - b)(a^2 + ab + b^2)$, and that $a^2 - b^2$ is equivalent to $(a - b)(a + b)$; hence, the above value of x will be,

$$x = \frac{(a-b)(a^2 + ab + b^2)}{(a-b)(a+b)}$$

Now, if before making the hypothesis $a = b$, we suppress the common factor $a - b$, the value of x becomes,

$$x = \frac{a^2 + ab + b^2}{a + b}$$

an expression which, under the hypothesis that $a = b$, is reduced to

$$x = \frac{3a^2}{2a} = \frac{3a}{2}$$

Take, as a second example, the expression,

$$x = \frac{a^2 - b^2}{(a-b)^2} = \frac{(a+b)(a-b)}{(a-b)(a-b)}$$

making $a = b$, the value of x becomes $x = \frac{0}{0}$, in consequence of the exist-

* It must, however, be remarked, that there are questions of such a nature, that *infinity* may be considered as the true answer of the problem. We shall find examples of this in *Trigonometry*, and in *Analytical Geometry*.

ence of the common factor $a-b$; but if, in the first instance, we suppress the common factor $a-b$, the value of x becomes,

$$x = \frac{a+b}{a-b}$$

an expression which, under the hypothesis that $a = b$, is reduced to

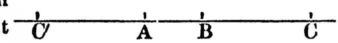
$$x = \frac{2a}{0} = \infty$$

From this it appears, that *the symbol* $\frac{0}{0}$ *in algebra sometimes indicates the existence of a factor common to the two terms of the fraction which is reduced to that form.* Hence, before we can pronounce with certainty upon the true value of such a fraction, we must ascertain whether its terms involve a common factor. If none such be found to exist, then we conclude that the equation in question is really *indeterminate.* If a common factor be found to exist, we must suppress it, and then make anew the particular hypothesis. This will now give us the true value of the fraction, which may present itself under one of the three forms $\frac{A}{B}$, $\frac{A}{0}$, $\frac{0}{0}$

In the first case, the equation is *determinate;* in the second, it is *impossible in finite numbers;* in the third, it is *indeterminate.*

156. We shall conclude this discussion with the following problem, which will serve as an illustration of the various singularities which may present themselves in the solution of a simple equation.

Problem.



Two couriers set off at the same time from two points, A and B, in the same straight line, and travel in the same direction A C. The courier who sets out from A travels m miles an hour, the courier who sets out from B travels n miles an hour; the distance from A to B is a miles. At what distance from the points A and B will the couriers be together?

Let C be the point where they are together, and let x and y denote the distances AC and BC, expressed in miles.

We have manifestly for the first equation

$$x - y = a \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since m and n denote the number of miles travelled by each in an hour, that is the respective velocities of the two couriers, it follows that the time required to traverse the two spaces x and y, must be designated by $\frac{x}{m}$, $\frac{y}{n}$; these two periods, moreover, are equal, hence we have for our second equation

$$\frac{x}{m} = \frac{y}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The values of x and y, derived from equations (1) and (2), are

$$x = \frac{a\,m}{m - n}, \quad y = \frac{a\,n}{m - n}$$

1°. So long as we suppose $m > n$, or $m - n$ positive, the problem will be solved without embarrassment. For in that case, we suppose the courier who starts from A to travel faster than the courier who starts from B, he must therefore overtake him eventually, and a point C can always be found where they will be together.

2°. Let us now suppose $m < n$, or $m - n$ negative, the values of x and y are both negative, and we have

$$x = -\frac{a\,m}{n - m} \qquad y = -\frac{a\,n}{n - m}$$

the solution, therefore, in this case, points out that some absurdity must exist in the conditions of the problem. In fact, if we suppose $m < n$, we suppose that the courier who sets out from A travels slower than the courier who sets out from B; hence the distance between them augments every instant, and it is impossible that the couriers can ever be together, if they travel in the direction A C. Let us now substitute $-x$ for $+x$, and $-y$ for $+y$, in equations (1) and (2); when modified in this manner they become

$$\left.\begin{aligned} y - x &= a \\ \frac{x}{m} &= \frac{y}{n} \end{aligned}\right\}$$

equations which, when resolved, give

$$x = \frac{a\,m}{n - m}, \quad y = \frac{a\,n}{n - m}$$

in which the value of x and y are positive.

These values of x and y give the solution, not of the proposed problem, which is absurd under the supposition that $m < n$, but of the following:

Two couriers set out at the same time from the points A and B, and *travel in the direction* B C′, &c. (the rest as before;) the values of x and y mark the distances AC′, BC′, of the points C′, where the couriers are together, from the points of departure A and B.

3°. Let us next suppose $m = n$; the values of x and y in this case become

$$x = \frac{a\,m}{0}, \quad y = \frac{a\,n}{0}$$

or,

$$x = \infty, \quad y = \infty$$

that is to say, x and y each represent *infinity*. In fact, if we suppose $m = n$ we suppose the courier who sets out from A to travel exactly at the same rate as the courier who sets out from B; consequently, the original distance a by which they are separated will always remain the same, and if the couriers travel *for ever* they can never be together. Here also the conditions of the problem are *absurd*, although the result is not susceptible of the same modification as in the last case.

4°. Let us suppose $m = n$, and also $a = 0$; the values of x and y in this case become

$$x = \frac{0}{0}, \quad y = \frac{0}{0}$$

that is to say, the problem is *indeterminate*, and admits of an infinite number of solutions. In fact, if we suppose $a = 0$, we suppose that the couriers start from the same point, and if we at the same time suppose $m = n$, or that they trăvel equally fast, it is manifest that *they must always be together*, and consequently *every point* in the line AC satisfies the conditions of the problem.

5°. Finally, if we suppose $a = 0$, and m *not* $= n$, the values of x and y in this case become

$$x = 0 \quad y = 0$$

In fact, if we suppose the couriers to set out from the same point, and to travel with different velocities, it is manifest that the point of departure is the only point in which they can be together.

ON QUADRATIC EQUATIONS

157. *Quadratic equations*, or *equations of the second degree*, are divided into two classes.

I. Equations which involve the square only of the unknown quantity. These are termed *pure quadratics*. Of this description are the equations,

$$a x^2 = b; \; 3 x^2 + 12 = 150 - x^2; \; \frac{x^2}{3} - \frac{5}{12} + 3 x^2 = \frac{7}{24} + 2 x^2 + \frac{259}{24}$$

they are sometimes called *quadratic equations of two terms*, because, by transposition and reduction, they can always be exhibited under the general form

$$a x^2 = b.$$

Thus, the third of the equations given above

$$\frac{x^2}{3} - \frac{5}{12} + 3 x^2 = \frac{7}{24} + 2 x^2 + \frac{259}{24},$$

when cleared of fractions becomes

$$8 x^2 - 10 + 72 x^2 = 7 + 48 x^2 + 259,$$

transposing and reducing,

$$32 x^2 = 276$$

which is of the form

$$x^2 = b.$$

II. Equations which involve both the square and the simple power of the unknown quantity. These are termed *adfected*, or *complete quadratics*; of this description are the equations,

$$a x^2 + b x = c; \quad x^2 - 10 x = 7; \quad \frac{5 x^2}{6} - \frac{x}{2} + \frac{3}{4} = 8 - \frac{2 x}{3} - x^2 + \frac{273}{12}$$

they are sometimes called *quadratic equations of three terms*, because, by transposition and reduction, they can always be exhibited under the general form

$$a x^2 + b x = c.$$

Thus, the third of the equations given above,

$$\frac{5 x^2}{6} - \frac{x}{2} + \frac{3}{4} = 8 - \frac{2 x}{3} - x^2 + \frac{273}{12},$$

when cleared of fractions becomes

$$10 x^2 - 6 x + 9 = 96 - 8 x - 12 x^2 + 273,$$

or, transposing and reducing,

$$22 x^2 + 2 x = 360,$$

which is of the form

$$a x^2 + b x = c.$$

SOLUTION OF PURE QUADRATICS CONTAINING ONE UNKNOWN QUANTITY.

158. The solution of the equation

$$a x^2 = b$$

presents no difficulty. Dividing each member by a, it becomes

$$x^2 = \frac{b}{a}$$

whence

$$x = \pm \sqrt{\frac{b}{a}}.$$

If $\frac{b}{a}$ be a particular number, either integral or fractional, we can extract its square root, either exactly, or approximately, by the rules of arithmetic. If $\frac{b}{a}$ be an algebraic expression, we must apply to it the rules established for the extraction of the square root of algebraic quantities.

It is to be remarked, that since the square both of $+ m$, and $- m$, is $+ m^2$; so, in like manner, both $\left(+\sqrt{\frac{b}{a}}\right)^2$, and $\left(-\sqrt{\frac{b}{a}}\right)^2$, is $+ \frac{b}{a}$ Hence the above equation is susceptible of two solutions, or has *two roots*, that is, there are two quantities which, when substituted for x in the original equation, will render the two members identical; these are,

$$x = +\sqrt{\frac{b}{a}}, \quad \text{and} \quad x = -\sqrt{\frac{b}{a}},$$

for, substitute each of these values in the original equation $a x^2 = b$

it becomes, $\quad a \times \left(+\sqrt{\frac{b}{a}}\right)^2 = b$, or, $a \times \frac{b}{a} = b$, i. e. $b =$

and, $\quad a \times \left(-\sqrt{\frac{b}{a}}\right)^2 = b$, or, $a \times \frac{b}{a} = b$, i. e. $b =$

Example 1.

Find the values of x which satisfy the equation

$$4x^2 - 7 = 3x^2 + 9$$

Transposing and reducing,
$$\begin{aligned} x^2 &= 16 \\ \therefore x &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

hence the two values of x are $+4$, and -4, and either of these, if substituted for x in the original equation, will render the two members identical.

Example 2.

$$\frac{x^2}{3} - 3 + \frac{5x^2}{12} = \frac{7}{24} - x^2 + \frac{299}{24}.$$

Clearing of fractions, $8x^2 - 72 + 10x^2 = 7 - 24x^2 + 299$

Transposing and reducing,
$$\begin{aligned} 42x^2 &= 378 \\ x^2 &= \frac{378}{42} \\ &= 9 \\ \therefore x &= \pm 3 \end{aligned}$$

and the two values of x are $+3$, and, -3.

Example 3.

$$\begin{aligned} 3x^2 &= 5 \\ x^2 &= \frac{5}{3} \\ x &= \pm\sqrt{\frac{5}{3}} \\ &= \frac{\pm\sqrt{15}}{3} \end{aligned}$$

Since 15 is not a perfect square we can only approximate to the two values of x.

Example 4.

$$\frac{x}{\sqrt{r^2 + x^2} - x} = m$$

Clearing of fractions, $x = m\sqrt{r^2 + x^2} - mx$

$\therefore (m+1)x = m\sqrt{r^2 + x^2}$

Squaring, $(m^2 + 2m + 1)x^2 = m^2(r^2 + x^2)$

$\therefore (2m+1)x^2 = m^2 r^2$

$$x = \pm \frac{mr}{\sqrt{2m+1}}$$

Example 5.

$$\frac{m + x + \sqrt{2mx + x^2}}{m + x - \sqrt{2mx + x^2}} = n$$

Render the denominator rational by multiplying both terms of the fraction by the numerator, the equation then becomes,

$$\frac{(m + x + \sqrt{2mx + x^2})^2}{m^2} = n$$

Extracting the root,

$m + x + \sqrt{2mx + x^2} = \pm m\sqrt{n}$

Transposing, $\sqrt{2mx + x^2} = \pm m\sqrt{n} - (m + x)$

Squaring, $2mx + x^2 = m^2 n \mp 2m\sqrt{n}(m + x) + (m + x)^2$

Transposing and reducing,

$\pm 2m\sqrt{n}(m + x) = m^2(1 + n)$

$$\therefore m + x = \frac{m(1+n)}{\pm 2\sqrt{n}}$$

$$x = \frac{m(1+n)}{\pm 2\sqrt{n}} - m$$

$$= \pm m \cdot \frac{(\sqrt{n} \pm 1)^2}{2\sqrt{n}}$$

Ex. 6. $11(x^2 - 4) = 5(x^2 + 2)$

Ans. $x = + 3.$

Ex. 7. $\dfrac{x+7}{x^2-7x} - \dfrac{x-7}{x^2+7x} - \dfrac{7}{x^2-73} = o$

Ans. $x = + 9.$

Ex. 8. $\dfrac{m + \sqrt{m^2 - x^2}}{x} = \dfrac{x}{n}$

Ans. $x = \pm \sqrt{2mn - n^2}$

Ex. 9. $\dfrac{\sqrt{m^2 - x^2} - \sqrt{n^2 + x^2}}{\sqrt{m^2 - x^2} + \sqrt{n^2 + x^2}} = \dfrac{p}{q}$

Ans. $x = + \sqrt{\dfrac{m^2(p-q)^2 - n^2(p+q)^2}{2(p^2 + q^2)}}$

Ex. 10 $\dfrac{\sqrt{p + x} + \sqrt{p - x}}{\sqrt{x}} - \sqrt{\dfrac{2}{q}} = o$

Ans. $x = \pm 2\sqrt{pq - q^2}$

159. In the same manner we may solve all equations whatsoever, of any degree, which involve only one power of the unknown quantity, that is, all equations which are included under the general form,

$$a x^n = b$$

For, dividing each member of the equation by a, it becomes,

$$x^n = \frac{b}{a}$$

Extracting the n^{th} root on both sides,

$$x = \sqrt[n]{\frac{b}{a}}$$

If n be an even number, then the radical must be affected with the double sign $\pm$, for in that case, both $\left(+\sqrt[n]{\frac{b}{a}}\right)^n$, and, $\left(-\sqrt[n]{\frac{b}{a}}\right)^n$, will equally produce $\frac{b}{a}$.

Example 11.

$$5x^6 - 57 = 2x^6 + 135$$
$$3x^6 = 192$$
$$x^6 = 64$$
$$x = \sqrt[6]{64} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{\pm 8} = \pm 2$$

Here $+2$ and -2 are two of the roots of the above equation.

Example 12.

$$\frac{\sqrt{p+x}}{p} + \frac{\sqrt{p+x}}{x} = \frac{\sqrt{x}}{q}$$
$$(p+x)\sqrt{p+x} = \frac{p x \sqrt{x}}{q}$$

Or,

$$(p+x)^{\frac{3}{2}} = x^{\frac{3}{2}} \cdot \frac{p}{q}$$

Squaring,

$$(p+x)^3 = x^3 \cdot \frac{p^2}{q^2}$$

Extracting the cube root,

$$p + x = x\sqrt[3]{\frac{p^2}{q^2}}$$
$$\therefore x = \frac{p}{\sqrt[3]{\frac{p^2}{q^2}} - 1}$$

Example 13.

$$\frac{p}{q}\,x^{\frac{p}{q}-1} = \frac{r}{s}\,x^{\frac{r}{s}-1}$$

$$\text{Ans. } x = \left(\frac{q\,r}{p\,s}\right)^{\frac{q\,s}{p\,s - q\,r}}$$

160. We have seen, that an equation of the form $a\,x^2 = b$, has *two roots*, or that there are two quantities which, when substituted for x in the original equation, will render the two members identical. In like manner, we shall find that every equation which involves x in the third power, has *three roots;* an equation which contains x^4 has *four roots;* and it is a general proposition in the theory of equations, that *an equation has as many roots as it has dimensions.*

161. The above method of solving the equation $a\,x^n = b$, will give us only *one* of the n roots of the equation if n be an odd number, and *two* roots if n be an even number. Such a solution must, therefore, be considered imperfect, and we must have recourse to different processes to obtain the remaining roots. This, however, is a subject which we cannot here discuss.

SOLUTION OF COMPLETE QUADRATICS, CONTAINING ONE UNKNOWN QUANTITY.

162. In order to solve the general equation,

$$a x^2 + b x = c$$

let us begin by dividing both members by a, the coefficient of x^2, the equation then becomes,

$$x^2 + \frac{b}{a}\,x = \frac{c}{a}$$

or,

$$x^2 + p\,x = q$$

putting, for the sake of simplicity,

$$\frac{b}{a} = p, \quad \frac{c}{a} = q.$$

This being premised, if we can by any transformation render the first member of the above equation $x^2 + p\,x$ the perfect square of a binomial, a simple extraction of the square root will reduce the equation in question to a simple equation.

But we have already seen, that the square of a binomial $x + a$, or $x^2 + 2\,a\,x + a^2$, is composed of the square of the first term, plus twice the product of the first term by the second, plus the square of the second term.

Hence, considering $x^2 + p\,x$ as the two first terms of the square of a binomial, and consequently $p\,x$ as twice the product of the first term of the binomial

by the second, it is evident that the second term of this binomial must be $\frac{p}{2}$, for $2 \times \frac{p}{2} \times x = p\,x$.

In order, therefore, that the above expression may be transformed to a perfect square, we must add to it the square of this second term $\frac{p}{2}$, that is, *the square of half the coefficient of the simple power of* x; it thus becomes

$$x^2 + p\,x + \frac{p^2}{4}$$

which is the square of $x + \frac{p}{2}$. But since we have added $\frac{p^2}{4}$ to the left hand member of the equation, in order that the equality between the two members may not be destroyed, we must add the same quantity to the right hand member also, the equation thus transformed will be

$$x^2 + p\,x + \frac{p^2}{4} = \frac{p^2}{4} + q$$

Or, $$\left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} + q$$

Extracting the root, $$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} + q}$$

Transposing, $$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$$

$$= \frac{-p \pm \sqrt{p^2 + 4q}}{2}$$

If the original equation had been

$$x^2 - p\,x = q$$

or being transformed, it would have become

$$x^2 - p\,x + \frac{p^2}{4} = \frac{p^2}{4} + q$$

Or, $$\left(x - \frac{p}{2}\right)^2 = \frac{p^2}{4} + q$$

And ∴ $$x = \frac{p \pm \sqrt{p^2 + 4q}}{2}$$

We affix the double sign $\pm\sqrt{\frac{p^2}{4} + q}$, because the square both of $+\sqrt{\frac{p^2}{4} + q}$, and also of $-\sqrt{\frac{p^2}{4} + q}$, is $+\left(\frac{p^2}{4} + q\right)$, and every quadratic equation, must, therefore, have two roots.

From what has just been said, we deduce the following general

RULE FOR THE SOLUTION OF A COMPLETE QUADRATIC EQUATION.

1. *Transpose all the known quantities, when necessary, to one side of the equation, arrange all the terms involving the unknown quantity on the other side, and reduce the equation to the form* $a x^2 + b x = c$.

2. *Divide each side of the equation by the coefficient of* x^2.

3. *Add to each side of the equation the square of half the coefficient of the simple power of* x.

That member of the equation which involves the unknown quantity will thus be rendered a perfect square, and extracting the root on both sides, the equation will be reduced to one of the first degree, which may be solved in the usual manner.

Example 1.

$$12 x - 210 = 205 - 3 x^2 + 5$$

Transposing and reducing, $3 x^2 + 12 x = 420$
Dividing by the coefficient of x^2, $x^2 + 4 x = 140$

Completing the square by adding to each side the square of half the coefficient of the second term,

$$x^2 + 4 x + 4 = 140 + 4$$

Or,

$$(x + 2)^2 = 144$$

Extracting the root, $x + 2 = \pm \sqrt{144}$

$$= \pm 12$$

$$\therefore x = - 2 \pm 12$$

Hence,

$$\begin{cases} x = - 2 + 12 = 10 \\ x = - 2 - 12 = - 14 \end{cases}$$

Either of these two numbers, when substituted for x in the original equation, will render the two members identical.

Example 2.

$$2 x^2 + 34 = 20 x + 2$$

Transposing and reducing, $2 x^2 - 20 x = - 32$
Dividing by 2, $x^2 - 10 x = - 16$
Completing the square, $x^2 - 10 x + 25 = 25 - 16$

Or,

$$(x-5)^2 = 9$$

Extracting the root,

$$x - 5 = \pm \sqrt{9}$$

$$x = 5 \pm 3$$

Hence,

$$\begin{cases} x = 5 + 3 = 8 \\ x = 5 - 3 = 2 \end{cases}$$

Example 3.

$$3x^2 - 2x = 65$$

Dividing by 3,

$$x^2 - \frac{2}{3}x = \frac{65}{3}$$

Completing the square,

$$x^2 - \frac{2}{3}x + \left(\frac{1}{3}\right)^2 = \frac{65}{3} + \left(\frac{1}{3}\right)^2$$

Or,

$$\left(x - \frac{1}{3}\right)^2 = \frac{196}{9}$$

$$\therefore x - \frac{1}{3} = \pm\sqrt{\frac{196}{9}}$$

$$= \pm \frac{14}{3}$$

$$x = \frac{1}{3} \pm \frac{14}{3}$$

Hence,

$$\begin{cases} x = \frac{1+14}{3} = 5 \\ x = \frac{1-14}{3} = -4\frac{1}{3} \end{cases}$$

Example 4.

$$x^2 + x - 2 = 0$$

Transposing,

$$x^2 + x = 2$$

The coefficient of x in this case is 1, $\therefore$ in order to complete the square, we must add to each side $\left(\frac{1}{2}\right)^2$, or $\frac{1}{4}$.

$$\therefore x^2 + x + \frac{1}{4} = 2 + \frac{1}{4}$$

$$\left(x + \frac{1}{2}\right)^2 = \frac{9}{4}$$

$$x + \frac{1}{2} = \pm \frac{3}{2}$$

$$\therefore x = 1, \text{ and } x = -2$$

Example 5.

$$6x - 30 = 3x^2$$

Transposing, $-3x^2 + 6x = 30$

Changing the sign on both sides, $3x^2 - 6x = -30$

Dividing by 3, $x^2 - 2x = -10$

Completing the square, $x^2 - 2x + 1 = 1 - 10$

Or

$$(x-1)^2 = -9$$

$$\therefore\ x - 1 = \pm\sqrt{-9}$$

Hence,

$$\begin{cases} x = 1 + \sqrt{-9} \\ x = 1 - \sqrt{-9} \end{cases}$$

In the above example, the values of x contain imaginary quantities, and the roots of the equation are therefore said to be impossible.

Example 6.

$$\frac{5}{6}x^2 - \frac{1}{2}x + \frac{3}{4} = 8 - \frac{2}{3}x - x^2 + \frac{273}{12}$$

Clearing of fractions,

$$10x^2 - 6x + 9 = 96 - 8x - 12x^2 + 273$$

Transposing and reducing,

$$22x^2 + 2x = 360$$

Dividing both members by 22,

$$x^2 + \frac{2}{22}x = \frac{360}{22}$$

Adding $\left(\frac{1}{22}\right)^2$ to both members,

$$x^2 + \frac{2}{22}x + \left(\frac{1}{22}\right)^2 = \frac{360}{22} + \left(\frac{1}{22}\right)^2$$

Extracting the root,

$$x + \frac{1}{22} = \pm\sqrt{\frac{360}{22} + \left(\frac{1}{22}\right)^2}$$

$$= \pm\sqrt{\frac{7921}{(22)^2}}$$

$$= \pm\frac{89}{22}$$

Hence,

$$\therefore \begin{cases} x = -\frac{1}{22} + \frac{89}{22} = 4 \\ x = -\frac{1}{22} - \frac{89}{22} = -\frac{45}{11} \end{cases}$$

Example 7.

$$a x^2 - \frac{a c}{a + b} = c x - b x^2$$

Transposing,

$$(a + b) x^2 - c x = \frac{a c}{a + b}$$

Dividing by $a + b$,

$$x^2 - \frac{c}{a + b} \cdot x = \frac{a c}{(a + b)^2}$$

Completing the square,

$$x^2 - \frac{c}{a + b} \cdot x + \frac{c^2}{4(a + b)^2} = \frac{a c}{(a + b)^2} + \frac{c^2}{4(a + b)^2}$$

Or,

$$\left\{x - \frac{c}{2(a + b)}\right\}^2 = \frac{c^2 + 4 a c}{4(a + b)^2}$$

acting the root,

$$x - \frac{c}{2(a + b)} = \pm \frac{\sqrt{c^2 + 4 a c}}{2(a + b)}$$

$$\therefore x = \frac{c \pm \sqrt{c^2 + 4 a c}}{2(a + b)}$$

The two values of x here are,

$$x = \frac{c + \sqrt{c^2 + 4 a c}}{2(a + b)}, \quad x = \frac{c - \sqrt{c^2 + 4 a c}}{2(a + b)}$$

Example 8.

$$a^2 + b^2 - 2bx + x^2 = \frac{m^2 x^2}{n^2}$$

Transposing,

$$(n^2 - m^2) x^2 - 2bn^2 x = -n^2 (a^2 + b^2)$$

Dividing by the coefficient of x^2,

$$x^2 - \frac{2bn^2}{n^2 - m^2} x = -n^2 \cdot \frac{a^2 + b^2}{n^2 - m^2}$$

Completing the square,

$$x^2 - \frac{2bn^2}{n^2 - m^2} x + \left(\frac{bn^2}{n^2 - m^2}\right)^2 = \left(\frac{bn^2}{n^2 - m^2}\right)^2 - \frac{n^2(a^2 + b^2)}{n^2 - m^2}$$

Or,

$$\left\{x - \frac{bn^2}{n^2 - m^2}\right\}^2 = \frac{n^2}{n^2 - m^2} \left\{\frac{b^2 n^2}{n^2 - m^2} - (a^2 + b^2)\right\}$$

$$= \frac{n^2}{(n^2 - m^2)^2} \left\{m^2 (a^2 + b^2) - n^2 a^2\right\}.$$

Extracting the root,

$$x - \frac{bn^2}{n^2 - m^2} = \pm \frac{n}{n^2 - m^2} \sqrt{m^2 (a^2 + b^2) - n^2 a^2}.$$

$$x = \frac{n}{n^2 - m^2} \left\{bn \pm \sqrt{m^2(a^2 + b^2) - n^2 a^2}\right\}$$

The two values of x are

$$x = \frac{n}{n^2 - m^2} \left\{bn + \sqrt{m^2 (a^2 + b^2) - n^2 a^2},\right\}$$

$$x = \frac{n}{n^2 - m^2} \left\{bn - \sqrt{m^2 (a^2 + b^2) - n^2 a^2}\right\}$$

Ex. 9. $x^2 + 6x = 27$

Ans. $x = 3$, $x = -9$.

Ex. 10. $x^2 - 7x + 3\frac{1}{4} = 0$

Ans. $x = 6\frac{1}{2}$, $x = \frac{1}{2}$.

Ex. 11. $622x - 15x^2 = 6384$

Ans. $x = 22\frac{4}{5}$, $x = 18\frac{2}{3}$.

Ex. 12. $8x^2 - 7x + 34 = 0$

Ans. $x = \frac{7 + \sqrt{-1039}}{16}$, $x = \frac{7 - \sqrt{1039}}{16}$

Ex. 13. $3x^2 + x = 7$

Ans. $x = \frac{-1 + \sqrt{85}}{6}$, $x = \frac{-1 - \sqrt{85}}{6}$

Ex. 14. $\frac{x}{3} - 4 - x^2 + 2x - \frac{4x^2}{5} = 45 - 3x^2 + 4x$

Ans. $x = 7.12\ldots\ldots$, $x = -5.73\ \ldots\ldots$

Ex. 15. $3x - \frac{6x^2 - 40}{2x - 1} - \frac{3x - 10}{9 - 2x} = 2$

Ans. $x = \frac{23}{2}$, $x = 4$

Ex. 16. $\frac{90}{x} - \frac{90}{x + 1} - \frac{27}{x + 2} = 0$

Ans. $x = 4$, $x = -\frac{5}{3}$

Ex. 17. $abx^2 + \frac{3a^2 x}{c} = \frac{6a^2 + ab - 2b^2}{c^2} - \frac{b^2 x}{c}$

Ans. $x = \frac{2a - b}{ac}$, $x = -\frac{3a + 2b}{bc}$

Ex. 18. $mx^2 - 2mx\sqrt{n} = nx^2 - mn$

Ans. $x = \dfrac{\sqrt{mn}}{\sqrt{m}+\sqrt{n}}$, $x = \dfrac{\sqrt{mn.}}{\sqrt{m}-\sqrt{n}}$

Ex. 19. $4a^2x^2 + 4a^2c^2x + 4abd^2x - 9cd^2x^2 + (ac^2 + bd^2)^2 = 0$

Ans. $x = -\dfrac{ac^2+bd^2}{2a+3d\sqrt{c}}$, $x = -\dfrac{ac^2+bd^2}{2a-3d\sqrt{c}}$

Ex. 20. $\dfrac{5a+10ab^2}{9b^2-3a^2b^2}x^2 - \left(\dfrac{5\sqrt{a+b}}{3b^3} + \dfrac{(1+2b^2)cd\sqrt{c}}{3-a^2}\right)x + \dfrac{cd}{ab}\sqrt{(a+b)c} = 0$

Ans. $x = \dfrac{(3-a^2)\sqrt{a+b}}{ab(1+2b^2)}$, $x = \dfrac{3b^2cd\sqrt{c}}{5a}$

163. The above rule will enable us to solve, not only quadratic equations, but all equations which can be reduced to the form

$$x^{2n} + px^n = q;$$

that is, all equations which contain only two powers of the unknown quantity, and in which one of these powers is double of the other.

For, if in the above equation we assume $y = x^n$, then $y^2 = x^{2n}$, and it becomes

$$y^2 + py = q;$$

Solving this according to the rule,

$$y = \frac{-p \pm \sqrt{p^2+4q}}{2}$$

Putting for y its value,

$$x^n = \frac{-p \pm \sqrt{p^2+4q}}{2}$$

Extracting the nth root on both sides,

$$x = \sqrt[n]{\frac{-p \pm \sqrt{p^2+4q}}{2}}$$

Example 1.

$$x^4 - 25x^2 = -144$$

Assume $x^2 = y$, the above becomes

$$y^2 - 25y = -144$$

Whence $\quad y = 16, \quad y = 9$

But since $\quad x^2 = y \quad \therefore \quad x = \pm\sqrt{y}$

$\therefore \quad x = \pm\sqrt{16}, \quad x = \pm\sqrt{9}$

Thus the four values of x are $+4$, -4, $+3$, -3.

Example 2.

$$x^4 - 7x^2 = 8$$

Assume $x^2 = y$, $\quad y^2 - 7y = 8$

Whence $\quad y = 8, \quad y = -1$

And since $\quad x^2 = y \quad \therefore \quad x = \pm\sqrt{y}$

Whence the four roots of the equation are $\pm\sqrt{8}$, $\pm\sqrt{-1}$, the two last of which are impossible roots.

Example 3.

Let $$x^6 - 2x^3 = 48$$
Assume $x^3 = y$, the above becomes
$$y^2 - 2y = 48$$
Whence $$y = 8, \text{ or } -6$$
But since $$x^3 = y \quad \therefore \quad x = \sqrt[3]{y}$$
Hence two of the roots of the above equation are $+\sqrt[3]{8}$ and $-\sqrt[3]{6}$; the remaining four roots cannot be determined by this process.

Example 4.

Let $$2x - 7\sqrt{x} = 99$$
Or, $$2x - 7x^{\frac{1}{2}} = 99$$
This equation manifestly belongs to this class, for the exponent of x in the first term is 1, and in the second term half as great, or $\frac{1}{2}$;

In this case assume $\sqrt{x} = y$ the equation becomes,

$$2y^2 - 7y = 99$$
Whence $$y = 9, \quad y = -\frac{11}{2}$$
But since $$\sqrt{x} = y$$
$$\therefore \quad x = 81, \quad x = \frac{121}{4}$$

To account for the two values of x in this equation, it must be observed that one belongs to $+\sqrt{x}$, the other to $-\sqrt{x}$.

This will appear clearly in the following example.

Example 5.

$$ax = b + \sqrt{cx}$$

Solving this equation in the same manner as the preceding, we shall find

$$x = \frac{2ab + c + \sqrt{4abc + c^2}}{2a^2}, \qquad x = \frac{2ab + c - \sqrt{4abc + c^2}}{2a^2}$$

if we substitute these two values of x in the original equation, we shall find that the first only will verify it; the second belongs to the equation

$$ax = b - \sqrt{cx}$$

these two equations, multiplied together, produce the complete quadratic equation
$$a^2x^2 - (2ab + c)x + b^2 = 0$$

whose roots are the two values of x given above.

164. Many other equations of degrees higher than the second may be solved by completing the square; although, it must be remarked, we can seldom obtain *all* the roots in this manner. The transformations to which we subject equations of this nature, in order that the rule may become applicable, depend upon

various algebraic artifices, for which no general rule can be given. The following examples will serve to give the student some idea of the course he must pursue · a little practice will soon render him dextrous in the employment of such devices.

Example 6.

Let $$\sqrt{x+12} + \sqrt[4]{x+12} = 6$$

Assume, $x + 12 = y$; the equation then becomes,

$$y^{\frac{1}{2}} + y^{\frac{1}{4}} = 6$$

which evidently belongs to the same class as the previous examples; completing the square, we shall have,

$$y^{\frac{1}{4}} = 2, \text{ or, } -3$$

Raising both sides of the equation to the power of 4,

$$y = 16, \text{ or } 81$$
$$\therefore \quad x, \text{ or } y - 12 = 4, \text{ or } 69.$$

Example 7.

Let $$2x^2 + \sqrt{2x^2+1} = 11$$

Add 1 to each member of the equation, it becomes,

$$2x^2 + 1 + \sqrt{2x^2+1} = 12$$

Assume $2x^2 + 1 = y$, then,

$$y + y^{\frac{1}{2}} = 12$$

Completing the square, and solving, we find,

$$y^{\frac{1}{2}}, \text{ or, } \sqrt{2x^2+1} = 3, \text{ and, } -4$$
$$2x^2 + 1 = 9, \text{ and, } 16$$
$$x^2 = 4, \text{ and, } \frac{15}{2}$$

Hence, $x = +2, -2, +\sqrt{\frac{15}{2}}, -\sqrt{\frac{15}{2}}$

It may be remarked, that it is in general unnecessary to substitute y, which has been done in the above examples for the sake of perspicuity alone

Example 8.

Let, $$\left(x + \frac{8}{x}\right)^2 + x = 42 - \frac{8}{x}$$

Transposing, $$\left(x + \frac{8}{x}\right)^2 + \left(x + \frac{8}{x}\right) = 42$$

Considering $x + \frac{8}{x}$ as one quantity, and completing the square.

$$\left(x + \frac{8}{x}\right)^2 + \left(x + \frac{8}{x}\right) + \frac{1}{4} = \frac{169}{4}$$

$$\therefore \quad x + \frac{8}{x} = -\frac{1}{2} \pm \frac{13}{2}$$
$$= 6, \text{ and, } -7$$

Hence, we have the two equations,

$$x^2 - 6x = -8$$
$$x^2 + 7x = -8$$

Solving the first in the usual manner, we find,

$$x = 4, \text{ and, } 2$$

and by the second, we have,

$$x = \frac{-7 + \sqrt{17}}{2}, \text{ and, } \frac{-7 - \sqrt{17}}{2}$$

which are the four roots of the proposed equation. If we had reduced this equation, by performing the operations indicated, instead of employing the above artifice, it would have become,

$$x^4 + x^3 - 26x^2 + 8x + 64 = 0$$

a complete equation of the fourth degree.

Ex. 9. $x^4 + 4x^2 = 12$

Ans. $x = \pm\sqrt{2}$

Ex. 10 $x^6 - 8x^3 - 513 = 0$

Ans. $x = 3$

Ex. 11. $x^4 - (2bc + 4a^2)x^2 + b^2c^2 = 0$

Ans. $x = \pm\sqrt{bc + 2a^2 \pm 2a\sqrt{bc + a^2}}$

Ex. 12. $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$

Ans. $x = 243$, and, $x = \sqrt[3]{(-28)^5}$

Ex. 13. $a x^{\frac{6}{4}} + b x^{\frac{3}{4}} - c = 0$

Ans. $x = \left(\frac{\pm\sqrt{b^2 + 4ac} - b}{2a}\right)^{\frac{4}{3}}$

Ex. 14. $x^2 - x + 5\sqrt{2x^2 - 5x + 6} = \frac{3x + 33}{2}$

Ans. $x = \frac{5 \pm \sqrt{1.529}}{4}$, and, $x = 3$, and, $-\frac{1}{2}$

Ex. 15. $\frac{x + \sqrt{x^2 - a^2}}{x - \sqrt{x^2 - a^2}} = \frac{x}{a}$

Ans. $x = \frac{2}{8}(\pm\sqrt{-7} - 3)$

ON THE SOLUTION OF QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

165. An equation containing two unknown quantities, is said to be of the *second degree* when it involves terms in which *the sum of the exponents of the unknown quantities is equal to 2, but never exceeds 2.* Thus,

$$3x^2 - 4x + y^2 - xy - 5y + 6 = 0, \quad 7xy - 4x + y = 0$$

are equations of the second degree.

It follows from this, that every equation of the second degree containing two unknown quantities, is of the form,

$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

where a, b, c, ... represent known quantities, either numerical or algebraical.

Let it be required to determine the values of x and y, which satisfy the equations.

$$\left.\begin{array}{l} ay^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots\ldots\ldots(1) \\ a'y^2 + b'xy + c'x^2 + d'y + e'x + f' = 0 \ldots\ldots\ldots\ldots(2) \end{array}\right\}$$

Arranging these two equations according to the powers of y, they become,

$$\left.\begin{array}{l} ay^2 + (bx + d)y + (cx^2 + ex + f) = 0 \ldots\ldots\ldots\ldots \\ a'y^2 + (b'x + d')y + (c'x^2 + e'x + f') = 0 \ldots\ldots\ldots\ldots \end{array}\right\}$$

$$\text{Put } bx+d=h \;;\; cx^2+ex+f=k$$
$$b'x+d'=h' \;;\; c'x^2+e'x+f'=k'.$$

$$\therefore ay^2+hy+k=0 \;.\;.\;.\;.\;.\;.\;.\;.\;.\; (3)$$
$$a'y^2+h'y+k'=0 \;.\;.\;.\;.\;.\;.\;.\;.\;.\; (4).$$

Multiply (3) and (4) by a' and a respectively, and also by k' and k; then

$$aa'y^2+a'hy+a'k=0 \;.\;.\;.\; (5) \qquad ak'y^2+hk'y+kk'=0 \;.\;.\;.\; (7)$$
$$aa'y^2+ah'y+ak'=0 \;.\;.\;.\; (6) \qquad a'ky^2+h'ky+kk'=0 \;.\;.\;.\; (8).$$

Subtracting (6) from (5), and also (7) from (8), we have

$$(a'h-ah')y+a'k-ak'=0 \;.\;.\;.\; (9)$$
$$(a'k-ak')y+h'k-hk'=0 \;.\;.\;.\; (10).$$

Multiplying (9) by $h'k-hk'$, and (10) by $a'k-ak'$, we have

$$(a'h-ah')(h'k-hk')y+(a'k-ak')(h'k-hk')=0 \;.\;.\;.\;.\; (11)$$
$$(a'k-ak')^2y+(a'k-ak')(h'k-hk')=0 \;.\;.\;.\;.\; (12)$$
$$\therefore (a'h-ah')(h'k-hk')=(a'k-ak')^2 \;.\;.\;.\;.\; (13).$$

Substituting the values of h, h', k, k', in equation (13), we have

$$\left\{(a'b-ab')x+a'd-ad'\right\}.\left\{(b'c-bc')x^3+(b'e-be'-c'd+cd')x^2+(b'f-bf'+de'-de')x+d'f-df'\right\}$$
$$=\left\{(a'c-ac')x^2+(a'e-ae')x+a'f-af'\right\}^2$$

Hence, by multiplying and expanding, the final equation in x is of the fourth degree; but the general form includes a variety of equations, according to the values of the coefficients a, b, c, &c.; and when d, e, f, d', e', f', are each

$=0$, the solution may be obtained by quadratics, the resulting equation in x being

$$\{(ab-ab')x+a'd-ad'\}\cdot\{(b'c-bc')x-(c'd-cd')\}=(a'c-ac')^2x^2.$$

Hence, in general, *the solution of two equations of the second degree, containing two unknown quantities, depends upon the solution of an equation of the fourth degree containing one unknown quantity.*

Although the principle already established will not enable us to solve equations of this description *generally*, yet there are many particular cases in which they may be reduced either to pure or adfected quadratics, and the roots determined in the ordinary manner.

Example 1.

Required the values of x and y, which satisfy the equations,

$$\left\{\begin{array}{rcl} x+y & = & p \ldots\ldots\ldots(1) \\ xy & = & q^2 \ldots\ldots\ldots(2) \end{array}\right\}$$

Squaring (1),	$x^2+2xy+y^2$	$=$	$p^2 \ldots\ldots\ldots(3)$
Multiply (2) by 4,	$4xy$	$=$	$4q^2 \ldots\ldots\ldots(4)$
Subtract (4) from (3),	$x^2-2xy+y^2$	$=$	p^2-4q^2
or,	$(x-y)^2$	$=$	p^2-4q^2
Extract the root,	$x-y$	$=$	$\pm\sqrt{p^2-4q^2} \ldots\ldots\ldots(5)$
But by (1),	$x+y$	$=$	p
Add (1) to (5),	$2x$	$=$	$p\pm\sqrt{p^2-4q^2}$
Subtract (5) from (1),	$2y$	$=$	$p\mp\sqrt{p^2-4q^2}$

Hence, the corresponding values of x and y will be,

$$\left.\begin{array}{l} x=\dfrac{p+\sqrt{p^2-4q^2}}{2} \\ y=\dfrac{p-\sqrt{p^2-4q^2}}{2} \end{array}\right\} \text{ and, } \left.\begin{array}{l} x=\dfrac{p-\sqrt{p^2-4q^2}}{2} \\ y=\dfrac{p+\sqrt{p^2-4q^2}}{2} \end{array}\right\}$$

Example 2

$$\left\{\begin{array}{rcl} x+y & = & a \ldots\ldots\ldots(1) \\ x^2+y^2 & = & b^2 \ldots\ldots\ldots(2) \end{array}\right\}$$

Square (1)	$x^2+2xy+y^2$	$=$	a^2
But by (2)	$x^2 \qquad +y^2$	$=$	b^2
Subtracting,	$2xy$	$=$	$a^2-b^2 \ldots\ldots\ldots(3)$
Subtract (3) from (2)	$x^2-2xy+y^2$	$=$	$2b^2-a^2$
or,	$(x-y)^2$	$=$	$2b^2-a^2$
Extracting the root.	$x-y$	$=$	$\pm\sqrt{2b^2-a^2}$
But by (1)	$x+y$	$=$	a
$\therefore$ adding and subtracting,	$2x$	$=$	$a\pm\sqrt{2b^2-a^2}$
	$2y$	$=$	$a\mp\sqrt{2b^2-a^2}$.

Hence the corresponding values of x and y will be

$$\left.\begin{aligned} x &= \frac{a + \sqrt{2b^2 - a^2}}{2} \\ y &= \frac{a - \sqrt{2b^2 - a^2}}{2} \end{aligned}\right\} \text{ and, } \left.\begin{aligned} x &= \frac{a - \sqrt{2b^2 - a^2}}{2} \\ y &= \frac{a + \sqrt{2b^2 - a^2}}{2} \end{aligned}\right\}$$

Example 3.

$$\left\{\begin{aligned} x + y &= m \text{ ....(1)} \\ x^3 + y^3 &= n^3 \text{ ....(2)} \end{aligned}\right\}$$

Cube (1)	$x^3 + 3x^2y + 3xy^2 + y^3$	$= m^3$
But by (2)	$x^3 \qquad\qquad + y^3$	$= n^3$
Subtraction,	$3x^2y + \quad 3xy^2$	$= m^3 - n^3$
Or,	$3xy\ (x + y)$	$= m^3 - n^3$
Substitute for $(x + y)$ its value derived from (1)	$3xy\ .\ m$	$= m^3 - n^3$
	$\therefore\ xy$	$= \dfrac{m^3 - n^3}{3m}$
	$\therefore\ 4xy$	$= \dfrac{4(m^3 - n^3)}{3m}$ ....(3)
Squaring (1),	$x^2 + 2xy + y^2$	$= m^2$
But by (3)	$4xy$	$= \dfrac{4(m^3 - n^3)}{3m}$
Subtracting,	$x^2 - 2xy + y^2$	$= m^2 - \dfrac{4(m^3 - n^3)}{3m}$
Or,	$(x - y)^2$	$= \dfrac{4n^3 - m^3}{3m}$
	$\therefore\ x - y$	$= \pm\sqrt{\dfrac{4n^3 - m^3}{3m}}$
But by (1)	$x + y$	$= m$
	$\therefore\ 2x$	$= m \pm \sqrt{\dfrac{4n^3 - m^3}{3m}}$
	$2y$	$= m \mp \sqrt{\dfrac{4n^3 - m^3}{3m}}$

Hence two corresponding values of x and y are

$$\left.\begin{aligned} x &= \frac{m}{2} + \sqrt{\frac{4n^3 - m^3}{12m}} \\ y &= \frac{m}{2} - \sqrt{\frac{4n^3 - m^3}{12m}} \end{aligned}\right\} \text{ and } \left.\begin{aligned} x &= \frac{m}{2} - \sqrt{\frac{4n^3 - m^3}{12m}} \\ y &= \frac{m}{2} + \sqrt{\frac{4n^3 - m^3}{12m}} \end{aligned}\right\}$$

Example 4.

$$\left\{\begin{aligned} x^{\frac{3}{2}} + x^{\frac{3}{4}} y^{\frac{3}{4}} + y^{\frac{3}{2}} &= a \text{ ....(1)} \\ x^3 + x^{\frac{3}{2}} y^{\frac{3}{2}} + y^3 &= b \text{ ....(2)} \end{aligned}\right\}$$

Square (1), $x^3 + x^{\frac{3}{2}} y^{\frac{3}{2}} + y^3 + 2x^{\frac{3}{2}} . x^{\frac{3}{4}} y^{\frac{3}{4}} + 2x^{\frac{3}{2}} y^{\frac{3}{2}} + 2y^{\frac{3}{2}} . x^{\frac{3}{4}} y^{\frac{3}{4}} = a^2$

But by (2), $x^3 + x^{\frac{3}{2}} y^{\frac{3}{2}} + y^3 = b$

Subtracting, $2x^{\frac{3}{2}} . x^{\frac{3}{4}} y^{\frac{3}{4}} + 2x^{\frac{3}{2}} y^{\frac{3}{2}} + 2y^{\frac{3}{2}} x^{\frac{3}{4}} y^{\frac{3}{4}} = a^2 - b$

$$\text{Or,}\quad 2x^{\frac{3}{4}}y^{\frac{3}{4}}\left(x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}\right) = a^2-b$$

$$\therefore\ 2x^{\frac{3}{4}}y^{\frac{3}{4}}a = a^2-b$$

$$\therefore\ x^{\frac{3}{4}}y^{\frac{3}{4}} = \frac{a^2-b}{2a}\ \ldots(3)$$

But by (1), $x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}} = a$

And by (3), $x^{\frac{3}{4}}y^{\frac{3}{4}} = \frac{a^2-b}{2a}$

Adding, $x^{\frac{3}{2}}+2x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}} = a+\frac{a^2-b}{2a}$

$$\text{Or,}\quad \left(x^{\frac{3}{4}}+y^{\frac{3}{4}}\right)^2 = \frac{3a^2-b}{2a}$$

$$\therefore\ x^{\frac{3}{4}}+y^{\frac{3}{4}} = \pm\sqrt{\frac{3a^2-b}{2a}}\ \ldots\ldots\ldots\ldots\ldots(4)$$

Again, from (1) $x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}} = a$

And from (3) $3x^{\frac{3}{4}}y^{\frac{3}{4}} = \frac{3(a^2-b)}{2a}$

Subtracting, $x^{\frac{3}{2}}-2x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}} = a-\frac{3(a^2-b)}{2a}$

$$\text{Or,}\quad \left(x^{\frac{3}{4}}-y^{\frac{3}{4}}\right)^2 = \frac{3b-a^2}{2a}$$

$$\therefore\ x^{\frac{3}{4}}-y^{\frac{3}{4}} = \pm\sqrt{\frac{3b-a^2}{2a}}\ \ldots\ldots\ldots\ldots(5)$$

But by (4) $x^{\frac{3}{4}}+y^{\frac{3}{4}} = \pm\sqrt{\frac{3a^2-b}{2a}}$

∴ adding and subtracting,

$$x^{\frac{3}{4}} = \frac{\pm\sqrt{\frac{3a^2-b}{2a}}\pm\sqrt{\frac{3b-a^2}{2a}}}{2}$$

$$y^{\frac{3}{4}} = \frac{\pm\sqrt{\frac{3a^2-b}{2a}}\mp\sqrt{\frac{3b-a^2}{2a}}}{2}$$

Hence the corresponding values of x and y are

$$x = \left\{\frac{\pm\sqrt{3a^2-b}+\sqrt{3b-a^2}}{\sqrt{8a}}\right\}^{\frac{4}{3}} \qquad x = \left\{\frac{\pm\sqrt{3a^2-b}-\sqrt{3b-a^2}}{\sqrt{8a}}\right\}^{\frac{4}{3}}$$

and,

$$y = \left\{\frac{\pm\sqrt{3a^2-b}-\sqrt{3b-a^2}}{\sqrt{8a}}\right\}^{\frac{4}{3}} \qquad y = \left\{\frac{\pm\sqrt{3a^2-b}+\sqrt{3b-a^2}}{\sqrt{8a}}\right\}^{\frac{4}{3}}$$

The following require the completion of the square:

Example 5.

$$\left\{\begin{aligned} x+y+x^2+y^2 &= a \ \ldots\ldots\ldots\ldots(1)\\ x-y+x^2-y^2 &= b \ \ldots\ldots\ldots\ldots(2)\end{aligned}\right\}$$

Add (1) and (2), $2x^2+2x = a+b\ldots\ldots\ldots\ldots(3)$

Subtract (2) from (1), $2y^2+2y = a-b\ldots\ldots\ldots\ldots(4)$

Equations (3) and (4) are common adfected quadratics; solving these in the usual manner we find

$$\left.\begin{aligned} x &= \frac{-1 \pm \sqrt{1+2a+2b}}{2} \\ y &= \frac{-1 \pm \sqrt{1+2a-2b}}{2} \end{aligned}\right\}$$

Example 6.

$$\left\{\begin{aligned} x+y &= 6 \ldots\ldots\ldots\ldots(1) \\ x^4+y^4 &= 272 \ldots\ldots\ldots\ldots(2) \end{aligned}\right\}$$

Raise (1) to the power of 4.

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4 = 1296$$

But from (2), $x^4 + y^4 = 272$

Subtracting, $4x^3y+6x^2y^2+4xy^3 = 1024$

Or, $2xy(2x^2+3xy+2y^2) = 1024 \ldots\ldots(3)$

But by (1), $2xy(2x^2+4xy+2y^2) = 144xy \ldots\ldots(4)$

Subtracting (3) from (4), $2x^2y^2 = 144xy - 1024$

Transposing and dividing by (2), $x^2y^2 - 72xy = -512$

Completing the square, $x^2y^2 - 72xy + 1296 = 1296 - 512$

Or, $(xy-36)^2 = 784$

$\therefore \; xy - 36 = \pm\sqrt{784}$

$xy = 36 \pm 28$

$= 64$, and, 8

First, let us suppose $xy = 8$.

By (1), $x^2+2xy+y^2 = 36$

And, $4xy = 32$

Subtracting, $x^2-2xy+y^2 = 4$

$\therefore \; x-y = \pm 2$

But, $x+y = 6$

$\therefore$ adding and subtracting,

$$\left.\begin{aligned} x &= 4 \\ y &= 2 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} x &= 2 \\ y &= 4 \end{aligned}\right\}$$

Secondly, let us take the other value of xy, or 64.

By (1), $x^2+2xy+y^2 = 36$

$4xy = 256$

Subtracting, $x^2-2xy+y^2 = -220$

$\therefore \; x-y = \pm\sqrt{-220}$

But, $x+y = 6$

$\therefore$ adding and subtracting,

$$\left.\begin{aligned} x &= \frac{6+\sqrt{-220}}{2} \\ y &= \frac{6-\sqrt{-220}}{2} \end{aligned}\right\} \text{ and, } \left\{\begin{aligned} x &= \frac{6-\sqrt{-220}}{2} \\ y &= \frac{6+\sqrt{-220}}{2} \end{aligned}\right\}$$

Hence in the above equations, two of the roots of x and y are possible, and two impossible.

Ex. 7. $\left.\begin{array}{rl} 2x+3y &= 118 \ldots\ldots\ldots\ldots\ldots\ldots (1) \\ 5x^2-7y^2 &= 4333 \ldots\ldots\ldots\ldots\ldots\ldots (2) \end{array}\right\}$

Ans. $\left.\begin{array}{rl} x &= 35 \\ y &= 16 \end{array}\right\}$ and, $\left.\begin{array}{rl} x &= -229\frac{6}{17} \\ y &= 192\frac{4}{17} \end{array}\right\}$

Ex. 8. $\left.\begin{array}{rl} 8x+23y &= 2x^3+2y^3 \ldots\ldots\ldots\ldots (1) \\ 34y+6x^2-5y^2 &= 13xy+24 \ldots\ldots\ldots\ldots (2) \end{array}\right\}$

Ans. $\left.\begin{array}{rl} x &= 3 \\ y &= 2 \end{array}\right\}$ $\left.\begin{array}{rl} x &= \dfrac{-181}{133} \\ y &= \dfrac{34}{133} \end{array}\right\}$ $\left.\begin{array}{rl} x &= \dfrac{55\mp\sqrt{1114}}{26} \\ y &= \dfrac{-9\pm 3\sqrt{1114}}{26} \end{array}\right\}$

Ex. 9. $\left.\begin{array}{rl} (x-y)(x^2-y^2) &= a \ldots\ldots\ldots\ldots\ldots\ldots (1) \\ (x+y)(x^2+y^2) &= b \ldots\ldots\ldots\ldots\ldots\ldots (2) \end{array}\right\}$

Ans. $x = \dfrac{\sqrt{2b-a}\pm\sqrt{a}}{2\sqrt[6]{2b-a}}$, $y = \dfrac{\sqrt{2b-a}\mp\sqrt{a}}{2\sqrt[6]{2b-a}}$

Ex. 10. $\left.\begin{array}{rl} \dfrac{xyz}{x+y} &= a \ldots\ldots\ldots\ldots\ldots\ldots (1) \\ \dfrac{xyz}{y+z} &= b \ldots\ldots\ldots\ldots\ldots\ldots (2) \\ \dfrac{xyz}{x+z} &= c \ldots\ldots\ldots\ldots\ldots\ldots (3) \end{array}\right\}$

Ans. $x = \pm\sqrt{\dfrac{2abc(ab+bc-ac)}{(ab+ac-bc)(bc+ac-ab)}}$

$y = \pm\sqrt{\dfrac{2abc(bc+ac-ab)}{(ab+ac-bc)(ab+bc-ac)}}$

$z = \pm\sqrt{\dfrac{2abc(ab+ac-bc)}{(ab+bc-ac)(bc+ac-ab)}}$

PROBLEMS WHICH PRODUCE QUADRATIC EQUATIONS.

Problem 1.

166. To find a number such, that twice its square, augmented by three times the number, is equal to 65.

Let x be the number required, we have for the equation of the problem

$$2x^2+3x = 65$$

Solving the equation, $x = -\frac{3}{4}\pm\sqrt{\frac{65}{2}+\frac{9}{16}} = -\frac{3}{4}\pm\frac{23}{4}$

Hence $x = 5; \quad x = -\frac{13}{2}$

The first of these two values satisfies the conditions of the problem, as stated in the enunciation; for, in fact,

$$2(5)^2 + 3 \times 5 = 2 \times 25 + 15$$
$$= 65$$

In order to interpret the meaning of the second value, let us observe, that if we substitute $-x$ for $+x$ in the equation $2x^2 + 3x = 65$, the coefficient of $3x$ alone will change its sign, for $(-x)^2 = (+x)^2 = x^2$. Hence the value of x will no longer be

$$x = -\frac{3}{4} \pm \frac{23}{4}$$

but will become
$$x = +\frac{3}{4} \pm \frac{23}{4}$$

Hence
$$x = \frac{13}{2}; \quad x = -5$$

where the values of x differ from those already found in sign alone.

Hence we may conclude, that the negative solution $-\frac{13}{2}$, considered without reference to its sign, is the solution of the following problem:

To find a number such, that twice its square, *diminished* by three times the number, is equal to 65.

In fact we have,

$$2\left(\frac{13}{2}\right)^2 - 3 \times \frac{13}{2} = \frac{169}{2} - \frac{39}{2}$$
$$= 65$$

Problem 2.

A tailor bought a certain number of yards of cloth for 12 pounds. If he had paid the same sum for 3 yards less of the same cloth, then the cloth would have cost 4 shillings a yard more. Required the number of yards purchased.

Let x be the number of yards purchased,

Then $\frac{240}{x}$ is the price of one yard, expressed in shillings;

If he had paid the same sum for 3 yards less, in that case the price of each would be represented by $\frac{240}{x-3}$;

But by the conditions of the problem, this last price is greater than the former by 4 shillings; hence the equation of the problem will be,

$$\frac{240}{x-3} = \frac{240}{x} + 4$$

Or,
$$x^2 - 3x = 180$$

Whence
$$x = \frac{3}{2} \pm \sqrt{\frac{9}{4} + 180} = \frac{3}{2} \pm \frac{27}{2}$$
$$\therefore \quad x = 15; \quad x = -12$$

The value of $x = 15$ satisfies the conditions of the problem, for

$$\frac{240}{15} = 16; \qquad \frac{240}{12} = 20$$

the price of each yard in the first case being 16 shillings, and in the latter case 20, which exceeds the former by 4 shillings.

With regard to the second solution, we can form a new enunciation to which it will correspond. Resuming the original equation, and changing x into $-x$ it becomes

$$\frac{240}{-x-3} = \frac{240}{-x} + 4 .$$

Or,

$$\frac{240}{x+3} = \frac{240}{x} - 4$$

an equation which may be considered as the algebraic representation of the following problem:

A tailor bought a certain number of yards of cloth for 12 pounds. If he had paid the same sum for 3 yards *more*, then the cloth would have cost 4 shillings a yard *less*. Required the number of yards purchased.

The above equation when reduced becomes

$$x^2 + 3x = 180$$

instead of $x^2 - 3x = 180$, as in the former case; solving the above, we find

$$x = 12; \qquad x = -15.$$

The two preceding problems illustrate the principle explained with regard to problems of the first degree.

Problem 3.

A merchant purchased two bills; one for £8776, payable in 9 months, the other for £7488, payable in 8 months. For the first he paid £1200 more than for the second. Required the rate of interest allowed.

Let x represent the interest of £100 for 1 month,

Then, $12x$, $9x$, $8x$, severally represent the interest of £100, for 1 year, 9 months, 8 months,

And, $100 + 9x$, $100 + 8x$, represent what a capital of £100 will become at the end of 9 and of 8 months, respectively.

Hence, in order to determine the actual value of the two bills, we have the following proportions:

$$100 + 9x : 100 :: 8776 : \frac{8776 \times 100}{100 + 9x}$$

$$100 + 8x : 100 :: 7488 : \frac{7488 \times 100}{100 + 8x}$$

the fourth terms of the above proportions express the sum paid by the merchant for each of the bills.

Hence by the conditions of the problem,

$$\frac{877600}{100 + 9x} - \frac{748800}{100 + 8x} = 1200$$

Or, dividing each member by 400,

$$\frac{2194}{100 + 9x} - \frac{1872}{100 + 8x} = 8$$

Clearing of fractions and reducing,

$$216x^2 + 4396x = 2200$$

Whence

$$x = -\frac{2198}{216} \pm \sqrt{\frac{2200}{216} + \left(\frac{2198}{216}\right)^2}$$

$$= \frac{-2198 \pm \sqrt{5306404}}{216}$$

$$\therefore \quad 12x = \frac{-2198 \pm \sqrt{5306464}}{18}$$

$$= \frac{-2198 \pm 2303.5.........}{18}$$

$$\therefore \quad 12x = 5.86........; \text{ and, } 12x = -250.08........$$

The positive solution, $12x = 5.86$ ......, represents the required rate of interest per cent. per annum.

With regard to the negative solution, it can only be considered as connected with the other by the same equation of the second degree. If we resume the original equation, and substitute $-x$ for $+x$, we shall find great difficulty in reconciling this new equation with an enunciation analogous to that of the proposed problem.

Problem 4.

A man purchased a horse, which he afterwards sold, to disadvantage, for 24 pounds. His loss per cent, by this bargain, upon the original price of the horse, is expressed by the number of pounds which he paid for the horse. Required the original price.

Let x be the number of pounds which he paid for the horse

Then, $x - 24$ will represent his loss;

But, by the conditions of the problem, his loss per cent. is represented by the number of units in x;

His loss per cent. on one pound is $\frac{x}{100}$

his loss per cent on x pounds must be $\frac{x^2}{100}$, or x times as great.

This gives us the equation

$$\overline{100}$$

$$x = 50 \pm \sqrt{100} = 50 \pm 10$$

$$\text{Hence,} \quad x = 60; \quad x = 40$$

Both these solutions equally fulfil the conditions of the problem.

Let us suppose, in the first place, that he paid 60 pounds for the horse; since he sold it for 24; his loss was 36. On the other hand, by the enunciation, his loss was 60 per cent. on the original price; *i. e.* $\frac{60}{100}$ of 60, or $\frac{60 \times 60}{100} = 36$; thus 60 satisfies the conditions.

In the second place, let us suppose that he paid 40 pounds; his loss in this was 16. On the other hand, his loss ought to be 40 per cent. on the original price; *i. e.* $\frac{40}{100}$ of 40, or $\frac{40 \times 40}{100} = 16$; thus 40 also satisfies the conditions.

General Discussion of the Equation of the Second Degree.

167. In the problems of the second degree which we have hitherto solved, the given quantities have been expressed by particular numbers. But, in order that we may be enabled to resolve general problems, and to interpret the various results at which we may arrive, from assigning particular values to the given quantities, we must resume the equation of the second degree under its most general form, and examine the circumstances which arise from making every possible hypothesis, with regard to the coefficients. Such is the object of the discussion of the equation of the second degree.

168. Before commencing this discussion, we shall notice another method of solving the equation of the second degree, which leads to important consequences.

We have seen that every equation of the second degree may be reduced to the form

$$x^2 + px + q = 0 \text{ .... } (1)$$

where p and q are given quantities, numerical or algebraical, integral or fractional, positive or negative.

This being premised, transposing q and adding $\frac{p^2}{4}$ to each member, in order to render the first member a perfect square, the equation becomes

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} - q$$

or,

$$\left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} - q$$

Whatever may be the value of $\frac{p^2}{4} - q$, we can always represent its square root by m, and the equation then becomes,

$$\left(x + \frac{p}{2}\right)^2 = m^2$$

or,

$$\left(x + \frac{p}{2}\right)^2 - m^2 = 0$$

The first member of this equation, being the difference of two squares, may be put under the form,

$$\left(x + \frac{p}{2} + m\right)\left(x + \frac{p}{2} - m\right)$$

which gives the new equation,

$$\left(x + \frac{p}{2} + m\right)\left(x + \frac{p}{2} - m\right) = 0 \text{ .... } (2)$$

an equation in which the first member is the product of two factors, and the second member is 0. We may render this product $=0$, and consequently satisfy equation (2), in two different ways.

Either, by putting $x+\frac{p}{2}-m=0$, whence, $x=-\frac{p}{2}+m$

Or, by putting $x+\frac{p}{2}+m=0$, whence, $x=-\frac{p}{2}-m$

That is, substituting for m its value, $\begin{cases} x=-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q} \\ x=-\frac{p}{2}-\sqrt{\frac{p^2}{4}-q} \end{cases}$

It is, moreover, manifest that we cannot render the first member of equation (2) equal to 0, unless by putting for x some quantity which shall render one of the two factors which compose the expression equal to 0.

Hence, since equation (2) is a consequence of equation (1), and reciprocally, it follows, that,

Every equation of the second degree admits of two values of the unknown quantity, and not more than two.

These values possess some remarkable properties.

I. Since the equation,

$$x^2+px+q=0$$

may, by a series of transformations, be reduced to the form,

$$(x+\frac{p}{2}+m)\ (x+\frac{p}{2}-m)=0$$

being equal to $\sqrt{\frac{p^2}{4}-q}$, it follows, that,

The first member, x^2+px+q, *of every equation of the second degree, whose second member is* 0, *is composed of the product of two binomial factors of the first degree in* x, *having* x *for a common term, and for their second terms the two values of* x *with their signs changed.*

This property has caused the name of *roots* of the equation to be given to the two values of the unknown quantity, for if we know the values of the unknown quantity we can determine the equation. Thus,

Take the equation $x^2+3x-28=o$, which, when solved, gives,

$$x=4;\quad x=-7$$

the first member of the above equation results from the product $(x-4)\ (x+7)$; in fact.

$$\begin{aligned}(x-4)\ (x+7) &= x^2-4x+7x-28\\ &= x^2+3x-28\end{aligned}$$

II. If we represent the two roots of the equation by α and β, by the preceding property, we have,

$$x^2+px+q=(x-\alpha)\ (x-\beta)$$

Now,

$$\alpha=-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}$$

$$\beta = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}$$

Adding, $\quad \alpha + \beta = -\frac{p}{2} - \frac{p}{2} \quad = -p$

Multiplying, $\quad \alpha\beta = \frac{p^2}{4} - (\frac{p^2}{4} - q) = +q$

Hence, it appears that,

1°. *The algebraic sum of the two roots is equal to the coefficient of the second term of the equation with its sign changed.*

2°. *The product of the two roots is equal to the last term of the equation.*

DISCUSSION.

169. Let us resume the general equation,

$$x^2 + px + q = 0$$

which, when solved, gives,

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Let us make different hypotheses, successively, with regard to the coefficients.

I. Let q be positive, and $< \frac{p^2}{4}$, and let p be positive.

The equation in this case, the coefficients being written with their proper signs, will be of the form,

$$x^2 + px + q = 0$$

which, when solved, gives,

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

q being less than $\frac{p^2}{4}$, $\frac{p^2}{4} - q$ is a positive quantity, and the root $\sqrt{\frac{p^2}{4} - q}$ can always be extracted, either exactly or approximately, and must be some quantity less than $\frac{p}{2}$; hence, in this case, *the two values of* x *will both be negative.*

II. Let q be positive, and $< \frac{p^2}{4}$, and let p be negative,

Here the equation will be,

$$x^2 - px + q = 0$$

Whence, $\quad x = +\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$

reasoning as above, it is manifest, that in this case *the two values of* x *will both be positive.*

III. IV. Let q be positive, and $> \frac{p^2}{4}$, and let p be either positive or negative.

Here the equation will be,

$$x^2 \pm px + q = 0$$

Whence, $$x = \mp\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

q being greater than $\frac{p^2}{4}$, the quantity under the radical will be negative, and consequently, in each of the above cases (*i. e.* both when p is positive and when p is negative,) *the two values of* x *will be imaginary.*

In fact, if we examine the general equation, we shall find that the conditions are absurd, for transposing q, and completing the square, we have,

$$x^2 \pm px + \frac{p^2}{4} = \frac{p^2}{4} - q$$

but since $\frac{p^2}{4} - q$ is, by hypothesis, a negative quantity, we may represent it by $-m$, where m is some positive quantity,

$$x^2 \pm px + \frac{p^2}{4} = -m$$

$$\left(x \pm \frac{p}{2}\right)^2 + m = 0$$

that is, the sum of two quantities, each of which is essentially positive, is equal to 0, a manifest absurdity. Solving the equation,

$$x = \mp\frac{p}{2} \pm \sqrt{-m}$$

and the symbol $\sqrt{-m}$, which denotes absurdity, serves to distinguish this case. Hence, *when the roots are imaginary, the problem to which the equation corresponds is absurd.*

We still say, however, that the equation has two roots, for subjecting these values of x to the same calculations as if they were real, that is, substituting them for x in the proposed equations, we shall find that they render the two members identical.

V. VI. VII. VIII. Let q be negative, and either $>$ or $< \frac{p^2}{4}$, p either positive or negative,

Here the equation will be,

$$x^2 \pm px - q = 0$$

Whence, $$x = \mp\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}$$

Since $\frac{p^2}{4} + q$ is always a positive quantity, the root $\sqrt{\frac{p^2}{4} + q}$ can always be extracted, either exactly or approximately, and must be some quantity greater than $\frac{p}{2}$; consequently, in each of the four above cases (*i. e.* whether q be greater or less than $\frac{p^2}{4}$, and whether p be positive or negative), *one value of* x *will be positive, and the other negative.*

IX. Let q be positive, and $= \frac{p^2}{4}$, p positive,

Resuming the equation,

$$x^2 + px + q = 0$$

Whence,

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

The radical $\sqrt{\frac{p^2}{4} - q}$ vanishes upon the supposition that $q = \frac{p^2}{4}$ and the two values of x are reduced to $-\frac{p}{2}$; in this case, we say that *the two roots are equal.*

If we take the original equation, and substitute $\frac{p^2}{4}$ for q, it becomes,

$$x^2 + px + \frac{p^2}{4} = 0$$

or,

$$\left(x + \frac{p}{2}\right)^2 = 0$$

In this case, the first member of the equation is *the product of two equal factors,* and we therefore conclude that the two roots are equal, for each of the factors when equated to 0, will give the same value of x.

X. Upon the same hypothesis, if p be negative, the two roots will be equal and each $= +\frac{p}{2}$.

XI. Let $q = 0$, and let p be positive.
Here the two values of x, in the solution of the general equation, will be reduced to

$$x = -\frac{p}{2} + \frac{p}{2} = 0, \text{ and, } x = -\frac{p}{2} - \frac{p}{2} = -p$$

In this case, the general equation is of the form,

$$x^2 + px = 0$$

or,

$$x(x + p) = 0$$

an equation which can be verified, either by putting $x = 0$, or, $x + p = 0$.

XII. Upon the same hypothesis, if p be negative, the two roots of the equation will be $x = 0$, $x = +p$.

XIII. Let q be negative, and let $p = 0$.
The general equation will become,

$$x^2 - q = 0$$

whence

$$x = \pm\sqrt{q}$$

that is, in this case the two values of x will be equal, and have opposite signs.

XIV. Let q be positive, and $p = 0$.
This case is the same as the last, with this difference, that the two values of x will be imaginary; for we shall have

$$x^2 + q = 0$$

$$x = \pm\sqrt{-q}$$

The two last cases belong to the class of equations which we have treated under the title of *Pure Quadratics*.

XV. Let $q = 0$, and $p = 0$.
The equation will then be reduced to

$$x^2 = 0$$

and the two values of x will then be each $= 0$.

We may exhibit the results at which we have arrived in the following table:

The general form of the equations, the coefficients being considered independently of their signs, is

$$x^2 + px + q = 0$$

I. II. Let q be positive and $< \frac{p^2}{4}$

$$\begin{cases} \text{I.} & p \text{ positive, } x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}, \text{ both values negative.} \\ \text{II.} & p \text{ negative, } x = +\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}, \text{ both values positive.} \end{cases}$$

III. IV. Let q be positive and $> \frac{p^2}{4}$

$$\left.\begin{cases} \text{III.} & p \text{ positive, } x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}, \\ \text{IV.} & p \text{ negative, } x = +\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}, \end{cases}\right\} \text{both values imaginary}$$

V. VI. Let q be negative and $< \frac{p^2}{4}$

$$\begin{cases} \text{V.} & p \text{ positive, } x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}, \\ \text{VI.} & p \text{ negative, } x = +\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}, \end{cases}$$

VII. VIII. Let q be positive and $> \frac{p^2}{4}$

one value positive, and one negative.

$$\begin{cases} \text{VII.} & p \text{ positive, } x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}, \\ \text{VIII.} & p \text{ negative, } x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}, \end{cases}$$

IX. X. Let $q = \frac{p^2}{4}$

$$\left.\begin{cases} \text{IX.} & p \text{ positive, } x = -\frac{p}{2}. \\ \text{X.} & p \text{ negative, } x = +\frac{p}{2} \end{cases}\right\} \text{the two values equal}$$

XI. XII. Let $q = 0$,

$$\begin{cases} \text{XI.} \quad p \text{ positive}, \; x = -\frac{p}{2} \pm \frac{p}{2}, \text{ one value } = -p, \text{ the other } = 0. \\ \text{XII.} \quad p \text{ negative}, \; x = +\frac{p}{2} \pm \frac{p}{2}, \text{ one value } = +p, \text{ the other } = 0. \end{cases}$$

XIII. Let q be negative,

$\{$ XIII. $p = 0, \quad x = \pm\sqrt{q}$, the two values equal with opposite signs.

XIV. Let q be positive,

$\{$ XIV. $p = 0, \quad x = \pm\sqrt{-q}$, both values imaginary.

XV. Let $q = 0$,

$\{$ XV. $p = 0, \quad x = 0$, both values equal to 0.

XVI. One case, attended with remarkable circumstances, still remains to be examined. Let us take the equation,

$$ax^2 + bx - c = 0$$

Whence,

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

Let us suppose that, in accordance with a particular hypothesis made on the given quantities in the equation, we have $a = 0$; the expression for x then becomes

$$x = \frac{-b \pm b}{0}, \text{ whence } \begin{cases} x = \frac{0}{0} \\ x = \frac{-2b}{0} \end{cases}$$

the second of the above values is under the form of infinity, and may be considered as an answer, if the problem proposed be such as to admit of infinite solutions.

We must endeavour to interpret the meaning of the first, $\frac{0}{0}$.

In the first place, if we return to the equation $ax^2 + bx - c = 0$, we perceive that the hypothesis $a = 0$ reduces it to $bx + c$, whence we derive $x = \frac{c}{b}$, a *finite* and *determinate* expression, which must be considered as representing the true value of $\frac{0}{0}$ in the case before us.

That no doubt may remain on this subject, let us assume the equation

$$ax^2 + bx - c = 0$$

and put $x = \frac{1}{y}$, the expression will then become

$$\frac{a}{y^2} + \frac{b}{y} - c = 0$$

Whence,

$$cy^2 - by - a = 0$$

Let $a = 0$, this last equation will become

$$cy^2 - by \quad = 0$$

from which we have the two values $y = 0$, $y = \frac{b}{c}$; substituting these values in $x = \frac{1}{y}$, we deduce

$$1^{\circ}. \quad x = \frac{1}{0}; \qquad 2^{\circ}. \quad x = \frac{c}{b}$$

If, in addition to the hypothesis $a = 0$, we have also $b = 0$, the value $x = \frac{c}{b}$ becomes $\frac{c}{0}$, or *infinite.*

In fact, the equation $cy^2 - by - a = 0$, under this double hypothesis, is reduced to $cy^2 = 0$, an equation in which the values of y are equal, and each $= 0$. Hence the two corresponding values of x will both be *infinite.*

If we suppose $a = 0$, $b = 0$, $c = 0$, the proposed equation will become altogether indeterminate.

170. Let us now proceed to illustrate the principles established in this general discussion, by applying them to different problems.

Problem 5.

To find in a line A B which joins two lights of different intensities, a point which is illuminated equally by each.

P_2 A P_1 B P_3

(It is a principle in Optics that the intensities of the same light at different distances are inversely as the squares of the distances.)

Let a be the distance A B between the two lights,
... b be the intensity of the light A at the distance of one foot from A,
... c be the intensity of the light B at the distance of one foot from B,
.. P_1 be the point required,
... $AP_1 = x$, $\quad\therefore\quad BP_1 = a - x$.

By the optical principle above enunciated, since the intensity of A at the distance of 1 foot is b, its intensity at the distance of 2, 3, 4, ... feet, must be $\frac{b}{4}, \frac{b}{9}, \frac{b}{16}$; hence the intensity of A at the distance of x feet must be $\frac{b}{x^2}$. In the same manner, the intensity of B at the distance $a - x$ must be $\frac{c}{(a-x)^2}$: but according to the conditions of the question, these two intensities are equal, hence we have for the equation of the problem,

$$\frac{b}{x^2} = \frac{c}{(a-x)^2}$$

Solving this equation, and reducing the result to its most simple form,

$$x = \frac{a\sqrt{b}}{\sqrt{b} \pm \sqrt{c}}$$

We shall now proceed to discuss these two values:

$$\left.\begin{array}{l} 1^\circ. \ \ldots\ldots x = \dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}} \\ 2^\circ. \ \ldots\ldots x = \dfrac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}} \end{array}\right\} \text{whence,} \left\{\begin{array}{l} a-x = \dfrac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}} \\ a-x = \dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}} \end{array}\right.$$

I. Let $b > c$.

The first value of x, $\dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, is positive, and less than a, for $\dfrac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is a proper fraction; hence, this value gives for the point equally illuminated, a point P_1, situated between the points A and B. We perceive, moreover, that the point P_1 is nearer to B than to A; for, since $b > c$, we have, $\sqrt{b}+\sqrt{b} > \sqrt{b}+\sqrt{c}$, or, $2\sqrt{b} > \sqrt{b}+\sqrt{c}$, and $\therefore \dfrac{\sqrt{b}}{\sqrt{b}+\sqrt{c}} > \dfrac{1}{2}$, and consequently, $\dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}} > \dfrac{a}{2}$. This is manifestly the result at which we ought to arrive, for we here suppose the intensity of A to be greater than that of B.

The corresponding value of $a-x$, $\dfrac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$, is positive, and less than $\dfrac{a}{2}$.

The second value of x, $\dfrac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, is positive, and greater than a, for $\sqrt{b} > \sqrt{b}-\sqrt{c}$, $\therefore \dfrac{\sqrt{b}}{\sqrt{b}-\sqrt{c}} > 1$, and $\therefore \dfrac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}} > a$. This second value gives a point P_2, situated in the production of A B, and to the right of the two lights. In fact, we suppose that the two lights give forth rays in all directions, there may therefore be a point in the production of A B equally illuminated by each, but this point must be situated in the production of A B to the right, in order that it may be nearer to the less powerful of the two lights.

It is easy to perceive why the two values thus obtained are connected by the same equation. If, instead of assuming A P_1 for the unknown quantity x, we take A P_2, then B $P_2 = x-a$, thus we have the equation $\dfrac{b}{x^2} = \dfrac{c}{(x-a)^2}$; but since $(x-a)^2$ is identical with $(a-x)^2$, the new equation is the same as that already established, and which consequently ought to give A P_2 as well as A P_1.

The second value of $a-x$, $\dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, is negative, as it ought to be, because $x > a$; but changing the signs of the equation $a-x = \dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, we find $x-a = \dfrac{a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, and this value of $x-a$ represents the absolute length of B P_2.

II. Let $b < c$.

The first value of x, $\dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is positive, and less than $\dfrac{a}{2}$, for $\sqrt{b}+\sqrt{c} > \sqrt{b}+\sqrt{b}$, $\therefore \sqrt{b}+\sqrt{c} > 2\sqrt{b}$, $\therefore \dfrac{\sqrt{b}}{\sqrt{b}+\sqrt{c}} < \dfrac{1}{2}$, $\therefore \dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}} < \dfrac{a}{2}$.

The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$, is positive, and greater than $\frac{a}{2}$.

Hence, the point P_1 is situated between the points A and B, and is nearer to A than to B. This is manifestly the true result, for the present hypothesis supposes that the intensity of B is greater than the intensity of A.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, or $\frac{-a\sqrt{b}}{\sqrt{c}-\sqrt{b}}$ is essentially negative. In order to interpret the signification of this result, let us resume the original equation and substitute $-x$ for $+x$, it thus becomes $\frac{b}{x^2}=\frac{c}{(a+x)^2}$. But since $(a-x)$ expresses in the first instance the distance of B from the point required, $a+x$ ought still to express this same distance, and therefore the point required must be situated to the left of A, in P_3, for example. In fact, since the intensity of the light B, is, under the present hypothesis, greater than the intensity of A, the point required must be nearer to A than to B.

The corresponding value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, or $\frac{a\sqrt{c}}{\sqrt{c}-\sqrt{b}}$, is positive, and the reason of this is, that, x being negative, $a-x$ expresses, in reality, an *arithmetical sum.*

III. Let $b=c$.

The first two values of x and of $a-x$ are reduced to $\frac{a}{2}$, which gives the bisection of A B for the point equally illuminated by each light, a result which is manifestly true, upon the supposition that the intensity of the two lights is the same.

The other two values are reduced to $\frac{a\sqrt{b}}{0}$, that is, they become *infinite*, that is to say, the second point equally illuminated is situated at a distance from the points A and B greater than any which can be assigned. This result perfectly corresponds with the present hypothesis; for if we suppose the difference $b-c$, without vanishing altogether, to be exceedingly small, the second point equally illuminated, exists, but at a great distance from the two lights; this is indicated by the expression $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, the denominator of which is exceedingly small in comparison with the numerator if we suppose b very nearly equal to c. In the extreme case, when $b=c$, or $\sqrt{b}=\sqrt{c}$, the point required no longer exists, or is situated at an infinite distance.

IV. Let $b=c$, and $a=0$.

The first system of values of x and $a-x$ in this case become 0, and the second system $\frac{0}{0}$. This last result is here the symbol of indetermination; for if we recur to the equation of the problem

$$\frac{b}{x^2}=\frac{c}{(a-x)^2}$$

Or,

$$(b-c)x^2 - 2abx = -a^2b$$

it becomes under the present hypothesis

$$0.x^2 - 0.x = 0$$

an equation which can be satisfied by the substitution of any number whatever for x. In fact, since the two lights are supposed to be equal in intensity and to be placed at the same point, they must illuminate *every point* in the line A B *equally*.

The solution 0, given by the first system, is one of those solutions, *infinite in number*, of which we have just spoken.

V. Let $a = 0$, b not being $= c$.

Each of the two systems in this case is reduced to 0, which proves, that in this case, there is only one point equally illuminated, viz. *the point in which the two lights are placed.*

The above discussion affords an example of the precision with which algebra answers to all the circumstances included in the enunciation of a problem.

We shall conclude this subject by solving one or two problems which require the introduction of more than one unknown quantity.

Problem 6

To find two numbers such that when multiplied by the numbers a and b respectively, the sum of the products may be equal to $2s$, and the product of the two numbers equal to p.

Let x and y be the two numbers sought, the equations of the problem will be

$$ax + by = 2s \ldots\ldots (1)$$
$$xy = p \ldots\ldots (2)$$

From (1)

$$y = \frac{2s - ax}{b}$$

Substituting this value in (2) and reducing, we have

$$ax^2 - 2sx + bp = 0$$

Whence,

$$x = \frac{s}{a} \pm \frac{1}{a}\sqrt{s^2 - apb}$$

And, ∴

$$y = \frac{s}{b} \mp \frac{1}{b}\sqrt{s^2 - apb}$$

The problem is, we perceive, susceptible of two direct solutions, for s is manifestly $> \sqrt{s^2 - abp}$; but in order that these solutions may be real we must have $s^2 >$ or $= abp$.

Let $a = b = 1$; in this case the values of x and y are reduced to

$$x = s \pm \sqrt{s^2 - p} \quad , \quad y = s \mp \sqrt{s^2 - p}$$

Here we perceive, that the two values of y are equal to those of x taken in an inverse order; that is to say, if $s+\sqrt{s^2-p}$ represent the value of x, then $s-\sqrt{s^2-p}$ will represent the corresponding value of y, and reciprocally.

We explain this circumstance by observing, that in this particular case the equations of the problem are reduced to $x+y=2s$, $xy=p$ and the question then becomes, Required two numbers whose sum is $2s$, and whose product is p, or, in other words, *To divide a number* 2 s *into two parts, such that their product may be equal to* p.

Problem 7.

To find four numbers in proportion, the sum of the extremes being $2s$, the sum of the means $2s'$, and the sum of the squares of the four terms $4c^2$.

Let a, x y, z, represent the four terms of the proportion; by the conditions of the question, and the fundamental property of proportions, we shall have as the equations of the problem

$$a+z = 2s \ldots\ldots (1)$$
$$x+y = 2s' \ldots\ldots (2)$$
$$xy = az \ldots\ldots (3)$$
$$a^2+x^2+y^2+z^2 = 4c^2 \ldots\ldots (4)$$

Squaring (1) and (2) and adding the results,

$$a^2+x^2+y^2+z^2+2az+2xy = 4(s^2+s'^2)$$

But by (4), $a^2+x^2+y^2+z^2 = 4c^2$

Subtracting, $2az+2xy = 4(s^2+s'^2-c^2)$

$\therefore$ by (3), $4az = 4(s^2+s'^2-c^2) = 4xy$.(5)

Squaring (1), $a^2+2az+z^2 = 4s^2$

But by (5) $4az = 4(s^2+s'^2-c^2)$

Subtracting, $a^2-2az+z^2 = 4(c^2-s'^2)$

Extracting the root, $a-z = \pm 2\sqrt{c^2-s'^2}$

But by (1) $a+z = 2s$

$\therefore$ adding and subtracting, $a = s\pm\sqrt{c^2-s'^2}$

$z = s\mp\sqrt{c^2-s'^2}$

Precisely in the same manner we shall find $x = s'\pm\sqrt{c^2-s^2}$

$y = s'\mp\sqrt{c^2-s^2}$

The four numbers will therefore be

$$a = s+\sqrt{c^2-s'^2}, \quad x = s'+\sqrt{c^2-s^2}$$
$$z = s-\sqrt{c^2-s'^2}, \quad y = s'-\sqrt{c^2-s^2}$$

These four numbers constitute a proportion, for we have

$$az = (s+\sqrt{c^2-s'^2})(s-\sqrt{c^2-s'^2}) = s^2-c^2+s'^2$$
$$xy = (s'+\sqrt{c^2-s^2})(s'-\sqrt{c^2-s^2}) = s'^2-c^2+s^2$$

Prob. 8. What two numbers are those, whose sum is 20, and their product 36? Ans. 2 and 18.

Prob. 9. To divide the number 60 into two such parts, that their product may be to the sum of their squares, in the ratio of 2 to 5. Ans. 20 and 40.

Prob. 10. The difference of two numbers is 3, and the difference of their cubes is 117; what are those numbers? Ans. 2 and 5.

Prob. 11. A company at a tavern had £8 15*s.* to pay for their reckoning; but, before the bill was settled, two of them left the room, and then those who remained had 10*s.* a-piece more to pay than before: how many were there in company? Ans. 7.

Prob. 12. A grazier bought as many sheep as cost him £60, and, after reserving 15 out of the number, he sold the remainder for £54, and gained 2*s.* a head by them; how many sheep did he buy? Ans. 75.

Prob. 13. There are two numbers, whose difference is 15, and half their product is equal to the cube of the lesser number; what are those numbers? Ans. 3 and 18.

Prob. 14. A person bought cloth for £33 15*s.* which he sold again at £2. 8*s.* per piece, and gained by the bargain as much as one piece cost him; required the number of pieces? Ans. 15.

Prob. 15. What number is that, which, when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? Ans. 24.

Prob. 16. What two numbers are those, whose sum multiplied by the greater is equal to 77; and whose difference multiplied by the lesser is equal to 12? Ans. 4 and 7.

Prob. 17. To find a number such, that if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21. Ans. 7 or 3.

Prob. 18. To divide 100 into two such parts, that the sum of their square roots may be 14. Ans. 64 and 36.

Prob. 19. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference. Ans. 10 and 14.

Prob. 20. The sum of two numbers is 8, and the sum of their cubes is 152; what are the numbers? Ans. 3 and 5.

Prob. 21. The sum of two numbers is 7, and the sum of their 4th powers is 641; what are the numbers? Ans. 2 and 5.

Prob. 22. The sum of two numbers is 6, and the sum of their 5th powers is 1056; what are the numbers? Ans. 2 and 4.

Prob. 23. Two partners, A and B, gained £140 by trade; A's money was 3 months in trade, and his gain was £60 less than his stock; and B's money which was £50 more than A's, was in trade 5 months; what was A's stock? Ans. £100.

Prob. 24. To find two numbers such that the difference of their squares may

be equal to a given number, q^2; and when the two numbers are multiplied by the numbers a and b respectively, the difference of the products may be equal to a given number, s^2

Ans. $\dfrac{as^2 \pm b\sqrt{s^4-(a^2-b^2)q^2}}{a^2-b^2}$

$\dfrac{bs^2 \pm a\sqrt{s^4-(a^2-b^2)q^2}}{a^2-b^2}$.

Prob. 25. To divide two numbers, a and b, each into two parts, such that the product of one part of a by one part of b may be equal to a given number, p, and the product of the remaining parts of a and b equal to another given number, p'.

$$\text{Ans. } a = \frac{ab-(p'-p) \pm \sqrt{\{ab-(p'-p)\}^2-4abp}}{2b}$$

$$+\frac{ab+(p'-p) \mp \sqrt{\{ab-(p'-p)\}^2-4abp}}{2b}.$$

$$b = \frac{ab-(p'-p) \pm \sqrt{\{ab-(p'-p)\}^2-4abp}}{2a}$$

$$+\frac{ab+(p'-p) \mp \sqrt{\{ab-(p'-p)\}^2-4abp}}{2a}.$$

Prop. 26. To find a number such that its square may be to the product of the differences of that number, and two other given numbers, a and b, in the given ratio, $p : q$.

Ans. $\dfrac{(a+b)p \pm \sqrt{(a-b)^2p^2+4abpq}}{2(p-q)}$.

Prob. 27. A wine merchant sold 7 dozen of sherry and and 12 dozen of claret for 50*l.*; he sold 3 dozen more of sherry for 10*l.* than he sold of claret for 6*l.* Required the price of each.

Ans. Claret, 3*l.*; and sherry, 2*l.* per dozen.

Prob. 28. There is a number consisting of two digits, which, when divided by the sum of its digits, gives a quotient greater by 2 than the first digit; but if the digits be inverted, and the resulting number be divided by a number greater by unity than the sum of the digits, the quotient shall be greater by 2 than the former quotient. What is the number?

Ans. 24.

Prob. 29. A regiment of foot receives orders to send 216 men on garrison duty, each company sending the same number of men; but before the detachment marched, three of the companies were sent on another service, and it was then found that each company that remained would have to send 12 men additional, in order to make up the complement, 216. How many companies were in the regiment, and what number of men did each of the remaining companies send on garrison duty?

Ans. There were 9 companies; and each of the remaining 6 sent 36 men.

ON THE NATURE OF EQUATIONS.

171. The valuable improvements recently made in the process for the determination of the roots of equations of all degrees, render it indispensably necessary to present to the notice of the student a concise view of the present state of this interesting department of analytical investigation. The researches of Messrs. Atkinson and Horner on the method of continuous approximation to the roots of equations, and the beautiful *theorem* of M. Sturm for the complete separation of the real and imaginary roots, have given a fresh impulse to this branch of scientific research, and entirely changed the state of the subject of numerical equations. Indeed, the elegant process of Sturm for discovering the number of real roots, and their initial figures in any numerical equation, combined with the admirable method of continuous approximation as improved by Horner, fully complete the theory and numerical solution of equations of all degrees.

We do not intend to enter at great length into the theory of equations; but it is hoped that the portion of it which we have introduced into the present treatise will be discussed in a simple and perspicuous manner, and be found amply sufficient for most practical purposes.

Definitions.

1. An *equation* is an algebraical expression of equality between two quantities.

2. A *root of an equation* is that number, or quantity, which, when substituted for the unknown quantity in the equation, verifies that equation.

3. A *function* of a quantity is any expression involving that quantity; thus, $a x^2+b$, $a x^3+c x+d$, $\frac{a x^2+b}{c x+d}$, a^x are all functions of x; and also ax^2-by^2, $\sqrt{4x-5y}$, $\frac{2x+3y}{3x-2y}$, $y^2+y x+x^2+a^2+b+2$, are all functions of x and y. These functions are usually written $f(x)$, and $f(x, y)$.

Proposition I.

Any function of x, *of the form*

$$x^n+p x^{n-1}+q x^{n-2}+r x^{n-3}+\ldots\ldots$$

when divided by x—a, *will leave a remainder, which is the same function of* a *that the given polynomial is of* x.

Let $f(x)=x^n+p x^{n-1}+q x^{n-2}+\ldots\ldots$; and, dividing $f(x)$ by $x-a$, let Q denote the quotient thus obtained, and R the remainder which does not involve x; hence, by the nature of division, we have

$$f(x)=\mathrm{Q}\,(x-a)+\mathrm{R}.$$

Now this equation must be true for every value of x; hence, if $x=a$, we have

$$f(a)=0+\mathrm{R};$$

for R is altogether independent of x, and therefore the remainder R is the same function of a that the proposed polynomial is of x.

Examples.

(1.) What is the remainder of x^2-6x+7 divided by $x-2$, without actually performing the operation?

(2.) What is the remainder of $x^3-6x^2+8x-19$, divided by $x-3$?

(3.) What is the remainder of $x^4+6x^3+7x^2+5x-4$, divided by $x-5$?

(4.) What is the remainder of $x^3+p x^2+q x+r$, divided by $x-a$?

ANSWERS.

(1.) $R=2^2-6\times2+7=-1$. (3.) 1571.
(2.) $R=(-3)^3-6(-3)^2+8(-3)-19=-124$. (4.) a^3+pa^2+qa+r.

PROPOSITION II.

If a *is the root of the equation,*

$$x^n+A_1x^{n-1}+A_{11}x^{n-2}+ \ldots\ldots A_{n-2}x^2+A_{n-1}x+A_n=0$$

the first member of the equation is divisible by $x-a$.

Instead of deducing the remainder as in the last proposition, we shall actually perform the division, either by the usual way, or the preferable method of synthetic division, as it keeps the work within the breadth of the page.

By Synthetic Division.

$$\begin{array}{r|l} 1 & 1+A_1+A_{11}+\ldots\ldots A_{n-2}+A_{n-1}+A_n \\ +a & +a+aR_1+\ldots\ldots +aR_{n-2}+aR_{n-1} \\ \hline & 1+R_1+R_{11}+\ldots\ldots R_{n-2}+R_{n-1}+R_n. \end{array}$$

Hence the quotient is

$$x^{n-1}+R_1x^{n-2}+R_{11}x^{n-3}+ \ldots . R_{n-2}x+R_{n-1};$$

$$\begin{array}{llll} \text{where } R_1 & =A_1+a & R_{n-2} & =A_{n-2}+aR_{n-3} \\ R_{11} & =A_{11}+aR_1 & R_{n-1} & =A_{n-1}+aR_{n-2} \\ R_{111} & =A_{111}+aR_{11}, \&c. & R_n & =A_n+aR_{n-1}; \end{array}$$

and by successive substitutions we have the final remainder

$$\begin{aligned} R_n &= A_n+aR_{n-1} \\ &= A_n+a\{A_{n-1}+aR_{n-2}\} \\ &= A_n+a\ A_{n-1}+a^2\{A_{n-2}+aR_{n-3}\} \\ &= A_n+a\ A_{n-1}+a^2\ A_{n-2}+a^3\{A_{n-3}+aR_{n-4}\} \\ &\quad\vdots \\ &= A_n+a\ A_{n-1}+a^2\ A_{n-2}+\ldots a^{n-2}A_{11}+a^{n-1}A_1+a^n \end{aligned}$$

Now this remainder is the same function of a that the first member of the proposed equation is of x; and, therefore, since a is a root of the equation, the remainder vanishes, and the polynomial, or first member of the equation, is divisible by $x-a$.

Conversely, if the first member of an equation, f(x)=0, *be divisible by* x—a, *then* a *is a root of the equation.*

For, by the foregoing demonstration, the final remainder is $f(a)$; but since $f(x)$, or the first member of the equation, is divisible by $x-a$, the remainder must vanish; hence $f(a)=0$, and therefore, a being substituted for x in the equation $f(x)=0$, verifies the equation, and consequently a is a root of the equation.

PROPOSITION III.

Every equation containing but one unknown quantity has as many roots as there are units in the highest power of the unknown quantity.

Let $f(x)=0$ be an equation of the nth degree; then if a_1 be a root of this equation, we have, by last proposition,

$$(x-a_1)f_1(x)=f(x)=0,$$

where $f_1(x)$ represents the quotient arising from the division of $f(x)$ by $x-a_1$. Now if a_2 is also a root of the equation $f(x)=0$; it is obvious that

$f_1(x)$ must be divisible by $x-a_2$, for $x-a_1$ is not divisible by $x-a_2$; hence, if $f_2(x)$ represent the quotient of $f_1(x)$ divided by $x-a_2$, we have

$$(x-a_1)(x-a_2)f_2(x)=f(x)=0.$$

In like manner, if $a_3, a_4, a_5, \ldots\ldots a_n$ are roots of the equation, the polynomial $f(x)$ is divisible by $x-a_3, x-a_4, \ldots\ldots x-a_n$, and the equation will, therefore, assume the form

$$(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)=0;$$

and, consequently, there are as many roots as factors, that is, as units in the highest power of x, the unknown quantity; for the last equation will be verified by any one of the n conditions,

$$x=a_1,\ x=a_2,\ x=a_3,\ x=a_4,\ \ldots\ldots\ x=a_n:$$

and since the equation contains n factors, there are n roots.

Cor. When one root of an equation is known, the depressed equation containing the remaining roots is readily found by synthetic division; and if two or more roots are known, the equation containing the remaining roots is found by two or more corresponding divisions.

Examples.

(1.) One root of the equation $x^4-25x^2+60x-36=0$ is 3; find the equation containing the remaining roots.

$$\begin{array}{l}1+0\ -25\ +60-36\ (3\\ \quad\ \ 3\ +\ 9\ -48+36\\ \hline 1+3\ -16\ +12.\end{array}$$

Hence $x^3+3x^2-16x+12=0$

is the equation containing the remaining roots.

(2.) Two roots of the equation $x^4-12x^3+48x^2-68x+15=0$, are 3 and 5; find the quadratic containing the remaining roots.

$$\begin{array}{l}1-12\ +48-68+15\ (3\\ \quad\ \ 3\ -27+63-15\\ \hline 1-\ 9\ +21-\ 5\ (5\\ \quad\ \ 5\ -20\\ \hline 1-\ 4\ +1\end{array}$$

$$\therefore\ x^2-\ 4x+1=0$$

is the equation containing the two remaining roots.

(3.) One root of the cubic equation $x^3-6x^2+11x-6=0$ is 1; find the quadratic containing the other roots. Ans. $x^2-5x+6=0$.

(4.) Two roots of the biquadratic equation $4x^4-14x^3-5x^2+31x+6=0$ are 2 and 3; find the reduced equation. Ans. $4x^2+6x-1=0$.

(5.) One root of the cubic equation $x^3+3x^2-16x+12=0$ is 1; find the remaining roots. Ans. 2 and -6.

(6.) Two roots of the biquadratic equation $x^4-6x^3+24x-16=0$, are 2 and -2; find the other two roots. Ans. $3\pm\sqrt{5}$.

Proposition IV.

To form the equation whose roots are $a_1, a_2, a_3, a_4; \ldots\ldots a_n$.

The polynomial, $f(x)$, which constitutes the first member of the equation required, being equal to the continued product of $x-a_1, x-a_2, x-a_3, \ldots$ $x-a_n$ by the last proposition, we have

$$(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)=0;$$

and by performing the multiplication here indicated, we have, when

$$n=2 \qquad x^2\begin{array}{l|}-a_1\\-a_2\end{array}\; x+a_1a_2=0$$

$$n=3 \qquad x^3\begin{array}{l|}-a_1\\-a_2\\-a_3\end{array}\; x^2\begin{array}{l|}+a_1a_2\\+a_1a_3\\+a_2a_3\end{array}\; x-a_1a_2a_3=0$$

$$n=4 \qquad x^4\begin{array}{l|}-a_1\\-a_2\\-a_3\\-a_4\\ \\ \\ \end{array}\; x^3\begin{array}{l|}+a_1a_2\\+a_1a_3\\+a_2a_3\\+a_1a_4\\+a_2a_4\\+a_3a_4\end{array}\; x^2\begin{array}{l|}-a_1a_2a_3\\-a_1a_2a_4\\-a_1a_3a_4\\-a_2a_3a_4\\ \\ \\ \end{array}\; x+a_1a_2a_3a_4=0, \text{ and so on.}$$

By continuing the multiplication to the last, the equation will be found whose roots are those proposed; and from what has been done we learn that

(1) The coefficient of the *second* term in the resulting polynomial will be the sum of all the roots with their signs changed.

(2) The coefficient of the *third* term will be the sum of the products of every two roots with their signs changed.

(3) The coefficient of the *fourth* term will be the sum of the products of every three roots with their signs changed.

(4) The coefficient of the *fifth* term will be the sum of the products of every four roots with their signs changed, and so on; the *last* or *absolute* term being the product of all the roots with their signs changed.

Cor. 1. If the coefficient of the second term in any equation be 0, that is, if the second term be absent, the sum of the positive roots is equal to the sum of the negative roots.

Cor. 2. If the signs of the terms of the equation be all positive, the roots will be all negative, and if the signs be alternately positive and negative, the roots will be all positive.

Cor. 3. Every root of an equation is a divisor of the last or absolute term.

Cor. 4. No equation, whose coefficients are all integers, and that of the highest power of the unknown quantity unity, can have a fractional root. This will be obvious by transposing the absolute term in any equation, and substituting for the unknown quantity a fraction in its lowest terms, which will give a fraction in its lowest terms equal to an integer, showing that such equation cannot have a fractional root.

Cor. 5. In any equation whose roots are all real, and the last, or absolute term very small when compared with the coefficients of the other terms, then will the roots of such an equation be also very small.

Examples.

(1.) Form the equation whose roots are 2, 3, 5, and -6.

Here we have simply to perform the multiplication indicated in the equation,

$$(x-2)(x-3)(x-5)(x+6)=0;$$

and this is best done by detached coefficients in the following manner:—

$$\begin{array}{l} 1-2\ (-3 \\ \underline{\quad -3+6} \\ 1-5+6\ (-5 \\ \quad -5+25-30 \\ 1-10+31-30\ (6 \\ \qquad 6-60+186-180 \\ \hline 1-4-29+156-180. \end{array}$$

$\therefore x^4-4x^3-29x^2+156x-180=0$ is the equation sought.

(2.) Form the equation whose roots are 1, 2, and -3.

(3.) Form the equation whose roots are 3, -4, $2+\sqrt{3}$, and $2-\sqrt{3}$.

(4.) Form the equation whose roots are $3+\sqrt{5}$, $3-\sqrt{5}$, and -6.

(5.) Form the equation whose roots are 1, -2, 3, -4, 5, and -6.

(6.) Form the equation whose roots are $2+\sqrt{-1}$, $2-\sqrt{-1}$, and -3.

(7.) Form the equation whose roots are 2, 4, $\frac{1}{4}$, and $\frac{1}{2}$.

Answers.

(2.) $x^3-7x+6=0$.

(3.) $x^4-3x^3-15x^2+49x-12=0$.

(4.) $x^3-32x+24=0$.

(5.) $x^6+3x^5-41x^4-87x^3+400x^2+444x-720=0$.

(6.) $x^3-x^2-7x+15=0$.

(7.) $8x^4-54x^3+101x^2-54x+8=0$.

Proposition V.

If the signs of the alternate terms in an equation be changed, the signs of all the roots will be changed.

Let $x^n+A_1x^{n-1}+A_{11}x^{n-2}+\ .\ .\ .\ .\ .\ A_{n-1}x+A_n=0\ .\ .\ .\ .\ .\ (1)$

be an equation; then changing the signs of the alternate terms, we have

$$x^n-A_1x^{n-1}+A_{11}x^{n-2}-\ .\ .\ .\ .\ .\ \pm A_{n-1}x\mp A_n=0\ .\ .\ .\ .\ .\ (2).$$

$$\text{or}-x^n+A_1x^{n-1}-A_{11}x^{n-2}+\ .\ .\ .\ .\ .\ \mp A_{n-1}x\pm A_n=0\ .\ .\ .\ .\ .\ (3).$$

But equations (2) and (3) are identical, for the sum of the positive terms in each is equal to the sum of the negative terms, and therefore they are identical. Now if a be a root of eq (1), and if a and $-a$ be substituted for x in equation (1) and (2) respectively, the results will be the very same; and since the former is verified by such substitution, a being a root, the latter is also verified, and therefore $-a$ is a root of the identical equations (2) and (3).

Cor. If the signs of all the terms are changed the signs of the roots remain unchanged.

Examples.

(1.) The roots of the equation $x^3-6x^2+11x-6=0$ are 1, 2, 3. What are the roots of the equation $x^3+6x^2+11x+6=0$? Ans. -1, -2, -3.

(2.) The roots of the equation $x^4-6x^2+24x-16=0$ are 2, -2, $3\pm\sqrt{5}$. Express the equation whose roots are 2, -2, $-3+\sqrt{5}$, and $-3-\sqrt{5}$.

Ans. $x^4-6x^2-24x-16=0$

PROPOSITION VI.

Surds and impossible roots enter equations by pairs.

Let $x^n+A_1x^{n-1}+A_{11}x^{n-2}+ \ldots . A_{n-1}x+A_n=0$, be an equation, having a root of the form $a+b\sqrt{-1}$; then will $a-b\sqrt{-1}$ be also a root of the equation. For, let $a+b\sqrt{-1}$ be substituted for x in the equation, and we have

$$(a+b\sqrt{-1})^n+A_1(a+b\sqrt{-1})^{n-1}+ \ldots . A_{n-1}(a+b\sqrt{-1})+A_n=0.$$

Now, by expanding the several terms of this equation, we shall have a series of monomials, all of which will be real, except the odd powers of $b\sqrt{-1}$, which will be imaginary. Let P represent the real, and $Q\sqrt{-1}$ the imaginary terms of the expanded equation; then

$$P+Q\sqrt{-1}=0,$$

an equation which can exist only when $P=0$, and $Q=0$.

Again, let $a-b\sqrt{-1}$ be substituted for x in the proposed equation; then the only difference in the expanded result will be in the signs of the odd powers of $b\sqrt{-1}$, and the collected monomials, by the previous notation, will assume the form $P-Q\sqrt{-1}$; but we have seen that $P=0$, and $Q=0$;

$$\therefore\ P-Q\sqrt{-1}=0,$$

and hence $a-b\sqrt{-1}$ also verifies the equation, and is therefore a root.

In a similar manner, it is proved that if $a+\sqrt{b}$ be one root of an equation, $a-\sqrt{b}$ will also be a root of that equation.

Cor. 1. An equation which has impossible roots is divisible by $\{x-(a+b\sqrt{-1})\}\{x-(a-b\sqrt{-1})\}$ or $x^2-2ax+a^2+b^2$, and therefore every equation may be resolved into rational, simple, or quadratic factors.

Cor. 2. All the roots of an equation of an even degree may be impossible, but if they are not all impossible, the equation must have at least two real roots.

Cor. 3. The product of every pair of impossible roots being of the form a^2+b^2, is positive; and, therefore, the absolute term of an equation whose roots are all impossible must be positive.

Cor. 4. Every equation of an odd degree has at least one real root, and that root must, necessarily, have a contrary sign to that of the last term.

Cor. 5. Every equation of an even degree, whose last term is negative, has at least two real roots; the one positive, and the other negative.

PROPOSITION VII.

An equation cannot have a greater number of positive roots than there are variations of signs in the successive terms from + to —, or from — to +, nor can it have a greater number of negative roots than there are permanencies, or successive repetition of the same sign in the successive terms.

Let an equation have the following signs in the successive terms, viz.:—

$$+ - + - - - + + + -, \text{ or } + - - - + - + + +.$$

Now, if we introduce another positive root, we must multiply the equation by $x-a$, and the signs in the partial and final products will be

$$\begin{array}{ccccccccccc} + & - & + & - & - & - & + & + & + & - & \\ & - & + & - & + & + & + & - & - & - & + \\ \hline + & - & + & - & \pm & \pm & + & \pm & \pm & - & + \end{array} \qquad \begin{array}{cccccccccc} + & - & - & - & + & - & + & + & + & \\ & - & + & + & + & - & + & - & - & - \\ \hline + & - & \pm & \pm & + & - & + & \pm & \pm & - \end{array}$$

where the ambiguous sign $\pm$ indicates that the sign may be $+$ or $-$ according to the relative magnitudes of the quantities with contrary signs in the partial products, and where it will be observed the permanencies in the proposed equation are changed into signs of ambiguity; hence the permanencies, take the ambiguous sign as you will, are not increased in the final product of the introduction of the positive root $+a$; but the number of signs is increased by *one*, and therefore the number of variations must be increased by *one*. Hence it is obvious that the introduction of every positive root also introduces one additional variation of sign; and therefore the whole number of positive roots cannot exceed the number of variations of signs in the successive terms of the proposed equation.

Again, by changing the signs of the alternate terms, the roots will be changed from positive to negative, and *vice versa* (see Prop. V). Hence the permanencies in the proposed equation will be replaced by variations in the changed equation, and the variations in the former by permanencies in the latter; and since the changed equation cannot have a greater number of positive roots than there are variations of signs, the proposed equation cannot have a greater number of negative roots than there are permanencies of signs.

Examples.

(1.) The equation $x^6+3x^5-41x^4-87x^3+400x^2+444x-720=0$, has six real roots. How many are positive?

(2.) The equation $x^4-3x^3-15x^2+49x-12=0$, has four real roots. How many of these are negative?

TRANSFORMATION OF EQUATIONS.

Proposition I.

To transform an equation into another whose roots shall be the roots of the proposed equation increased or diminished by any given quantity.

Let $ax^n+A_1x^{n-1}+A_{11}x^{n-2}+ \ldots\ldots A_{n-1}x+A_n=0$, be an equation, and let it be required to transform it into an equation whose roots shall be the roots of this equation diminished by r.

This transformation might be effected by substituting $y+r$ for x in the proposed equation, and the resulting equation in y would be that required; but this operation is generally very tedious, and we must therefore have recourse to some more simple mode of forming the transformed equation. If we write $y+r$ for x in the proposed equation, it will obviously be an equation of the very same dimensions, and its form will evidently be

$$ay^n+B_1y^{n-1}+B_{11}y^{n-2}+ \ldots\ldots B_{n-1}y+B_n=0 \ldots\ldots (1).$$

But $y=x-r$, and therefore (1) becomes

$$a(x-r)^n+B_1(x-r)^{n-1}+ \ldots\ldots B_{n-1}(x-r)+B_n=0 \ldots\ldots (2);$$

which, when developed, must be identical with the proposed equation; for, since $y+r$ was substituted for x in the proposed, and then $x-r$ for y in (2), the transformed equation, we must necessarily have reverted to the original equation; hence we have

$a(x-r)^n+B_1(x-r)^{n-1}+ \, . \, . \, B_{n-1}(x-r)+B_n=ax^n+A_1x^{n-1}x+ \, . \, . \, A_{n-1}x+A_n.$
Now if we divide the first member by $x-r$, the remainder will evidently be B_n, and the quotient

$$a(x-r)^{n-1}+B_1(x-r)^{n-2}+ \, . \, . \, . \, . \, . \, B_{n-2}(x-r)+B_{n-1};$$

and since the second member is identical with the first, the very same quotient and remainder would arise by dividing this second member also by $x-r$; hence it appears that if the first member of the original equation be divided by $x-r$, the remainder will be the last or absolute term of the sought transformed equation.

Again, if we divide the quotient thus obtained, viz..

$$a(x-r)^{n-1}+B_1(x-r)^{n-2}+ \, . \, . \, . \, . \, B_{n-2}(x-r)+B_{n-1}$$

by $x-r$, the remainder will obviously be B_{n-1}, the coefficient of the term last but one in the transformed equation; and thus by successive divisions of the polynomial in the first member of the proposed equation by $x-r$, we shall obtain the whole of the coefficients of the required equation.

Rule.

Let the polynomial in the first member of the proposed equation be a function of x, and r the quantity by which the roots of the equation are to be diminished or increased; then divide the proposed polynomial by $x-r$, or $x+r$, according as the roots of the proposed are to be diminished or increased, and the quotient thus obtained by the same divisor, giving a second quotient, which divide by the same divisor, and so on till the division terminates; then will the coefficients of the transformed equation, beginning with the highest power of the unknown quantity, be the coefficient of the highest power of the unknown in the proposed equation, and the several remainders arising from the successive divisions taken in a reverse order, the first remainder being the last or absolute term in the required transformed equation.

Note. When there is an absent term in the equation, its place must be supplied with a cipher.

Examples.

(1.) Transform the equation $5x^4-12x^3+3x^2+4x-5=0$ into another whose roots shall be less than those of the proposed equation by 2.

$$\begin{array}{l}
x-2)5x^4-12x^3+3x^2+4x-5(5x^3-2x^2-x+2 \\
\quad\;\; 5x^4-10x^3 \\
\hline
\qquad\quad -2x^3+3x^2 \\
\qquad\quad -2x^3+4x^2 \\
\hline
\qquad\qquad\quad -x^2+4x \\
\qquad\qquad\quad -x^2+2x \\
\hline
\qquad\qquad\qquad\quad 2x-5 \\
\qquad\qquad\qquad\quad 2x-4 \\
\hline
\qquad\qquad\qquad\qquad -1. \text{ First remainder}
\end{array}$$

$$
\begin{array}{l}
x-2)5x^3-2x^2-x+2(5x^2+8x+15 \\
\quad 5x^3-10x^2 \\
\hline
\qquad\quad 8x^2-x \\
\qquad\quad 8x^2-16x \\
\hline
\qquad\qquad\quad 15x+2 \\
\qquad\qquad\quad 15x-30 \\
\hline
\qquad\qquad\qquad 32. \text{ Second remainder.}
\end{array}
$$

$$
\begin{array}{l}
x-2)5x^2+8x+15(5x+18 \\
\quad 5x^2-10x \\
\hline
\qquad 18x+15 \\
\qquad 18x-36 \\
\hline
\qquad\quad 51. \text{ Third remainder.}
\end{array}
\qquad
\begin{array}{l}
x-2)5x+18(5 \\
\quad 5x-10 \\
\hline
\qquad 28. \text{ Fourth remainder.}
\end{array}
$$

Therefore the transformed equation is

$$5y^4+28y^3+51y^2+32y-1=0.$$

This laborious operation can be avoided by *Horner's Synthetic Method* of division; and its great superiority over the usual method will be at once apparent by comparing the subsequent elegant process with the work above. Taking the same example, and writing the modified or changed term of the divisor $x-2$ on the right hand instead of the left, the whole of the work will be thus arranged:—

$$
\begin{array}{rrrrl}
5-12 & +8 & +4 & -5(2 & \\
10 & -4 & -2 & 4 & \\
\hline
-2 & -1 & 2 & -1 & \therefore B_{iv}=-1 \\
10 & 16 & 30 & & \\
\hline
8 & 15 & 32 & & \therefore B_{iii}=32 \\
10 & 36 & & & \\
\hline
18 & 51 & & & \therefore B_{ii}=51 \\
10 & & & & \\
\hline
28 & & & & \therefore B_{i}=28
\end{array}
$$

$\therefore$ $5y^4+28y^3+51y^2+32y-1=0$ is the required equation, as before.

(2.) Transform the equation $5y^4+28y^3+51y^2+32y-1=0$ into another having its roots greater by 2 than those of the proposed equation.

$$
\begin{array}{rrrrr}
5\ + & 28+ & 51\ + & 32 & -1(-2 \\
 & -10 & -36 & -30 & -4 \\
\hline
 & 18 & 15 & 2 & -5 \\
 & -10 & -16 & 2 & \\
\hline
 & 8 & -1 & 4 & \\
 & -10 & 4 & & \\
\hline
 & -2 & 3 & & \\
 & -10 & & & \\
\hline
 & -12 & & &
\end{array}
$$

$\therefore$ $5x^4-12x^3+3x^2+4x-5=0$ is the sought equation; which, from the transformations we have made, must be the original equation in Example 1.

(3.) Find the equation whose roots are less by 1·7 than those of the equation

$$x^3-2x^2+3x-4=0.$$

$$\begin{array}{rrrl}
1 & -2 & +3 & -4\ (1 \\
 & 1 & -1 & 2 \\ \hline
 & -1 & 2 & -2 \\
 & 1 & 0 & \\ \hline
 & 0 & 2 & \\
 & 1 & & \\ \hline
 & 1 & &
\end{array}$$

Now we know the equation whose roots are less by 1 than those of the given equation: it is $x^3+x^2+2x-2=0$; and by a similar process for ·7, remembering the localities of the decimals, we have the required equation; thus:—

$$\begin{array}{rrrl}
1 & +1 & +2 & -2\ \ (\cdot 7 \\
 & \cdot 7 & 1\cdot 19 & 2\cdot 233 \\ \hline
 & 1\cdot 7 & 3\cdot 19 & \cdot 233 \\
 & 7 & 1\cdot 68 & \\
 & 2\cdot 4 & 4\cdot 87 & \\
 & 7 & & \\ \hline
 & 3\cdot 1 & &
\end{array}$$

$\therefore\ y^3+3\cdot 1y^2+4\cdot 87y+\cdot 233=0$ is the required equation.

This latter operation can be continued from the former, without arranging the coefficients anew in a horizontal line, recourse being had to this second operation merely to show the several steps in the transformation, and to point out the equations at each step of the successive diminutions of the roots. Combining these two operations, then, we have the subsequent arrangement.

$$\begin{array}{rrrl}
1 & -2 & +3 & -4\ (1\cdot 7 \\
 & 1 & -1 & 2 \\
 & -1 & 2 & -2 \\
 & 1 & 0 & 2\cdot 233 \\ \hline
 & 0 & 2 & \cdot 233 \\
 & 1 & 1\cdot 19 & \\ \hline
 & 1\cdot 7 & 3\cdot 19 & \\
 & 7 & 1\cdot 68 & \\ \hline
 & 2\cdot 4 & 4\cdot 87 & \\
 & \cdot 7 & & \\ \hline
 & 3\cdot 1 & &
\end{array}
\qquad \text{or} \qquad
\begin{array}{rrrl}
1 & -2 & +3 & -4\ (1\cdot 7 \\
 & 1\cdot 7 & -\ \cdot 51 & 4\cdot 233 \\
 & -\ \cdot 3 & 2\cdot 49 & \cdot 233 \\
 & 1\cdot 7 & 2\cdot 38 & \\
 & 1\cdot 4 & 4\cdot 87 & \\
 & 1\cdot 7 & & \\
 & 3\cdot 1 & &
\end{array}$$

We have then the same resulting equation as before, and in the latter of these we have used 1·7 at once. It is always better, however, to reduce continuously as in the former, to avoid mistakes incident to the multiplier 1·7.

(4.) Find the equation whose roots shall be less by 1 than those of the equation

$$x^3-7x+7=0.$$

(5.) Find the equation whose roots shall be less by 3 than the roots of the equation

$$x^4-3x^3-15x^2+49x-12=0,$$

and transform the resulting equation into anotner whose roots shall be greater by 4.

(6.) Give the equation whose roots shall be less by 10 than the roots of the equation

$$x^4+2x^3+3x^2+4x-12340=0.$$

(7.) Give the equation whose roots shall be less by 2 than those of the equation

$$x^5+2x^3-6x^2-10x-8=0.$$

(8.) Give the equation whose roots shall each be less by $\frac{1}{2}$ than the roots of the equation

$$2x^4-6x^3+4x^2-2x+1=0 .$$

Answers.

(4.) $y^3+3y^2-4y+1=0$ . . . . whence $x=y+1$
(5.) $y^4+9y^3+12y^2-14y=0$ . . . . —— $x=y+3$
and $z^4-7z^3+66z-72=0$ . . . . —— $x=z-1$
(6.) $y^4+42y^3+663y^2+4664y=0$. . . . . —— $x=y+10$
(7.) $y^5+10y^4+42y^3+86y^2+70y-4=0$ . . . . —— $x=y+2$
(8.) $2y^4-2y^3-2y^2-\frac{3}{2}y+\frac{3}{8}=0$ . . . . —— $x=y+\frac{1}{2}$

Proposition II.

To transform an equation into another whose second term shall be removed.

Let the proposed equation be

$$x^n+A_1x^{n-1}+A_{11}x^{n-2}+ \ldots\ldots A_{n-1}x+A_n=0;$$

and by Prop. IV. we know that the sum of the roots of this equation is $-A_1$; therefore the sum of all the roots must be increased by A_1, in order that the transformed equation may want its second term; but there are n roots, and hence each root must be increased by $\frac{A_1}{n}$, and then the changed equation will have its second term absent. If the sign of the second term of the proposed equation be negative, then the sum of all the roots is $+A_1$; and in this case we must evidently diminish each root by $\frac{A_1}{n}$, and the changed equation will then have its second term entirely removed. Hence this

Rule. Find the quotient of the coefficient of the second term of the equation divided by the highest power of the unknown quantity, and decrease or increase the roots of the equation by this quotient, according as the sign of the second term is negative or positive.

Examples.

(1.) Transform the equation $x^3-6x^2+8x-2=0$ into another whose second term shall be absent.

Here $A_1=-6$, and $n=3$; $\therefore$ we must diminish each root by $\frac{6}{3}$ or 2.

$$\begin{array}{rrrl}
1 & -6 & +8 & -2\,(2 \\
 & 2 & -8 & 0 \\ \hline
 & -4 & 0 & -2 \\
 & 2 & -4 & \\ \hline
 & -2 & -4 & \\
 & 2 & & \\ \hline
 & 0 & &
\end{array}$$

$\therefore\ y^3-4y-2=0$ is the changed equation.

And since the roots are diminished we must have the relation $x=y+2$.

(2.) Transform the equation $x^4-16x^3-6x+15=0$ into another whose second term shall be removed.

(3.) Transform the equation $x^5+15x^4+12x^3-20x^2+14x-25=0$ into another whose second term shall be absent.

(4.) Change the equation $x^2+ax+b=0$ into another deficient of the second term.

(5.) Change the equation $x^3+ax^2+bx+c=0$ into another wanting the second term.

ANSWERS.

(2.) $y^4-96y^2-518y-777=0$. (4.) $z^2-\frac{a^2}{4}+b=0$.

(3.) $y^5-78y^3+412y^2-757y+401=0$. (5.) $z^3-\left(\frac{a^2}{3}-b\right)z+\frac{2a^3}{27}-\frac{ab}{3}+c=0$.

PROPOSITION III.

To transform an equation into another whose roots shall be the reciprocals of the roots of the proposed equation.

Let $ax^n+A_1x^{n-1}+A_{11}x^{n-2}+\ldots\ldots A_{n-1}x+A_n=0$ be the proposed equation, and put $y=\frac{1}{x}$; then $x=\frac{1}{y}$, and by writing $\frac{1}{y}$ for x in the proposed equation, multiplying by y^n, and reversing the order of the terms, we have the equation

$$A_ny^n+A_{n-1}y^{n-1}+A_{n-2}y^{n-2}+\ldots\ldots A_{11}y^2+A_1y+a=0,$$

whose roots are the reciprocals of the roots of the proposed equation.

Cor. 1. Hence an equation may be transformed into another whose roots shall be greater or less than the *reciprocals* of the roots of the proposed equation, simply by reversing the order of the coefficients, and then proceeding as in Proposition I., p. 280.

Cor. 2. If the coefficients of the proposed equation be the same, whether taken in reverse or direct order, then it is evident that the transformed equation will be the same as the original one; and, therefore, the roots of such equations must be of the form

$$r_1, \frac{1}{r_1}; r_2, \frac{1}{r_2}; r_3, \frac{1}{r_3}; r_4, \frac{1}{r_4}; \&c.$$

Cor. 3. If the coefficients of an equation of an odd degree be the same whether taken in direct or inverse order, but have contrary signs; then also the roots of the transformed equation will be the same as the roots of the proposed equation; for changing the signs of all the terms, the original and transformed equations will be identical, and the roots remain unchanged when the signs of *all* the terms are changed. And this will likewise be the case in an equation of an even degree, provided only the middle term be absent, in order that the transformed equation with all its signs changed may be identical with the original equation.

Equations whose coefficients are the same when taken either in direct or reverse order, are therefore called *recurring equations*, or, from the form of the roots, *reciprocal equations.*

Cor. 4. If the sign of the last term of a recurring equation of an odd degree be $+$, one of the roots of such equation will be -1, and if the sign of the

last term be $-$, one root will be $+1$. For the proposed equation, and the reciprocal have one root, the same in each, and 1 is the only quantity whose reciprocal is the same quantity; hence, since each of the other roots has the same sign as its reciprocal, the product of each root and its reciprocal must be positive; and therefore the last term of the equation, being the product of all the roots with their signs changed, must have a contrary sign to that of the root unity.

Hence a recurring equation of an odd degree may always be depressed to an equation of the next lower degree, by dividing it by $x+1$, or $x-1$, according as the sign of the last term is $+$ or $-$.

Cor. 5. A recurring equation of an even degree may always be depressed to another of half the dimensions. For let the equation be

$$x^{2n}+A_1x^{2n-1}+A_{11}x^{2n-2}+\ldots\ldots A_{11}x^2+A_1x+1=0;$$

dividing by x^n, and placing the first and last, the second and last but one, &c., in juxtaposition, we have

$$x^n+\frac{1}{x^n}+A_1\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+\ldots\ldots A_{n-1}\left(x+\frac{1}{x}\right)+A_n=0.$$

Assume $y=x+\frac{1}{x}=x+x^{-1}$, then we have

$$x+\frac{1}{x}=y \qquad\qquad \therefore x+\frac{1}{x}=y$$

$$\left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2 \qquad\qquad x^2+\frac{1}{x^2}=y^2-2$$

$$\left(x+\frac{1}{x}\right)^3=x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right) \qquad\qquad x^3+\frac{1}{x^3}=y^3-3y$$

$$\left(x+\frac{1}{x}\right)^4=x^4+\frac{1}{x^4}+4\left(x^2+\frac{1}{x^2}\right)+6 \qquad\qquad x^4+\frac{1}{x^4}=y^4-4(y^2-2)-6$$

&c. &c. &c. &c. $=y^4-4y^2+2$;

and the resulting equation is therefore of the form

$$y^n+B_1y^{n-1}+B_{11}y^{n-2}+\ldots\ldots B_{n-1}y+B=0;$$

and the original equation is reduced to an equation of half the dimensions.

Examples.

(1.) Transform the equation $x^3-7x+7=0$ into another whose roots shall be less than the reciprocals of those of the given equation by unity.

$$\begin{array}{rrrl}
7 & -7 & +0 & +1\ (1 \\
 & 7 & 0 & 0 \\ \hline
 & 0 & 0 & 1 \\
 & 7 & 7 & \\ \hline
 & 7 & 7 & \\
 & 7 & & \\ \hline
 & 14 & &
\end{array}$$

$\therefore 7z^3+14z^2+7z+1=0$ is the equation sought, where $z+1=\frac{1}{x}$, or $x=\frac{1}{z+1}$

(2.) Find the roots of the recurring equation

$$x^5-6x^4+5x^3+5x^2-6x+1=0.$$

By Cor. 4, this equation has one root $x=-1$, and the depressed equation is

$$x^4-7x^3+12x^2-7x+1=0.$$

Divide by x^2, and arrange the terms as in Cor. 5; then

$$x^2+\frac{1}{x^2}-7\left(x+\frac{1}{x}\right)+12=0.$$

Put $x+\frac{1}{x}=z$; then $x^2+\frac{1}{x^2}=z^2-2$; hence, by substitution,

$$z^2-2-7z+12=0;$$

$$\text{or, } z^2-7z+10=0;$$

and, resolving the quadratic, we get

$$z=\frac{7}{2}\pm\sqrt{\frac{49}{4}-10}$$

$$=\frac{7\pm 3}{2}$$

$$=5 \text{ or } z=2.$$

Hence $x+\frac{1}{x}=5$, and $x+\frac{1}{x}=2$, and the resolution of these two quadratics gives

$$x=\tfrac{1}{2}(5\pm\sqrt{21}) \text{ and } x=+1 \text{ or } +1,$$

and the five roots are

$$-1,\ +1,\ +1,\ \frac{5+\sqrt{21}}{2}, \text{ and } \frac{5-\sqrt{21}}{2};$$

where $\frac{5-\sqrt{21}}{2}=\frac{(5-\sqrt{21})}{2}\cdot\frac{5+\sqrt{21}}{5+\sqrt{21}}=\frac{25-21}{2(5+\sqrt{21})}=\frac{2}{5+\sqrt{21}}$, which is the reciprocal of the root $\frac{5+\sqrt{21}}{2}$.

(3.) Give the equation whose roots are the reciprocals of the roots of the equation

$$x^6-3x^5-2x^4+3x^3+12x^2+10x-8=0.$$

(4.) Find the roots of the recurring equation

$$5y^5-4y^4+3y^3-3y^2+4y-5=0.$$

(5.) Find the roots of the recurring equation

$$x^5+x^4+x^3+x^2+x+1=0.$$

Answers.

(3.) $8x^6-10x^5-12x^4-3x^3+2x^2+3x-1=0.$

(4.) $1,\ \frac{1+\sqrt{-3}}{2},\ \frac{1-\sqrt{-3}}{2},\ \frac{-3+4\sqrt{-1}}{5}$, and $\frac{-3-4\sqrt{-1}}{5}$.

(5.) $-1,\ \sqrt{\frac{-1+\sqrt{-3}}{2}},\ \sqrt{\frac{-1-\sqrt{-3}}{2}},\ -\sqrt{\frac{-1+\sqrt{-3}}{2}}$, and $-\sqrt{\frac{-1-\sqrt{-3}}{2}}$.

Proposition IV.

To transform an equation into another whose roots shall be any proposed multiple or submultiple of the roots of the given equation.

Let $x^n+A_1x^{n-1}+A_{11}x^{n-2}+ \ldots A_{n-1}x+A_n=0$ be any equation; then putting $y=mx$, we have $x=\frac{y}{m}$, and by substituting this value of x in the given equation, and multiplying each term by m^n, we have

$$y^n+mA_1y^{n-1}+m^2A_{11}y^{n-2}+ \ldots \ldots m^{n-1}A_{n-1}y+m^nA_n=0;$$

an equation whose roots are m times those of the proposed equation. Hence we have simply to multiply the second term of the given equation by m, the third by m^2, the fourth by m^3, and so on, and the transformation is effected.

Cor. 1. If the coefficient of the first term be m; then, suppressing m in the first term, making no change in the second, multiplying the third by m, the fourth by m^2, and so on, the resulting equation will have its roots m times those of the given equation.

Cor. 2. Hence, if an equation have fractional coefficients, it may be changed into another, having integer coefficients, by transforming the given equation into another whose roots shall be those of the proposed equation multiplied by the product of the denominators of the fractions.

Cor. 3. If the coefficients of the second, third, fourth, &c. terms of an equation be divisible by m, m^2, m^3, and so on, respectively, then m is a common measure of the roots of the equation.

Examples.

(1.) Transform the equation $2x^3-4x^2+7x-3=0$ into another whose roots shall be three times those of the proposed equation.

(2.) Transform the equation $4x^4-3x^3-12x^2+5x-1=0$ into another whose roots shall be four times those of the given equation.

(3.) Transform the equation $x^3+\frac{1}{3}x^2-\frac{1}{4}x+2=0$ into another whose roots shall be 12 times those of the given equation.

Answers.

(1.) $2x^3-12x^2+63x-81=0$.

(2.) $x^4-3x^3-48x^2+80x-64=0$.

(3.) $x^3+4x^2-36x+3456=0$.

Proposition V.

To transform an equation into another, whose roots shall be the squares of the roots of the proposed equation.

Let $x^n+A_1x^{n-1}+A_{11}x^{n-2}+A \ldots\ldots +A_{n-1}x+A_n=0$ be any equation, then $x^n-A_1x^{n-1}+A_{11}x^{n-2}- \ldots\ldots \pm A_{n-1}x\mp A_n=0$ is the equation whose roots are the roots of the former, with contrary signs (Prop. V. p. 278).

Let a_1, a_2, a_3, &c., be the roots of the former equations, and $-a_1$, $-a_2$, $-a_3$, &c., those of the latter; then we have

$(x^n+A_{11}x^{n-2}+\ldots.)+(A_1x^{n-1}+A_{111}x^{n-3}+\ldots)=(x-a_1)\,(x-a_2)\,(x-a_3)\ldots.$

$(x^n+A_{11}x^{n-2}+\ldots.)-(A_1x^{n-1}+A_{111}x^{n-3}+\ldots)=(x+a_1)\,(x+a_2)\,(x+a_3)\ldots.$

Hence, by multiplying these two equations, we have

$(x^n+A_{11}x^{n-2}+\ldots)^2-(A_1x^{n-1}+A_{111}x^{n-3}+\ldots)^2=(x^2-a_1^2)\,(x^2-a_2^2)\,(x^2-a_3^2)\ldots$

Or, $x^{2n}-(A_1^2-2A_{11})\,x^{2n-2}+(A_{11}^2-2A_1A_{111}+2A_{IV})\,x^{2n-4}-$. &c. $=(x^2-a_1^2)$ $(x^2-a_2^2)\,(x^2-a_3^2)\ldots.$ by actually squaring and arranging according to the powers of x. Now, for x^2 write y; and we have

$$y^n-(A_1^2-2A_{11})y^{n-1}+(A_{11}^2-2A_1A_{111}+2A_{IV})y^{n-2}-\&c. =(y-a_1^2)\,(y-a_2^2)\,(y-a_3^2)\ldots$$

$\therefore y^n-(A_1^2-2A_{11})y^{n-1}+(A_{11}^2-2A_1A_{111}+2A_{IV})y^{n-2}-\ldots.=0$ is an equation whose roots are the squares of the roots of the given equation.

Examples.

(1.) Transform the equation $x^3+3x^2-6x-8=0$ into another, whose roots are the squares of those of the proposed equation.

Here $x^3-6x=-3x^2+8$ by transposition, and by squaring we have

$$x^6-12x^4+36x^2=9x^4-48x^2+64$$
$$\therefore x^6-21x^4+84x^2-64=0$$

Or $y^3-21y^2+84y-64=0$ is the required equation.

The roots of the given equation are $-1, -4, 2$; and those of the transformed equation are 1, 4, 16.

(2.) Transform the equation $x^3-x^2-7x+15=0$.

(3.) Transform the equation $x^4-6x^3+5x^2+2x-10=0$.

(4.) Transform the equation $x^4-4x^3-8x+32=0$.

(5.) Transform the equation $x^4-3x^3-15x^2+49x-12=0$.

Answers.

(2.) $y^3-15y^2+79y-225=0$.
(3.) $y^4-26y^3+29y^2-104y+100=0$.
(4.) $y^4-16y^3-64y+1024=0$.
(5.) $y^4-39y^3+495y^2-2041y+144=0$.

Proposition VI.

If the real roots of an equation, taken in the order of their magnitudes, be

$$a_1, a_2, a_3, a_4, a_5, \ldots\ldots\ldots$$

where a *is the greatest,* a_2 *the next, and so on; then if a series of numbers,*

$$b_1, b_2, b_3, b_4, b_5, \ldots\ldots\ldots$$

in which b_1 *is greater than* a_1, b_2 *a number between* a_1 *and* a_2, b_3 *a number between* a_2 *and* a_3, *and so on, be substituted for* x *in the proposed equation, the results will be alternately positive and negative.*

The polynomial in the first member of the proposed equation is the product of the simple factors,

$$(x-a_1)\,(x-a_2)\,(x-a_3)\,(x-a_4)\ldots\ldots$$

and quadratic factors, involving the imaginary roots; but the quadratic factors have always a positive value for every real value of x; therefore we may omit

these positive factors; and substituting for x the proposed series of values, b_1, b_2, b_3, &c., we have these results:

$$
\begin{array}{lcl}
(b_1-a_1)\,(b_1-a_2)\,(b_1-a_3)\,(b_1-a_4)\ldots\ldots = +.+.+.+\ldots\ldots = + \\
(b_2-a_1)\,(b_2-a_2)\,(b_2-a_3)\,(b_2-a_4)\ldots\ldots = -.+.+.+\ldots\ldots = - \\
(b_3-a_1)\,(b_3-a_2)\,(b_3-a_3)\,(b_3-a_4)\ldots\ldots = -.-.+.+\ldots\ldots = + \\
(b_4-a_1)\,(b_4-a_2)\,(b_4-a_3)\,(b_4-a_4)\ldots\ldots = -.-.-.+\ldots\ldots = - \\
\qquad\&c. \qquad\qquad \&c. \qquad\qquad \&c.
\end{array}
$$

Cor. 1. If two numbers be successively substituted for x in any equation, and give results with different signs, then between these numbers there must be *one, three, five,* or some *odd* number of roots.

Cor. 2. If the results of the substitution in Cor. 1 are affected with like signs, then between these numbers there must be *two, four,* or some *even* number of roots, or *no* root between these numbers.

Cor. 3. If any quantity q, and every quantity greater than q, renders the result positive, then q is greater than the greatest root of the equation.

Cor. 4. Hence, if the signs of the alternate terms be changed, and if p, and every quantity greater than p, renders the result positive, then $-p$ is less than the least root.

Example.

Find the initial figure in one of the roots of the equation

$$x^3-4x^2-6x+8=0.$$

Here one value of x does not differ greatly from unity, for the value of the given polynomial, when $x=1$, is -1, and when $x=\cdot 9$, it is found thus ·

$$
\begin{array}{rrrl}
1-4 & -6 & +8 & (\cdot 9 \\
\cdot 9 & -2\cdot 79 & -7\,911 & \\
\hline
-3\cdot 1 & -8\cdot 79 & +\cdot 089 & \therefore\ V=+\cdot 089.
\end{array}
$$

Hence the former value being negative, and the latter positive, the initial figure of one root is $\cdot 9$.

Proposition VII.

Given an equation of the n*th degree, to determine another of the* (n—1)*th degree, such that the real roots of the former shall be limits to those of the latter.*

Let $a_1, a_2, a_3, a_4, \ldots\ldots a_n$ be the roots taken in order of the equation

$$x^n+A_1x^{n-1}+A_{11}x^{n-2}+\ \ldots\ldots\ A_{n-1}x+A_n=0;$$

then diminishing the roots of this equation by r (Prop. I, p. 280), we have the following process, viz.:

$$
\begin{array}{cccccc}
1+A_1+ & A_{11}+ & \ldots\ldots & A_{n-2}+ & A_{n-1}+ & A_n\ (r \\
r & rB_1 & & rB_{n-3} & rB_{n-2} & rB_{n-1} \\
\hline
B_1 & B_{11} & & B_{n-2} & B_{n-1} & B_n \\
r & rC_1 & & rC_{n-3} & rC_{n-2} & \\
\hline
C_1 & C_{11} & & C_{n-2} & C_{n-1} &
\end{array}
$$

Whence $C_{n-1}=A_{n-1}+rB_{n-2}+rC_{n-2}$

$$\begin{aligned}
&=A_{n-1}+r(A_{n-2}+rB_{n-3})+r(A_{n-2}+rB_{n-3}+rC_{n-3})\\
&=A_{n-1}+2rA_{n-2}+2r^2B_{n-3}+r^2C_{n-3}\\
&=A_{n-1}+2rA_{n-2}+2r^2(A_{n-3}+rB_{n-4})+r^2(A_{n-3}+rB_{n-4}+rC_{n-4})\\
&=A_{n-1}+2rA_{n-2}+3r^2A_{n-3}+3r^3B_{n-4}+r^3C_{n-4}\\
&\quad\vdots\qquad\vdots\qquad\qquad\vdots\qquad\vdots\\
&=A_{n-1}+2rA_{n-2}+3r^2A_{n-3}+\ldots\ldots\ldots(n-1)r^{n-2}A_1+nr^{n-1}
\end{aligned}$$

Or, $C_{n-1}=nr^{n-1}+(n-1)A_1r^{n-2}+(n-2)A_{11}r^{n-3}+\ldots 2A_{n-2}r+A_{n-1}\ldots$ (1)

Again, the roots of the transformed equation will evidently be

$$a_1-r,\ a_2-r,\ a_3-r,\ a_4-r,\ \ldots\ a_n-r,$$

and as we have found the coefficient, C_{n-1}, of the last term but one, in the transformed equation, by one process, we shall now find the same coefficient, C_{n-1}, by another process (Prop. IV, p. 277;) hence we have

$$\left.\begin{aligned}
C_{n-1}=&(r-a_1)(r-a_2)(r-a_3)\ldots\ldots\ldots \text{ to } (n-1) \text{ factors}\\
&+(r-a_1)(r-a_2)(r-a_4)\ldots\ldots\ldots \quad\text{do.}\\
&+(r-a_1)(r-a_3)(r-a_4)\ldots\ldots\ldots \quad\text{do.}\\
&\quad\vdots\qquad\vdots\qquad\vdots\\
&+(r-a_2)(r-a_3)(r-a_4)\ldots\ldots\ldots \quad\text{do.}
\end{aligned}\right\}\ldots\ldots (2)$$

Now, these two expressions which we have obtained for C_{n-1} are equal to one another, and therefore whatever changes arise by substitution in the one, the same changes will be produced, by a like substitution, in the other; hence, substituting a_1, a_2, a_3, &c., successively for r in the second member of equation (2), we have these results:

$$\begin{aligned}
&(a_1-a_2)(a_1-a_3)(a_1-a_4)\ldots\ldots = +.+.+\ldots\ldots = +\\
&(a_2-a_1)(a_2-a_3)(a_2-a_4)\ldots\ldots = -.+.+\ldots\ldots = -\\
&(a_3-a_1)(a_3-a_2)(a_3-a_4)\ldots\ldots = -.-.+\ldots\ldots = +\\
&\qquad\qquad \&c. \qquad\qquad\qquad \&c. \qquad\qquad \&c.
\end{aligned}$$

But when a series of quantities, a_1, a_2, a_3, a_4, &c., are substituted for the unknown quantity in any equation, and give results which are alternately + and —, then, by Prop. VI, these quantities taken in order, are situated in the successive intervals of the real roots of the proposed equation; hence, making $C_{n-1}=0$, and changing r into x, we have from equation (1)

$$nx^{n-1}+(n-1)A_1x^{n-2}+(n-2)A_{11}x^{n-3}+\ldots 2A_{n-2}x+A_{n-1}=0\ \ldots (3)$$

an equation whose roots are therefore limits to those of the original equation,

$$x^n+A_1x^{n-1}+A_{11}x^{n-2}+\ \ldots\ A_{n-1}x+A_n=0,$$

and the manner of deriving it from the proposed equation is evident.

Let a_1, a_2, a_3, a_4, &c., be the roots of the proposed equation, and b_1, b_2, b_3, &c., those of the derived equation (3), ranged in the order of magnitude; then the roots of both the given and the derived equation will be represented in order of magnitude by the following arrangement, viz.:

$$a_1,\ b_1,\ a_2,\ b_2,\ a_3,\ b_3,\ a_4,\ b_4,\ a_5,\ b_5,\ \&c.\ldots$$

Cor. 1. If $a_2=a_1$, then $r-a_1$, will be found as a factor in each of the groups of factors in equation (2), which has been shown to be the limiting equation (3), and therefore the limiting equation, and the original equation, will obviously have a common measure of the form $x-a_1$.

Cor. 2. If $a_3=a_2=a_1$, then $(r-a_1).(r-a_1)$ will occur as a common factor in each group of factors in (2); that is, the limiting equation (3) is divisible by $(x-a_1)^2$; and therefore the proposed equation and the limiting equation have a common measure of the form $(x-a_1)^2$.

Cor. 3. If the proposed equation, have also $a_5=a_4$, then it will have a common measure with the limiting equation of the form $(x-a_1)^2\ (x-a_4)$, and so on.

Scholium. When therefore we wish to ascertain whether a proposed equation has *equal roots*, we must first find the limiting equation, and then find the greatest common measure of the polynomials in the first members of these two equations. If the greatest common measure be of the form

$$(x-a_1)^p\ (x-a_2)^q\ (x-a_3)^r \ldots\ldots$$

then the proposed equation will have $(p+1)$ roots$=a_1$, $(q+1)$ roots$=a_2$, $(r+1)$ roots$=a_3$, &c. The equation may then be depressed to another of lower dimensions.

BUDAN'S CRITERION

For determining the number of imaginary roots in any equation.

172. If the real positive roots of an equation, taken in the order of their magnitudes, be $a_1, a_2, a_3, a_4 \ldots\ldots a_n$, where a_1 is the smallest, and if we diminish the roots of the equation by a number h greater than a_1, but less than a_2, then the roots will be a_1-h, a_2-h, a_3-h, $\ldots\ a_n-h$, and the first of these will now be negative. But the number of positive roots is exactly equal to the number of variations of sign in the terms of the equation, when the roots are all real; and as we have changed one positive root into a negative one, the transformed equation must have one variation less than the proposed equation.

Again, by reducing all the roots by k, a number greater than a_2, but less than a_3, we shall have two negative roots, a_1-k, a_2-k, in the transformed equation, and therefore we shall have two variations of sign less than in the proposed equation; for two positive roots have been reduced so as to become negative ones. Hence it is obvious, that if we reduce the roots by a number greater than a_n, all the positive roots will become negative, and the transformed equation, having all its roots negative, will have the signs of all its terms positive (Prop. IV. p. 277), and all the variations have entirely disappeared.

We see, then, that if the roots of an equation be reduced until the signs of all the terms of the transformed equation be $+$, we have employed a greater number than the greatest positive root of that equation; and therefore its reciprocal must be less than the smallest real root of the reciprocal equation. Now, if we take the reciprocal equation, and reduce its roots by the reciprocal of the former number, we should have as many positive roots *left* in this transformed reciprocal equation as there were positive roots in the proposed equation, unless the equation has imaginary roots; hence the number of variations *lost* in the former case should be exactly equal to the number *left* in the latter, when the roots are all real; and, consequently, if this condition be not fulfilled, the difference of these numbers indicates the number of imaginary roots. To explain this reasoning more clearly, we shall suppose that an

equation has three positive roots; as, for instance, 1, 2·5, and 3. Now, if the roots of the proposed equation be reduced by 4, a number greater than 3, the greatest positive root, the three positive roots in the original equation will evidently be changed into three negative ones in the transformed one, and hence *three* variations must be lost. Again; the equation whose roots are the reciprocals of the proposed equation, must have three positive roots, 1, $\frac{2}{5}$, and $\frac{1}{3}$; and it is evident that if we reduce the roots of the reciprocal equation by $\frac{1}{4}$, the reciprocal of the former reducing number 4, we shall not change the character of the three positive roots, because $\frac{1}{4}$ is less than the least of them, and $1-\frac{1}{4}$, $\frac{2}{5}-\frac{1}{4}$, $\frac{1}{3}-\frac{1}{4}$, are all positive; hence the *three* variations introduced by the three positive roots must still be found in the transformed reciprocal equation, and therefore *three* variations are left in the latter transformation, indicating no imaginary roots. The theorem may, therefore, be stated thus:—

If, in transforming an equation by any number r, there be n variations *lost*, and if in transforming the reciprocal equation by $\frac{1}{r}$ (the reciprocal of r,) there be m variations *left*, then there will be at least $n-m$ imaginary roots in the interval 0, r.

For there are as many positive roots in the interval 0, r, of the direct equation, as there are between $\frac{1}{r}$ and $\frac{1}{0}$ of the reciprocal equation; hence, if n, the number of variations *lost* in the transformation of the direct equation by r, be greater than m, the number of variations *left* in the transformation of the reciprocal equation by $\frac{1}{r}$, there will be a contradiction with respect to the character of a number of the roots, equal to the difference $n-m$. Hence these roots are imaginary.

Example.

Find the number of imaginary roots of the equation

$$x^4-x^3+2x^2+x-4=0.$$

Direct.

1	−1	+2	+1	−4	(1
	1	0	2	3	
	0	2	3	−1	
	1	1	3		
	1	3	6		
	1	2			
	2	5			
	1				
	3				

Reciprocal.

−4	+1	+2	−1	+1	(1
	−4	−3	−1	−2	
	−3	−1	−2	−1	
	−4	−7	−8		
	−7	−8	−10		
	−4	−11			
	−11	−19			
	−4				
	−15				

Here *two* variations are *lost* in the transformation of the direct equation, and *no* variations are *left* in the transformation of the reciprocal equation; therefore, this equation has at least *two* imaginary roots; and it has *only* two, for the sign of the absolute term is negative, implying the existence of two real roots; the one positive, and the other negative.

Degua's Criterion.

173. In any equation, if we have a cipher-coefficient, or term wanting, and if the cipher-coefficient be situated between two terms having the same sign, there will be two imaginary roots in that equation.

Let the order of the signs be

$$+ + - 0 - + - -;$$

and for 0 writing + or — we have either

$$+ + - + - + - -, \text{ or } + + - - - + - -.$$

In the former of these we find *two* permanencies and *five* variations, and in the latter we have *four* permanencies and only *three* variations; hence, if the roots are all real, we must, in the former case, have *five* positive and *two* negative roots, and in the latter, *three* positive and *four* negative roots (Prop. VII. p. 279); hence we have *two* roots, both *positive* and *negative*, at the same time, and therefore these two roots cannot be *real* roots. These two roots, which involve the absurdity of being both positive and negative at the same time, must therefore be *imaginary* roots.

In nearly the same manner it may be shown that

(1.) If between terms having *like* signs, $2n$ or $2n-1$ cipher-coefficients intervene, there will be $2n$ imaginary roots indicated thereby.

(2.) If between terms having *different* signs, $2n+1$, or $2n$ cipher-coefficients intervene, there will be $2n$ imaginary roots indicated thereby.

Ex. The equation $x^4-x^3+6x^2+24=0$ has two imaginary roots; for the absent term is preceded and succeeded by terms having like signs, and the equation $x^3\pm1$ having the coefficients $1\pm0\pm0\pm1$ has also two imaginary roots.

EXAMPLES FOR PRACTICE.

(1.) How many imaginary roots are in the equation

$$x^5+x^3-2x^2+2x-1=0\,?$$

(2.) Has the equation $x^4-2x^2+6x+10=0$ any imaginary roots?

174. The most satisfactory and unfailing criterion for the determination of the number of imaginary roots in any equation is furnished by the admirable theorem of Sturm; which gives the precise number of real roots, and consequently the exact number of imaginary ones; since both the real and imaginary roots are together equal to the number denoted by the degree of the proposed equation.

PROPOSITION VIII.

To find the number of real and imaginary roots in any proposed equation.

The acknowledged difficulty which has hitherto been experienced in the important problem of the separation of the real and imaginary roots of any proposed equation, is now completely removed by the recent valuable researches of the celebrated M. STURM; and we shall now explain the theorem by which this desirable object has been so fully accomplished.

THEOREM OF STURM.

Let $X=Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots\ldots+Hx+K=0$ be any equation which has no equal roots, and let

$$X_1=nAx^{n-1}+(n-1)Bx^{n-2}+(n-2)Cx^{n-3}+\ldots\ldots+H$$

be the derived function, arising by multiplying each term of the equation

$X=0$, by its exponent, and then diminishing the exponent by unity. Divide X by X_1 until the remainder be of a lower degree than the divisor, and call the remainder $-X_2$, or changing the signs of all the terms in the remainder, we shall have $X_2 =$ modified remainder. Proceed in the same manner with the functions X_1 and X_2, and call the modified remainder X_3, and so on, as in the marginal scheme, until the division terminates by leaving a final remainder independent of x; and let this remainder, having its sign changed, be called X_{m+1}. Then we have the series of functions.

$$\begin{array}{l} X_1)X \quad (Q_1 \\ \quad \underline{X_1Q_1} \\ \quad X - X_1Q_1 = -X_2 \end{array}$$

$$\begin{array}{l} X_2)X_1 \quad (Q_2 \\ \quad \underline{X_2Q_2} \\ \quad X_1 - X_2Q_2 = -X_3 \end{array}$$

$$\begin{array}{l} X_3)X_2 \quad (Q_3 \\ \quad \underline{X_3Q_3} \\ \quad X_2 - X_3Q_3 = -X_4 \end{array}$$

$$X,\ X_1,\ X_2,\ X_3,\ X_4,\ \ldots\ldots\ X_{m+1};$$

which are of continually decreasing dimensions in x, and X_{m+1} is altogether independent of x.

Now, if p and q be any two numbers of which p is less than q, and if these numbers be substituted for x in the above series of functions, we shall have two series of signs, the one resulting from the substitution of p for x, giving h variations of sign, and the other from the substitution of q for x, giving k variations of sign; then the exact number of real roots of the proposed equation between the limits p and q will be $= h - k$.

In order to simplify the demonstration of this beautiful theorem, we shall premise one or two Lemmas.

Lemma 1. *Two consecutive functions cannot both vanish for the same value of x.*

From the process above described for the determination of the successive functions, we have obviously these equations:—

$$\begin{array}{ll} X \;\; = X_1\, Q_1 - X_2 & \ldots\ldots\ldots (1) \\ X_1 = X_2\, Q_2 - X_3 & \ldots\ldots\ldots (2) \\ X_2 = X_3\, Q_3 - X_4 & \ldots\ldots\ldots (3) \\ \vdots \quad\;\; \vdots \quad\;\; \vdots & \\ X_{m-1} = X_m\, Q_m - X_{m+1} & \ldots\ldots\ldots (m). \end{array}$$

Now, suppose $X_2=0$, and $X_3=0$; then by eq. (3) we have $X_4=0$; hence, since $X_3=0$, and $X_4=0$; then by eq. (4) we have $X_5=0$; and proceeding in this manner we shall find that $X_{m+1}=0$; but as the equation $X=0$ is supposed not to have equal roots, the polynomials X and X_1 have no common measure (Prop. VII), and therefore there must be a final remainder, X_{m+1}, totally independent of x, and must therefore remain unchanged for every value of x.

Lemma 2. *If one of the derived functions vanish for any particular value of x, the two adjacent functions have contrary signs for the same value of x.*

For by eq. (3) we have

$$X_2 = X_3Q_3 - X_4;$$

and if $X_3=0$; then $X_2=-X_4$, and therefore it is obvious that X_2 and X_4 must have contrary signs.

Demonstration of the Theorem.

Let p be nearer to $-\infty$ than any of the real roots of the equations

$$X=0,\ X_1=0,\ X_2=0,\ X_3=0, \ldots X_m=0;$$

and conceive p to increase continuously until it becomes 0, and then to go on increasing until it becomes equal to q, which we may suppose to be nearer $+\infty$ than any of the real roots of the preceding equations. Now, while p is less than any of the roots of these equations, no change of signs will occur by the substitution of p for x, in any of these functions (Prop. VI. Cor. 4;) but whenever p in its continuous progress towards q, arrives at a root of any of the derived equations, that function becomes zero, and neither the preceding nor succeeding function can vanish for the same value of x (Lemma 1), and these two adjacent functions have contrary signs (Lemma 2); hence the entire number of variations of sign is not affected by the vanishing of any of the derived functions. While, therefore, p advances in the scale of numbers by minute additions, it will pass successively over the roots of the proposed equation, as well as over those of the derived equations; and in passing from a number very little smaller to a number very little greater than a root of the equation $X=0$, the sign of X will be changed from + to — or from — to + (Prop. VI. Cor. 1); and the difference of these numbers may be made so small, that no change of signs can take place in the derived functions; hence the loss of a variation of sign arises from the change of sign of the function X. Again, when p becomes nearly equal to another root of $X=0$, the order of the signs of the derived functions may be changed, but the number of variations is not at all affected (Lemma 2); and, therefore, while p varies from a number very little smaller to a number very little greater than this root of $X=0$, there will be a loss of one variation of sign, arising from the change of the sign of X; and so on for the other roots of $X=0$. Whenever, then, the value of p passes over a root of the equation $X=0$, there is a loss of one variation of sign; and since a variation cannot be lost among the signs of the derived functions, nor can one be ever introduced, it is obvious that we are furnished with a simple and beautiful criterion for ascertaining the number of real roots between any two specified numbers, p and q. To illustrate this more fully, we shall suppose that the substitution of p and q for x in the series of functions, gives the two series of signs, viz.:—

X	X_1	X_2	X_3	X_4		X	X_1	X_2	X_3	X_4
+	—	—	+	+		+	+	+	+	+

Now there are two variations of sign in the former row of signs, and no variation in the latter; hence one variation is lost in the signs of the derived functions, and the sign of X remains unchanged; but a variation cannot be lost in the signs of the derived functions, on the supposition that *one* root lies between p and q; besides, the sign of X is unchanged; hence there must be a number, m, between p and q, which, substituted for x in the series of functions, gives the sign of X negative, and hence there must be one root between p and m, and another root between m and q. The loss of two variations of sign must, therefore, indicate the existence of two real roots between p and q; and, in like manner, the loss of three variations of sign indicates the existence of three roots in the interval, and so on. Hence, if the substitution of p for x gives h variations, and q for x gives k variations; then $h-k=$ number of real roots between p and q.

Since all the real roots are comprehended between the extreme values $-\infty$ and $+\infty$ we may readily ascertain the number of real roots by substituting $-\infty$ and $+\infty$ for x in the leading terms of the several functions, because the first term of each function must, for $x=\mp\infty$, be numerically greater than all the other terms in the function together; and hence the sign of the leading term will determine the sign of the whole function. Let h be the number of variations of sign arising from the substitution of $-\infty$ for x in the functions, and k the number for $+\infty$; then $h-k=$ the number of real roots in the equation, and $n-(h-k)=$ the number of imaginary roots. To determine the initial figures of the roots, we may substitute the successive numbers of the series

$$0, -1, -2, -3, -4, \ldots\ldots$$

till we have as many variations as $-\infty$ produced; and if we substitute the numbers of the series

$$0, 1, 2, 3, 4, \ldots\ldots$$

till we have as many variations as $+\infty$ produced, then the numbers which first produce the known number of variations, will be the limits of the roots of the equations, and the situation of the roots will be indicated by the signs arising from the substitution of the intermediate numbers.

175. When the equation has equal roots, one of the divisors will divide the preceding without a remainder, and the process will thus terminate without a remainder, independent of x. In this case, the last divisor is a common measure of X and X_1; and it has been shown (Prop. VI. Cor. 3, p. 292), that if $(x-a_1)(x-a_2)^2$ be the greatest common measure of X and X_1, then X is divisible by $(x-a_1)^2(x-a_2)^3$, and the depressed equation furnishes the distinct and separate roots of the equation; for Sturm's theorem takes no notice of the repetition of a root. The several functions may be divided by the greatest common measure so found, and the depressed functions employed for the determination of the distinct roots; but it is obvious that the original functions will furnish the separate roots just as well as the depressed ones, for the former differ only from the latter in being multiplied by a common factor; and whether the sign of this factor be $+$ or $-$, the number of variations of sign must obviously remain unchanged, since multiplying or dividing by a positive quantity does not affect the signs of the functions; and if the factor or divisor be negative, *all* the signs of the functions will be changed, and the number of variations of sign will remain precisely as before.

We shall now apply the theorem to a few examples.

Examples.

(1.) Find the number and situation of the roots of the equation

$$x^3-4x^2-6x+8=0*$$

* The process applied to the general cubic equation $x^3+ax^2+bx+c=0$, gives the following functions, viz.:

With the second term.

$$\left.\begin{array}{lcl} X & = & x^3+ax^2+bx+c \ldots\ldots \\ X_1 & = & 3x^2+2ax+b \ldots\ldots \\ X_2 & = & 2(a^2-3b)x+ab-9c \ldots \\ X_3 & = & -4a^3c+a^2b^2-18abc-4b^3-27c^2 \end{array}\right\}(1)$$

Without the second term, or $a=0$.

$$\left.\begin{array}{lcl} X & = & x^3+bx+c \ldots\ldots \\ X_1 & = & 3x^2+b \ldots\ldots \\ X_2 & = & -2bx-3c \ldots\ldots \\ X_3 & = & -4b^3-27c^2 \ldots\ldots \end{array}\right\}(2)$$

These functions in (1) and (2) will frequently be found useful in the application of Sturm's theorem to equations of the third degree, since the derived functions in any particular example may

Here we have $X = x^3-4x^2-6x+8$

$X_1=3x^2-8x-6$;

then, multiplying the polynomial X by 3, in order to avoid fractions,

$$
\begin{array}{rl}
3x^2-8x-6) & 3x^3-12x^2-18x+24\ (x-1 \\
 & 3x^3-\ 8x^2-\ 6x \\
\hline
 & -\ 4x^2-12x+24, \text{ multiply by } \tfrac{3}{4}; \\
\text{or,} & -\ 3x^2-\ 9x+18 \\
 & -\ 3x^2+\ 8x+\ 6 \\
\hline
 & -17x+12\ \therefore X_{11}=17x-12
\end{array}
$$

$$
\begin{array}{rl}
 & 3x^2-\ 8x-6 \\
 & 17 \\
\hline
17x-12) & 51x^2-136x-102\ (3x \\
 & 51x^2-\ 36x \\
\hline
 & -100x-102
\end{array}
$$

It is now unnecessary to continue the division further, since it is very obvious that the sign of the remainder, which is independent of x, is $-$; and, therefore, the series of functions are

$$
\begin{aligned}
X &= x^3-4x^2-6x+8 \\
X_1 &= 3x^2-8x-6 \\
X_{11} &= 17x-12 \\
X_{111} &= +
\end{aligned}
$$

Put $+\infty$ and $-\infty$ for x in the leading terms of these functions, and the signs of the results are

For $x=+\infty$, $+\ +\ +\ +$ no variation $\therefore k=0$

$x=-\infty$, $-\ +\ -\ +$ three variations $\therefore h=3$

$\therefore h-k=3-0=3$, the number of real roots in the proposed cubic equation.

Next, to find the situation of the roots we must employ narrower limits than $+\infty$ and $-\infty$. Commencing at zero, let us extend the limits both ways, and since the proposed equation has only one permanence of sign, one of the roots is negative, and the remaining roots are positive.

					Var.
For $x=0$ signs	+	−	−	+	2
$x=1$	−	−	+	+	1
$x=2$	−	−	+	+	1
$x=3$	−	−	+	+	1
$x=4$	−	+	+	+	1
$x=5$	−	+	+	+	1
$x=6$	+	+	+	+	0

					Var.
For $x=$ 0 signs	+	−	−	+	2
$x=-1$ ...	+	+	−	+	2
$x=-2$ ...	−	+	−	+	3

be found by substitution only. In order that all the roots of the equation $x^3+bx+c=0$ may be real, the first terms of the functions must be positive; hence $-2bx$ and $-4b^3-27c^2$ must be positive; and as $-27c^2$ is always negative, b must be negative, in order that $-4b^3$ and $-2b$ may be positive; therefore, when all the roots are real, $4b^3$ must be greater than $27c^2$, or $\left(\frac{b}{3}\right)^3$ greater than $\left(\frac{c}{2}\right)^2$. When, therefore, b is negative and $\left(\frac{b}{3}\right)^3 > \left(\frac{c}{2}\right)^2$, all the roots are real, a criterion which has been long known, and as simple as can be given.

We perceive, then, by the columns of variations, that the roots are between 0 and 1, 5 and 6, —1 and —2; hence the initial figures of the roots are —1, 0, and 5; and in order to narrow still further the limits of the root between 0 and 1, we shall resume the substitutions for x in the series of functions as before. But as the substitution of 1 for x, in the function X, gives a value nearly zero, we shall commence with 1, and descend in the scale of tenths, until we arrive at the first decimal figure of the root.

Let $x=$ 1 signs $- \; - \; + \; +$ one variation

$x=\cdot 9$ . . . . $+ \; - \; + \; +$ two variations;

hence the initial figures are —1, ·9, and 5.

(2.) Find the number and situation of the real roots of the equation

$$x^4+x^3-x^2-2x+4=0.$$

Here the several functions are

$$\begin{aligned} X &= x^4+x^3-x^2-2x+4 \\ X_1 &= 4x^3+3x^2-2x-2 \\ X_2 &= x^2+2x-6 \\ X_3 &= -x+1 \\ X_4 &= + \end{aligned}$$

Let $x=+\infty$ signs of leading terms $+ \; + \; + \; - \; +$ two variations

$x=-\infty$ . . . . $+ \; - \; + \; + \; +$ two variations;

and all the roots of the equation are imaginary.

(3.) Required the number and situation of the real roots of the equation

$$2x^4-11x^2+8x-16=0.$$

The first three functions are

$$\begin{aligned} X &= 2x^4-11x^2+8x-16 \\ X_1 &= 4x^3-11x+4 \\ X_2 &= 11x^2-12x+32; \end{aligned}$$

and the roots of the quadratic $11x^2-12x+32=0$ are imaginary; for $11\times32\times4$ is greater than 12^2; hence X_2 must preserve the same sign for every value of x, and the subsequent functions cannot change the number of variations, for a variation is only lost by the change of the sign of X. Hence,

For $x=+\infty$ signs $+ \; + \; +$ no variation

$x=-\infty$ . . . $+ \; - \; +$ two variations;

and the proposed equation has two real roots, the one positive, and the other negative, since the last term is negative. (Prop. VI, Cor. 5, p. 279.)

When $x=0$ signs $- \; + \; +$	$x=0$ signs $- \; + \; +$
$x=1$ . . . . $- \; - \; +$	$x=-1$ . . . . $- \; + \; +$
$x=2$ . . . . $- \; + \; +$	$x=-2$ . . . . $- \; - \; +$
$x=3$ . . . . $+ \; + \; +$	$x=-3$ . . . . $+ \; - \; +$

Hence the initial figures of the real roots are 2 and — 2.

When two roots are nearly equal to each other.

(4.) Find the roots of the equation

$$x^3+11x^2-102x+181=0.$$

The functions are

$$\begin{aligned} X &= x^3+11x^2-102x+181 \\ X_1 &= 3x^2+22x-102 \\ X_2 &= 122x-393 \\ X_3 &= +; \end{aligned}$$

and the signs of the leading terms are all +; hence the substitution of $-\infty$ and $+\infty$ must give three real roots.

To discover the situation of the roots, we make the substitutions

$x = 0$ which gives $+ - - +$ two variations
$x = 1$ . . . . $+ - - +$
$x = 2$ . . . . $+ - - +$
$x = 3$ . . . . $+ - - +$ two variations
$x = 4$ . . . . $+ + + +$ no variation;

hence the two positive roots are between 3 and 4, and we must therefore transform the several functions into others, in which x shall be diminished by 3. This is effected by Prop. 1, p. 280; and we get

$$\begin{aligned} Y &= y^3+20y^2-9y+1 \\ Y_1 &= 3y^2+40y-9 \\ Y_2 &= 122y-27 \\ Y_3 &= + \end{aligned}$$

Make the following substitutions in these functions, viz.:

$y = 0$ signs $+ - - +$ two variations
$y = \cdot 1$ . . . $+ - - +$
$y = \cdot 2$ . . . $+ - - +$ two variations
$y = \cdot 3$ . . . $+ + + +$ no variation;

hence the two positive roots are between 3·2 and 3·3, and we must again transform the last functions into others, in which y shall be diminished by ·2. Effecting this transformation, we have

$$\begin{aligned} Z &= z^3+20\cdot6z^2-\cdot88z+\cdot008 \\ Z_1 &= 3z^2+41\cdot2z-\cdot88 \\ Z_2 &= 122z-2\cdot6 \\ Z_3 &= +. \end{aligned}$$

Let $z = 0$ then signs are $+ - - +$ two variations
$z = \cdot01$ . . . . $+ - - +$ two variations
$z = \cdot02$ . . . . $- - - +$ one variation
$z = \cdot03$ . . . . $+ + + +$ no variation;

hence we have 3·21 and 3·22 for the positive roots, and the sum of the roots is -11; therefore $-11-3\cdot21-3\cdot22=-17\cdot4$ is the negative root.

When the equation has equal roots.

(5.) Find the number and situation of the real roots of the equation

$$x^5-7x^4+13x^3+x^2-16x+4=0.$$

By the usual process we find

$$\begin{aligned} X &= x^5-7x^4+13x^3+x^2-16x+4 \\ X_1 &= 5x^4-28x^3+39x^2+2x-16 \\ X_2 &= 11x^3-48x^2+51x+2 \\ X_3 &= 3x^2-8x+4 \\ X_4 &= x-2 \\ X_5 &= 0. \end{aligned}$$

Hence $x-2$ is a common measure of X and X_1; and if

$x = -\infty$ the signs are $- + - + -$ four variations
$x = -2$ . . . . $- + - + -$ four variations
$x = -1$ . . . . $0 + - + -$
$x = 0$ . . . . $+ - + + -$ three variations
$x = 1$ . . . . $- + + - -$ two variations
$x = 2$ . . . . $0\ 0\ 0\ 0\ 0$
$x = 3$ . . . . $- - + + +$ one variation
$x = 4$ . . . . $+ + + + +$ no variation.

Therefore we infer that there are four distinct and separate roots; one is -1, for X vanishes for this value of x; another between 0 and 1; a third is 2, and a fourth is between 3 and 4. The common measure $x-2$ indicates that the polynomial X is divisible by $(x-2)^2$; and hence there are two roots equal to 2 (Prop. VII, Cor. 1.)

Horner's Method of resolving Numerical Equations of all Orders.

176. The method of approximating to the roots of numerical equations of all orders, discovered by W. G. Horner, Esq., of Bath, is a process of very remarkable simplicity and elegance, consisting simply in a succession of transformations of one equation to another, each transformed equation as it arises having its roots less or greater than those of the preceding by the corresponding figure in the root of the proposed equation. We have shown how to discover the initial figures of the roots, by the theorem of Sturm; and by making the penultimate coefficient in each transformation available as a trial divisor of the absolute term, we are enabled to discover the succeeding figure of the root; and thus proceeding from one transformation to another, we are enabled to evolve, one by one, the figures of the root of the given equation, and push it to any degree of accuracy required.

General Rules.

1. Find the number and situation of the roots by *Sturm's* or *Budan's* theorem, and let the root required to be found be positive.

2. Transform the equation into another, whose roots shall be less than those of the proposed equation, by the initial figure of the root.

3. Divide the absolute term of the transformed equation by the *trial divisor*, or penultimate coefficient, and the next figure of the root will be obtained, by which diminish the root of the transformed equation as before, and proceed in this manner till the root be found to the required accuracy.

Note 1. When a negative root is to be found, change the signs of the alternate terms of the equation, and proceed as for a positive root.

Note 2. When three or four decimal places in the root are obtained, the operation may be contracted, and much labour saved, as will be seen in the following examples:

Examples.

(1.) Find all the roots of the cubic equation

$$x^3-7x+7=0.$$

By Sturm's theorem, the several functions are (Note, p. 297.)

$$X = x^3-7x+7$$
$$X_1=3x^2-7$$
$$X_2=2x-3$$
$$X_3= +$$

Hence, for $x=+\infty$ the signs are $+\ +\ +\ +$ no variation

$x=-\infty$ . . . . $-\ +\ -\ +$ three variations;

therefore the equation has *three* real roots, one negative, and two positive.

To determine the initial figures of these roots, we have

for $x=0$ signs	$+\ -\ -\ +$	for $x=\ \ 0$ signs	$+\ -\ -\ +$
$x=1$ . . .	$+\ -\ -\ +$	$x=-1$ . . .	$+\ -\ -\ +$
$x=2$ . . .	$+\ +\ +\ +$	$x=-2$ . . .	$+\ +\ -\ +$
		$x=-3$ . . .	$+\ +\ -\ +$
		$x=-4$ . . .	$-\ +\ -\ +$

hence there are two roots between 1 and 2, and one between -3 and -4.

But in order to ascertain the first figures in the decimal parts of the two roots situated between 1 and ·2, we shall transform the preceding functions into others, in which the value of x is diminished by unity. Thus for the function X, we have this operation:

1+0	—7	+7	(1
1	1	—6	
1	—6	1	
1	2		
2	—4		
1			
3			

And transforming the others in the same way, we obtain the functions

$Y = y^3 + 3y^2 - 4y + 1$; $Y_1 = 3y^2 + 6y - 4$; $Y_2 = 2y - 1$; $Y_3 = +$.

Let $y = \cdot 1$ then the signs are	+ — — +	two variations
$y = \cdot 2$	+ — — +	do.
$y = \cdot 3$	+ — — +	do.
$y = \cdot 4$	— — — +	one variation
$y = \cdot 5$	— — ∓ +	do.
$y = \cdot 6$	— + + +	do.
$y = \cdot 7$	+ + + +	no variation.

Therefore the initial figures of the three roots are 1·3, 1·6, and —3.

1 +0	— 7	+ 7 (1·356895867
1	1	— 6
1	— 6	* 1...
1	2	— 903
2	—*4 . .	*97...
1	99	—86625
*33	— 301	*10375...
3	108	—9048984
36	—*193	*1326016
3	1975	—1184430
*395	— 17325	141586
5	2000	—132923
400	—*15325 . .	8663
5	24336	—7382
*4056	— 1508164	1281
6	24372	—1181
4062	—*1483792	100
6	32514	—89
*|40|68	— 1480538	11
	32514	—10
	— 147728	1
	— 147692	
	36	
	— 1|4|7|6|5	

We have thus found one root $x=1{\cdot}356895867 \ldots\ldots.$, and the coefficients of the successive transformed equations are indicated by the asterisks in each column.

1 +0	− 7	+ 7 (1·692021471
1	1	− 6
1	− 6	1...
1	2	− 1104
2	− 4..	− 104...
1	216	100809
36	− 184	—3191...
6	252	3156888
42	68..	—34112
6	4401	31774
489	11201	—2338
9	4482	1589
498	15683..	—749
9	10144	635
5072	1578444	—114
2	10148	111
5074	1588592	3
2	1	
5076	15887	

Another root is $x=1{\cdot}692021471 \ldots\ldots.$

For the negative root, change the signs of the second and fourth terms.

1 − 0	− 7	− 7 (3·0489173396
3	9	+ 6
3	2	− 1......
3	18	814464
6	20....	− 185536...
3	3616	166382592.
904	203616	− 19153408
4	3632	18791228
908	207248..	—362180
4	73024	208875
9128	20797824	—153305
8	73088	146212
9136	20870912	—7093
8	8230	6266
9144	20879142	—827
	8230	626
	2088737	—201
	9	188
	2088746	— 13
	9	12
	208875	1

Hence the three roots of the proposed cubic equation are

$$x = 1{\cdot}356895867 \ldots\ldots$$
$$x = 1{\cdot}692021471 \ldots\ldots$$
$$x = -3{\cdot}048917339 \ldots\ldots$$

Note.—Since each successive figure in the decimal part of the root extends the right hand column *three* places of decimals, the middle column *two* places, and the left hand column *one*, therefore, in the contractions we must cut off *two* figures from the left hand column, *one* from the middle column, and none from the right hand column; and thus we cut off in effect *three* decimal places from each column.

(2.) Find the roots of the equation $x^3+11x^2-102x+181=0$.

We have already found the roots to be nearly 3·21, 3·22, and —17. (See Example 4, page 300.)

1 + 11	— 102	+ 181 (3·21312775
3	42	— 180
14	— 60	1...
3	51	— 992
17	— 9..	8...
3	404	— 6739
202	— 496	1261...
2	408	— 1217403
204	— 88..	43597
2	2061	— 34183
2061	— 6739	9414
1	2062	— 6787
2062	— 4677..	2627
1	61899	— 2372
20633	— 405801	255
3	61908	— 237
20636	— 34389 3	18
3	206 4	— 16
1·2 06 139	— 34182 9	2
	206 4	
	— 3397 6	
	4 1	
	— 3393 5	
	4 1	
	— 3 3 8 9	

In a similar manner, the two remaining roots will be found to be 3·22952121 and —17·44264896.

(3.) Given $x^4+x^3+x^2+3x-100=0$, to find the number and situation of the real roots.

By Sturm's method.

$$\begin{aligned} \text{Here we have } X &= x^4+x^3+x^2+3x-100 \\ X_1 &= 4x^3+3x^2+2x+3 \\ X_2 &= -5x^2-34x+1603 \\ X_3 &= -1132x+6059 \\ X_4 &= - \end{aligned}$$

Let $x=-\infty$ then signs are $+ - - + -$ three variations,
$x=+\infty$ . . . . $+ + - - -$ one variation;

hence two roots are real, and two imaginary; and the real roots must have contrary signs, for the last term of the equation is negative. To find the situation of the roots.

in	X $X_1 X_2 X_3 X_4$		in	X $X_1 X_2 X_3 X_4$
Let $x=0$ signs	$- + + + -$		Also $x=0$ signs	$- + + + -$
$x=1$ . . .	$- + + + -$		$x=-1$ . . .	$- 0 + + -$
$x=2$ . . .	$- + + + -$		$x=-2$ . . .	$- - - + + -$
$x=3$ . . .	$+ + + + -$		$x=-3$ . . .	$- - - + + -$
			$x=-4$ . . .	$+ - + + -$

In this example, the function X_1 vanishes for $x=-1$, and for the same value of x, the functions X and X_2 have contrary signs, agreeably to Lemma 2, and writing $+$ or $-$ for 0 gives the same number of variations. The initial figures of the root are, therefore, 2 and -3.

The same otherwise, by Budan's method.

If the roots of this equation be all real, the permanencies and variation indicate three negative roots and one positive root.

(1.) To find the positive root.

1+1+1+ 3−100 (2	1+1+ 1+ 3−100 (3
3+7+17− 66	4+13+42+ 26

In the transformation by 2, one variation is left, and in transforming by 3, there is no variation left; therefore, the positive root is between 2 and 3.

(2.) For the negative roots.

Direct Equation.	*Reciprocal Equation.*
1−1+1−3−100 (1	−100− 3+ 1− 1+ 1 (1
0+1−2−102	−103−102−103−102
1+2+0	
2+4	signs all −
3	

Here two variations are lost in the direct transformation, and no variations are left in the reciprocal transformation; therefore, the two roots in the interval 0 and −1 are imaginary.

1−1+1− 3−100 (3	1−1+ 1− 3−100 (4
2+7+18− 46	3+13+49+ 96

Hence the negative root is obviously situated between −3 and −4.

To find the negative root, we have the following operation:—

1 −1	+ 1	− 3	− 100(3·4335778633659 9
3	6	21	54
2	7	18	− 46....
3	15	66	416896
5	22	84...	− 43104....
3	24	20224	384456501
8	46··	104224	− 46583499....
3	4 56	22112	390491222121
114	50 56	126336...	− 75343767879
4	4 72	1816167	65189289046
118	55 28	128152167	− 10154478833
− 4	4 88	1827561	9128951421
122	60 16··	129979728...	− 1025527412
4	37 89	184012707	912928254
1 263	60 53 89	130163740707	− 112599158
3	37 98	184127241	104335040
1 266	60 91 87	13034786794 8	− 8264118
3	38 07	3071014 5	7825130
1 269	61 29 94..	13037857809 3	− 438988
3	381 6 9	3071332 5	391256
12 723	61 33 75 69	130409291 4 2	− 47732
3	381 7 8	43003 1	39126
12 726	61 37 57 47	13041359 17 3	− 8606
3	38 1 87	43003 1	7825
12 729	61 41 39 34	130417892 0	− 781
3	63 6	430 0	652
·12 732	61 42 02 9	130418322	− 129
	63 6	430	117
	61 42 66 5	13041875 2	− 12
	63 6	4 8	11
	61 43 30	1 3 0 4 1 8 8 0 4	1

For the positive root we have a similar operation.—

1 + 1 + 1 + 3 − 100(2·8028512181582;

but this we shall leave for the student to perform, and the two roots will be found to be

$$x = 2{\cdot}8028512181582\ldots$$
$$x = -3{\cdot}4335778633659\ldots$$

(4.) Find the roots of the equation $x^5+2x^4+3x^3+4x^2+5x-20=0$.

Here we have $X = x^5+2x^4+3x^3+4x^2+5x-20$

$$X_1 = 5x^4+8x^3+9x^2+8x+5$$
$$X_2 = -7x^3-21x^2-42x+255$$
$$X_3 = -13x+14$$
$$X_4 = -$$

For $x=-\infty$ we have signs $-+++-$ two variations;
$x=+\infty$ .... $++---$ one variation.

Hence the difference of variations of sign indicates the existence of one real and four imaginary roots, the real root being situated between 1 and 2.

1	+2	+3	+4	+5	−20 (1·125790··
	1	3	6	10	15
	3	6	10	15	−5.....
	1	4	10	20	387171
	4	10	20	35....	−112829
	1	5	15	37171	87005
	5	15	35...	387171	−25824
	1	6	2171	39414	22285
	6	21..	37171	42658 5	−3539
	1	71	2243	844	3136
	71	2171	39414	43502 5	−403
	1	72	2316	853 4	403
	72	2243	417 30	4435 6	
	1	73	4 7	21 5	
	73	2316	422 0	4457 1	
	1	74	4 7	21 5	
	74	··2 390	426 7	447 8	
	1		4 7	2	
	··75		·4 31	44 8	

hence the real root is nearly 1·125790; and by using another period of ciphers we should have the root correct to ten places of decimals, with very little additional labour.

Additional Examples for Practice.

(1.) Find all the roots of the equation $x^3-3x-1=0$.

(2.) Find all the roots of the equation $x^3-22x-24=0$.

(3.) Find the roots of the equation $x^3+x^2-500=0$.

(4.) Find the roots of the equation $x^3+x^2+x-100=0$.

(5.) Find the roots of the equation $2x^3+3x^2-4x-10=0$.

(6.) Find the roots of the equation $x^4-12x^2+12x-3=0$.

(7.) Find the roots of the equation $x^4-8x^3+14x^2+4x-8=0$.

(8.) Find the roots of the equation $x^4-x^3+2x^2+x-4=0$.

(9.) Find the roots of the equation $x^5-10x^3+6x+1=0$.

(10.) Find the roots of the equation $x^5+3x^4+2x^3-3x^2-2x-2=0$

(11.) Find all the roots of the equation

$$x^6+4x^5-3x^4-16x^3+11x^2+12x-9=0.$$

(12.) Given the equations

$$4x^2-5xy+6y^2-2x-10y+20=0$$
$$3x^2-8xy+4y^2-12x+9y-15=0$$

to find the values of x and y.

ANSWERS.

$$(1.)\begin{cases} x=+1\cdot8793852242 \\ x=-1\cdot532088886 \\ x=-\ \cdot347296355 \end{cases}$$

$$(2.)\begin{cases} x=+5\cdot1622776601 \\ x=-1\cdot1622776601 \\ x=-4 \end{cases}$$

(3.) $x=7\cdot61727975596$

(4.) $x=4\cdot2644299731$

(5.) $x=1\cdot62481908836$

$$(6.)\begin{cases} x=+2\cdot858083308163 \\ x=+\ \cdot606018306917 \\ x=+\ \cdot443276939605 \\ x=-3\cdot907378554685 \end{cases}$$

$$(7.)\begin{cases} x=+5\cdot2360679775 \\ x=+\ \cdot7639320225 \\ x=+2\cdot7320508075 \\ x=-\ \cdot7320508075 \end{cases}$$

$$(8.)\begin{cases} x=+1\cdot146994592 \\ x=-1\cdot090593586 \end{cases}$$

$$(9.)\begin{cases} x=-3\cdot0653157912983 \\ x=-\ \cdot6915762804900 \\ x=-\ \cdot1756747992883 \\ x=+\ \cdot8795087084144 \\ x=+3\cdot0530581626622 \end{cases}$$

(10.) $x=1\cdot059109003461882$

$$(11.)\begin{cases} x=-1;\ x=-3;\ x=1 \\ \qquad\qquad x=-3\ \ x=1 \\ \qquad\qquad\qquad\qquad x=1 \end{cases}$$

(12.) The biquadratic equation

$$480y^4-125y^3+3923y^2-7176y+25740=0,$$

contains the four values of y, and then x is found from the relation

$$x=-\frac{2(y^2-33y+60)}{17y+42}.$$

ON INEQUATIONS.

177. In discussing algebraical problems, it is frequently necessary to introduce *inequations*, that is, expressions connected by the sign $>$. Generally speaking, the principles already detailed for the transformation of equations, are applicable to inequations also. There are, however, some important exceptions which it is necessary to notice, in order that the student may guard against falling into error in employing the sign of inequality. These exceptions will be readily understood by considering the different transformations in succession.*

I. *If we add the same quantity to, or subtract it from, the two members of any inequation, the resulting inequation will always hold good, in the same sense as the original inequation.* That is, if

$$a > b,\ \text{then}\ a+a' > b+a',\ \text{and}\ a-a' > b-a'.$$

* *Example in Inequations.*—The double of a number diminished by 6 is greater than 24; and triple the number diminished by 6 is less than double the number increased by 10. Required a number which will fulfil the conditions.

Let x represent a number fulfilling the conditions of the question; then, in the language of inequations, we have $2x-6 > 24$, and $3x-6 < 2x+10$. From the former of these inequations we have $2x > 30$, or $x > 15$; and from the latter we get $3x-2x < 10+6$, or $x < 16$; therefore 15 and 16 are the limits, and any number between these limits will satisfy the conditions of the question. Thus, if we take the number 15·9 we have $15\cdot9\times2-6 > 24$ by 1·8, whilst $15\cdot9\times3-6 < 15\cdot9\times2+10$ by 1. Other examples may easily be formed

Thus,

$8 > 3$, we have still, $8 + 5 > 3 + 5$, and $8 - 5 > 3 - 5$.

So also,

$-3 < -2$, and we have still $-3 + 6 < -2 + 6$, and $-3 - 6 < -2 - 6$.

The truth of this proposition is evident from what has been said with reference to equations.

This principle enables us, as in equations, to transpose any term from one member of an inequation to the other, by changing its sign.

Thus, from the inequation,

$$a^2 + b^2 > 3b^2 - 2a^2$$

We deduce

$$a^2 + 2a^2 > 3b^2 - b^2$$

Or,

$$3a^2 > 2b^2.$$

II. *If we add together the corresponding members of two or more inequations which hold good in the same sense, the resulting inequation will always hold good in the same sense as the original individual inequations.*

That is, if

$$a > b, \ c > d, \ e > f$$

Then,

$$a + c + e > b + d + f.$$

III. *But if we subtract the corresponding members of two or more inequations which hold good in the same sense, the resulting inequation* WILL NOT ALWAYS *hold good in the same sense as the original inequations.*

Take the inequations $4 < 7$, $2 < 3$, we have still $4 - 2 < 7 - 3$, or $2 < 4$.

But take $9 < 10$ and $6 < 8$, the result is $9 - 6 >$ (not $<$) $10 - 8$, or $3 > 2$.

We must therefore avoid as much as possible making use of a transformation of this nature, unless we can assure ourselves of the sense in which the resulting inequality will subsist.

IV. *If we multiply or divide the two members of an inequation by a positive quantity, the resulting inequation will hold good in the same sense as the original inequation.*

Thus, if $a < b$, then $ma < mb$, $\frac{a}{m} < \frac{b}{m}$

$$-a > -b, \text{ then } -na > -nb, \ -\frac{a}{n} > -\frac{b}{n}$$

This principle will enable us to clear an inequation of fractions.

Thus if we have

$$\frac{a^2 - b^2}{2d} > \frac{c^2 - d^2}{3a}$$

Multiplying both members by $6ad$ it becomes

$$3a(a^2 - b^2) > 2d(c^2 - d^2)$$

But,

V. *If we multiply or divide the two members of an inequation by a negative quantity, the resulting inequation will hold in a sense opposite to that of the original inequation.*

Thus, if we take the inequation $8 > 7$, multiplying both members by -3, we have the opposite inequation, $-24 < -21$.

$$\text{Similarly, } 8 > 7, \text{ but } \frac{8}{-3} < \frac{7}{-3}, \text{ or } -\frac{8}{3} < -\frac{7}{3}.$$

VI. *We cannot change the signs of both members of an inequation, unless we reverse the sense of the inequation, for this transformation is manifestly the same thing as multiplying both members by* -1.

VII. *If both members of an inequation be positive numbers, we can raise them to any power without altering the sense of the inequation.* That is, if

$$a > b \text{ then } a^n > b^n.$$

Thus from $5 > 3$ we have $(5)^2 > (3)^2$ or $25 > 9$.
So also from $(a+b) > c$, we have $(a+b)^2 > c^2$.

But,

VIII. *If both members of an inequation be not positive numbers, we cannot determine,* a priori, *the sense in which the resulting inequation will hold good, unless the power to which they are raised be of an uneven degree.*

Thus, $-2 < 3$ gives $(-2)^2 < (3)^2$ or $4 < 9$
But, $-3 > -5$ gives $(-3)^2 < (-5)^2$ or $9 < 25$
Again, $-3 > -5$ gives $(-3)^3 > (-5)^3$ or $-27 > -125$.

In like manner,

IX. *We can extract any root of both members of an inequation without altering the sense of the inequation.* That is, if

$$a > b, \text{ then, } \sqrt[n]{a} > \sqrt[n]{b}.$$

If the root be of an even degree both members of the inequation must necessarily be positive, otherwise we should be obliged to introduce imaginary quantities, which cannot be compared with each other.

PROGRESSIONS.

ARITHMETICAL PROGRESSION.

178. **When** a series of quantities continually increase or decrease by the addition or subtraction of the same quantity, the quantities are said to be in *Arithmetical Progression.*

Thus the numbers 1, 3, 5, 7, ----- which differ from each other by the addition of 2 to each successive term, form what is called *an increasing arithmetical progression*, and the numbers 100, 97, 94, 91, ----- which differ from each other by the subtraction of 3 from each successive term, form what is called *a decreasing arithmetical progression.*

Generally, if a be the first term of an arithmetical progression, and δ the common difference, the successive terms of the series will be

$$a, \; a \pm \delta, \; a \pm 2\delta, \; a \pm 3\delta, \; \text{-----}$$

in which the positive or negative sign will be employed, according as the series is an increasing or decreasing progression.

Since the coefficient of δ in the *second* term is 1, in the *third* term 2, in the *fourth* term 3, and so on, the n^{th} term of the series will be of the form

$$a \pm (n-1)\delta.$$

In what follows we shall consider the progression as an increasing one, since all the results which we obtain can be immediately applied to a decreasing series by changing the sign of δ.

179. *To find the sum of* n *terms of a series in arithmetical progression.*

$$\begin{aligned} \text{Let } a &= \text{first term.} \\ \ldots\ l &= \text{last term.} \\ \ldots\ \delta &= \text{common difference.} \\ \ldots\ n &= \text{number of terms.} \\ \ldots\ S &= \text{sum of the series.} \end{aligned}$$

Then $\quad S = a + (a+\delta) + (a+2\delta) + \text{------} + l.$

Write the same series in a reverse order, and we have

$$S = l + (l-\delta) + (l-2\delta) + \text{------} + a$$

Adding, $\quad 2S = (a+l) + (a+l) + (a+l) + \text{------} + (a+l)$

$\quad = n(a+l)$ since the series consists of n terms.

$$\therefore \quad S = \frac{n(a+l)}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Or, since $l = a+(n-1)\delta$

$$S = \frac{2na+n(n-1)\delta}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence, if any three of the five quantities a, l, δ, n, S, be given, the remaining two may be found by eliminating between equations (1) and (2).

It is manifest from the above process that

The sum of any two terms which are equally distant from the extreme terms is equal to the sum of the extreme terms, and if the number of terms in the series be uneven, the middle term will be equal to one-half the sum of the extreme terms, or of any two terms equally distant from the extreme terms.

Example 1.

Required the sum of 60 terms of an arithmetical series, whose first term is 5 and common difference 10.

Here, $a=5$, $\delta=10$, $n=60$

$$\therefore \; l=a+(n-1)\delta=5+59\times10=595$$

$$\therefore \; S=\frac{(5+595)\times60}{2}$$

$$=600\times30=18000= \text{sum required.}$$

Example 2.

A body descends in vacuo through a space of $16\frac{1}{12}$ feet during the first second of its fall, but in each succeeding second $32\frac{1}{6}$ feet more than in the one immediately preceding. If a body fall during the space of 20 seconds, how many feet will it fall in the last second, and how many in the whole time?

Here, $a=\frac{193}{12}$; $\delta=\frac{386}{12}$, $n=20$

$$\therefore \; l = \frac{193}{12}+19\times\frac{386}{12}$$

$$= \frac{7527}{12} = 627\tfrac{1}{4} \text{ feet}$$

$$S = \frac{(193+7527)\times20}{2\times12}$$

$$= \frac{77200}{12}$$

$$= 6433\tfrac{1}{3} \text{ feet.}$$

Example 3.

To insert m arithmetical means between a and b.

Here we are required to form an arithmetical series of which the first and last terms, a and b, are given, and the number of terms $=m+2$; in order then to determine the series we must find the common difference.

Eliminating S by equations (1) and (2), we have

$$2a+(n-1)\delta = l + a$$

$$\delta = \frac{l-a}{n-1}$$

But here, $l = b$, $a = a$, $n = m+2$

$\therefore$ the required series will be

$$a+\left(a+\frac{b-a}{m+1}\right)+\left(a+\frac{2(b-a)}{m+1}\right)+\ldots\ldots+\left(a+\frac{m(b-a)}{m+1}\right)+\left(a+\frac{(m+1)(b-a)}{m+1}\right)$$

Or,

$$a+\frac{b+ma}{m+1}+\frac{2b+(m-1)a}{m+1}+\ldots\ldots+\frac{mb+a}{m+1}+b$$

Or,

$$a+\frac{b+ma}{m+1}+\frac{2b+(m-1)a}{m+1}+\ldots\ldots\frac{(m-1)b+2a}{m+1}+\frac{mb+a}{m+1}+b$$

Ex. 4. Required the sum of the odd numbers 1, 3, 5, 7, 9, &c. continued to 101 terms? Ans. 10201.

Ex. 5. How many strokes do the clocks of Venice. which go on to 24 o'clock, strike in the compass of a day? Ans. 300

Ex. 6. The first term of a decreasing arithmetical series is 10, the common difference $\frac{1}{3}$, and the number of terms 21; required the sum of the series? Ans. 140.

Ex. 7. One hundred stones being placed on the ground, in a straight line, at the distance of 2 yards from each other; how far will a person travel, who shall bring them one by one to a basket, which is placed 2 yards from the first stone? Ans. 11 miles and 840 yards.

GEOMETRICAL PROGRESSION.

180. A series of quantities, in which each is derived from that which immediately precedes it, by multiplication by a constant quantity, is called a *Geometrical Progression.*

Thus, the numbers 2, 4, 8, 16, 32, ... in which each is derived from the preceding by multiplying it by 2, form what is called an *increasing geometrical progression;* and the numbers 243, 81, 27, 9, 3, . . . in which each is derived from the preceding by multiplying it by the number $\frac{1}{3}$, form what is called a *decreasing geometrical progression.*

The common multiplier in a geometrical progression, is called the *common ratio.*

Generally, if a be the first term, and ϱ the common ratio, the successive terms of the series will be of the form,

$$a,\ a\varrho,\ a\varrho^2,\ a\varrho^3 - - - - -$$

The exponent of ϱ in the *second* term is 1, in the *third* term is 2, in the *fourth* term 3, and so on; hence, the n^{th} term of the series will be of the form,

$$a\varrho^{n-1}$$

181. *To find the sum of n terms of a series in Geometrical Progression.*

Let a = first term
... l = last term
... ϱ = common ratio
... n = number of terms
... S = sum of the series

Then,

$$S = a + a\varrho + a\varrho^2 + a\varrho^3 + \dots\dots + a\varrho^{n-1}$$

Multiply both sides of the equation by ϱ,

$$\therefore S\varrho = \quad a\varrho + a\varrho^2 + a\varrho^3 + \dots\dots + a\varrho^{n-1} + a\varrho^n$$

Subtract the first from the second,

$$S(\varrho - 1) = a\varrho^n - a$$

$$\therefore S = \frac{a(\varrho^n - 1)}{\varrho - 1} \dots\dots\dots (1)$$

Or, since,

$$l = a\varrho^{n-1}$$

$$S = \frac{\varrho l - a}{\varrho - 1} \dots\dots\dots (2)$$

If the series be a decreasing one, and consequently ϱ fractional, it will be convenient to change the signs of both numerator and denominator in the above expressions, which then become,

$$S = \frac{a(1 - \varrho^n)}{1 - \varrho}$$

$$S = \frac{a - \varrho l}{1 - \varrho}$$

Hence, it appears, that if any three of the five quantities, a, l, ϱ, n, S, be given, the remaining two may be found by eliminating between equations (1) and (2). It must be remarked, however, that when it is required to find ϱ from a, n, S given, or from n, l, S given, we shall obtain ϱ in an equation of the n^{th} degree, which cannot be solved generally.

Example 1.

Required the sum of 10 terms of the series 1, 2, 4, 8, . . .

Here, $a = 1$, $\varrho = 2$, $n = 10$

$$\therefore S = \frac{a(\varrho^n - 1)}{\varrho - 1}$$

$$= 2^{10} - 1$$

$$= 1023$$

Example 2.

Required the sum of 10 terms of the series $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$

$$\text{Here, } a = 1, \quad \varrho = \frac{2}{3}, \quad n = 10$$

$$\therefore S = \frac{a(1-\varrho^n)}{1-\varrho}$$

$$= \frac{1-\left(\frac{2}{3}\right)^{10}}{1-\frac{2}{3}}$$

$$= \frac{174075}{59049}$$

Example 3.

To insert m geometric means between a and b.

Here we are required to form a geometric series, of which the first and last terms, a and b, are given, and the number of terms $= m + 2$; in order, then, to determine the series, we must find the common ratio.

Eliminating S by equations (1) and (2),

$$a\varrho^n - a = \varrho l - a$$

$$\varrho = \sqrt[n-1]{\frac{l}{a}}$$

But here,

$$l = b, \quad n = m + 2$$

$$\therefore \quad \varrho = \sqrt[m+1]{\frac{b}{a}}$$

Hence, the series required will be,

$$a + a\sqrt[m+1]{\frac{b}{a}} + a\sqrt[m+1]{\frac{b^2}{a^2}} + \ldots + a\sqrt[m+1]{\frac{b^{m-1}}{a^{m-1}}} + a\sqrt[m+1]{\frac{b^m}{a^m}} + a\sqrt[m+1]{\frac{b^{m+1}}{a^{m+1}}}$$

or,

$$a + \sqrt[m+1]{a^m b} + \sqrt[m+1]{a^{m-1}b^2} + \ldots + \sqrt[m+1]{a^2 b^{m-1}} + \sqrt[m+1]{ab^m} + b$$

or,

$$a + a^{\frac{m}{m+1}} b^{\frac{1}{m+1}} + a^{\frac{m-1}{m+1}} b^{\frac{2}{m+1}} + \ldots + a^{\frac{2}{m+1}} b^{\frac{m-1}{m+1}} + a^{\frac{1}{m+1}} b^{\frac{m}{m+1}} + b$$

182. *To find the sum of an infinite series decreasing in geometrical progression.*

We have already found, that the sum of n terms of a decreasing geometrical series, is,

$$S = \frac{a - a\varrho^n}{1-\varrho}$$

which may be put under the form,

$$S = \frac{a}{1-\varrho} - \frac{a}{1-\varrho} \cdot \varrho^n$$

Since ϱ is a fraction, ϱ^n is less than unity, and the greater the number n, the smaller will be the quantity ϱ^n; if, therefore, we take a very great number of terms of a decreasing series, the quantity ϱ^n, and, consequently, the term $\frac{a\varrho^n}{1-\varrho}$, will be very small in comparison with $\frac{a}{1-\varrho}$; and if we take n greater than any assignable number, or make $n = \infty$, then ϱ^n will be smaller than any assignable number, and therefore may be considered $= 0$, and the second term in the above expression will vanish.

Hence, we may conclude, that the sum of an infinite series, decreasing in geometrical progression, is,

$$S = \frac{a}{1-\varrho}$$

Strictly speaking, $\frac{a}{1-\varrho}$ is the *limit* to which the sum of any number of terms approaches, and the above expression will approach more or less nearly to perfect accuracy, according as the number of terms be greater or smaller.

Thus, let it be required to find the sum of the infinite series

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \text{\&c.}$$

Here, $a = 1$, $\varrho = \frac{1}{3}$, $n = \infty$

$$\therefore \quad S = \frac{a}{1-\varrho}$$

$$= \frac{1}{1-\frac{1}{3}}$$

$$= \frac{3}{2}$$

The error which we should commit in taking $\frac{3}{2}$ for the sum of the first n terms of the above series, is determined by the quantity,

$$\frac{a\varrho^n}{1-\varrho} = \frac{3}{2}\left(\frac{1}{3}\right)^n$$

Thus, if $n = 5$, then, $\frac{3}{2}\left(\frac{1}{3}\right)^5 = \frac{1}{2 \cdot 3^4} = \frac{1}{162}$

.. $n = 6$, then, $\frac{3}{2}\left(\frac{1}{3}\right)^6 = \frac{1}{2 \cdot 3^5} = \frac{1}{486}$

Hence, if we take $\frac{3}{2}$ as the sum of 5 terms of the above series, the amount would be too great by $\frac{1}{162}$.

If we take $\frac{3}{2}$ as the sum of 6 terms, the amount would be too great by $\frac{1}{486}$, and so on.

HARMONICAL PROGRESSION.

183. A series of quantities are said to be in *harmonical progression*, when, if any three consecutive terms be taken, the first is to the third as the difference of the first and second to the difference of the second and third.

Thus, if $a, b, c, d,$ . . . . be a series of quantities in harmonical progression, we shall have,

$$a : c :: a - b : b - c; \quad b : d :: b - c : c - d; \text{ \&c.}$$

184. *The reciprocals of a series of terms in harmonical progression are in arithmetical progression.*

Let $a, b, c, d, e, f,$ . . . . be a series in harmonical progression,

Then, by definition,

$$a : c :: a - b : b - c; \ b : d :: b - c : c - d; \ c : e :: c - d : d - e; \text{ \&c.}$$

$$\therefore ab - ac = ac - bc, \ bc - bd = bd - dc, \ cd - ce = ce - ed, \text{ \&c.}$$

$$\therefore \frac{ab}{abc} - \frac{ac}{abc} = \frac{ac}{abc} - \frac{bc}{abc}, \ \frac{bc}{bcd} - \frac{bd}{bcd} = \frac{bd}{bcd} - \frac{dc}{bcd}, \ \frac{cd}{cde} - \frac{ce}{cde} = \frac{ce}{cde} - \frac{ed}{cde}$$

$$\text{or, } \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a} \quad \frac{1}{d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{b}, \quad \frac{1}{e} - \frac{1}{d} = \frac{1}{d} - \frac{1}{c}$$

from which it appears, that the quantities $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e},$ &c. are in arithmetical progression.

To insert m harmonic means between a and b.

Since the reciprocals of quantities in harmonical progression are in arithmetical progression, let us insert m arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$.

Generally in arithmetical progression,

$$l = a + (n - 1)\delta$$

$$\therefore \delta = \frac{l - a}{n - 1}$$

In this case $l = \frac{1}{b}$, $a = \frac{1}{a}$, $n = m + 2$, and $\therefore n - 1 = m + 1$.

The arithmetic series will be

$$\frac{1}{a} + \frac{a + mb}{(m+1)ab} + \frac{2a + (m-1)b}{(m+1)ab} + \ldots\ldots\ldots \frac{(m-1)a + 2b}{(m+1)ab} + \frac{ma + b}{(m+1)ab} + \frac{1}{b}$$

Therefore the harmonical series will be

$$a+\frac{(m+1)ab}{a+mb}+\frac{(m+1)ab}{2a+(m-1)b}+\dots\dots+\frac{(m+1)ab}{(m-1)a+2b}+\frac{(m+1)ab}{ma+b}+b$$

ON PERMUTATIONS AND COMBINATIONS.

185. The *Permutations* of any number of quantities signify the changes which these quantities may undergo with respect to their order.

Thus, if we take the quantities a, b, c; then, $a\ b\ c$, $a\ c\ b$, $b\ a\ c$, $b\ c\ a$, $c\ a\ b$, $c\ b\ a$, are the permutations of these three quantities taken *all together*; $a\,b$, $a\,c$, $b\,a$, $b\,c$, $c\,a$, $c\,b$, are the permutations of these quantities taken *two* and *two*; a, b, c, are the permutation of these quantities taken singly, or *one* and *one*, &c.

The problem which we propose to resolve is,

186. *To find the number of the permutations of* n *quantities, taken* p *and* p *together.*

Let $a, b, c, d, \dots\dots\dots k$, be the n quantities.

The number of the permutation of these n quantities taken singly, or one and one, is manifestly n.

The number of the permutations of these n quantities, taken two and two together, will be $n\ (n-1)$. For since there are n quantities,

$$a, b, c, d, \dots\dots\dots\dots\dots\dots\dots k$$

If we remove a there will remain $(n-1)$ quantities,

$$b, c, d, \dots\dots\dots\dots\dots\dots\dots k$$

Writing a before each of these $(n-1)$ quantities, we shall have

$$a\,b,\ a\,c,\ a\,d, \dots\dots\dots\dots\dots\dots a\,k$$

That is, $(n-1)$ permutations of the n quantities taken two and two, in which a stands first. Reasoning in the same manner for b, we shall have $(n-1)$ permutations of the n quantities taken two and two, in which b stands first, and so on for each of the n quantities in succession, hence the whole number of permutations will be

$$n\ (n-1)$$

The number of the permutations of n quantities taken three and three together is $n\ (n-1)\ (n-2)$. For since there are n quantities, if we remove a there will remain $(n-1)$ quantities; but, by the last case, writing $(n-1)$ for n, the number of the permutations of $(n-1)$ quantities taken two and two is $(n-1)$ $(n-2)$; writing a before each of these $(n-1)\ (n-2)$ permutations, we shall have $(n-1)\ (n-2)$ permutations of the n quantities taken three and three, in which a stands first. Reasoning in the same manner for b, we shall have

$(n-1)(n-2)$ permutations of the n quantities taken three and three in which b stands first, and so on for each of the n quantities in succession, hence the whole number of permutations will be

$$n(n-1)(n-2)$$

In like manner we can prove that the number of permutations of n quantities taken four and four will be

$$n(n-1)(n-2)(n-3)$$

Upon examining the above results, we readily perceive that a certain relation exists between the numerical part of the expressions, and the class of permutations to which they correspond.

Thus the number of permutations of n quantities, taken *two and two*, is

$n(n-1)$ which may be written under the form $n(n-2+1)$

Taken *three and three*, it is

$n(n-1)(n-2)$ which may be written under the form $n(n-1)(n-3+1)$

Taken *four and four*, it is

$n(n-1)(n-2)(n-3)$ which may be written under the form $n(n-1)$
$(n-2)(n-4+1)$

Hence from *analogy* we may conclude, that the number of permutations of n things, taken p and p together, will be

$$n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots(n-p+1)$$

In order to *demonstrate* this, we shall employ the same species of proof already exemplified in (Arts. 39 and 89), and show that, if the above law be assumed to hold good for any one class of permutations, it must necessarily hold good for the class next superior.

Let us suppose, then, that the expression for the number of the permutations of n quantities taken $(p-1)$ and $(p-1)$ together is

$$n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots\{n-(p-1)+1\}\ldots\ldots\ldots\ldots(A)$$

It is required to prove that the expression for the number of the permutations of n quantities, taken p and p together, will be

$$n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots(n-p+1)$$

Remove a one of the n quantities $a, b, c, d\ldots\ldots\ldots k$, then by the expression (A), writing $(n-1)$ for n, the number of the permutations of the $(n-1)$ quantities $b, c, d\ldots\ldots\ldots k$, taken $(p-1)$ and $(p-1$, will be

$$(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots\{(n-1)-(p-1)+1\}$$

Or,

$$(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots(n-p+1)$$

Writing a before each of these $(n-1)(n-2)(n-3)\ldots\ldots\ldots(n-p+1)$ permutations, we shall have $(n-1)(n-2)(n-3)\ldots\ldots\ldots(n-p+1)$ permutations of the n quantities. in which a stands first. Reasoning in the same manner

for b, we shall have $(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots(n-p+1)$ permutations of the n quantities in which b stands first, and so on for each of the n quantities in succession, hence the whole number of permutations will be

$$n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots(n-p+1)\ldots\ldots\ldots\ldots\ldots(1)$$

Hence it appears, that, if the above law of formation hold good for any one class of permutations, it must hold good for the class next superior; but it has been proved to hold good when $p=2$, or for the permutations of n quantities taken two and two, hence it must hold good when $p=3$, or for the permutation of n quantities taken three and three, $\therefore$ it must hold good when $p=4$, and so on. The law is, therefore, *general.*

Example.

Required the number of the permutations of the eight letters, a, b, c, d, e, f, g, h, taken 5 and 5 together.

Here

$n=8$, $p=5$, $n-p+1=4$ hence the above formula.

$$n(n-1)(n-2)\ldots\ldots(n-p+1) = 8\times7\times6\times5\times4 = 6720$$

the number required.

187. In formula (1) let $p=n$, it will then become

$$n(n-1)(n-2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots2\,.\,1$$

Or,

$$1\,.\,2\,.\,3\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(n-1)\,n\ldots\ldots\ldots\ldots\ldots(2)$$

Which expresses the number of the permutations of n quantities taken all together.*

Example.

Required the number of the permutations of the eight letters, a, b, c, d, e, f, g, h.

Here $n=8$, hence the above formula (2) in this case becomes,

$$1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8 = 40320$$

the number required.

188. The number of the permutations of n quantities, supposing them all different from each other, we have found to be

$$1\,.\,2\,.\,3\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(n-1)\,n$$

* Many writers on Algebra confine the term *permutations* to this class where the quantities are taken *all together*, and give the title of *arrangements*, or *variations* to the groupes of the n quantities when taken *two and two*, *three and three*, *four and four*, &c. The introduction of these additional designations appears unnecessary, but in using the word *permutations* absolutely, we must always be understood to mean those represented by formula (2), unless the contrary be specified.

But if the same quantity be repeated a certain number of times, then it is manifest that a certain number of the above permutations will become identical.

Thus, if one of the quantities be repeated α times, the number of identical permutations will be represented by $1 . 2 . 3 \ldots\ldots\ldots\ldots \alpha$, and hence, in order to obtain the number of permutations different from each other, we must divide (2) by $1 . 2 . 3 \ldots\ldots\ldots\ldots \alpha$, and it will then become

$$\frac{1 . 2 . 3 \ldots\ldots\ldots\ldots (n-1) n}{1 . 2 . 3 \ldots\ldots\ldots\ldots \alpha}$$

If one of the quantities be repeated α times, and another of the quantities be repeated β times, then we must divide by $1 . 2 \ldots\ldots\ldots\ldots \alpha \times 1 . 2 \ldots\ldots\ldots\ldots \beta$; and, in general, if among the n quantities there be α of one kind, β of another kind, γ of another kind, and so on, the expression for the number of the permutations different from each other of these n quantities will be

$$\frac{1 . 2 . 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n}{1 . 2 \ldots\ldots \alpha \times 1 . 2 \ldots\ldots \beta \times 1 . 2 \ldots\ldots \gamma, \, \&c.} \ldots\ldots\ldots\ldots (3)$$

Example I.

Required the numbers of the permutations of the letters in the word *algebra*.

Here $n = 7$, and the letter a is repeated twice, hence formula (3) becomes

$$\frac{1 . 2 . 3 . 4 . 5 . 6 . 7}{1 . 2} = 2520 \text{ the number required.}$$

Example 2.

Required the number of the permutations of the letters in the word *caifacaratadaddara*.

Here $n = 18$, a is repeated eight times, c twice, d thrice, r twice, hence the number sought will be

$$\frac{1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 . 10 . 11 . 12 . 13 . 14 . 15 . 16 . 17 . 18}{1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 \times 1 . 2 \times 1 . 2 . 3 \times 1 . 2} = 6616209600$$

Example 3.

Required the number of the permutations of the product $a^x \, b^y \, c^z$, written at full length.

Here $n = x + y + z$, the letter a is repeated x times, the letter b, y times and the letter c, z times; the expression sought will therefore be

$$\frac{1 . 2 . 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (x + y + z)}{1 . 2 . 3 \ldots\ldots x \times 1 . 2 . 3 \ldots\ldots y \times 1 . 2 . 3 \ldots\ldots z}$$

189. The *Combinations* of any number of quantities, signify the different collections which may be formed of these quantities, without regard to the *order* in which they are arranged in each collection.

Thus the quantities a, b, c, when taken *all together*, will form only one combination, abc; but will form six different permutations, abc, acb, bac, bca, cab, cba; taken *two and two* they will form the three combinations ab, ac, bc, and the six permutations ab. ba, ac, ca, bc, cb.

The problem which we propose to resolve is,

190. *To find the number of the combinations of* n *quantities, taken* p *and* p *together.*

Let the number of combinations required be x:

Suppose these x combinations to be formed and to be written one after the other, in a horizontal line; write below the first of these all the permutations of the p letters which it contains, and since the number of these is $1.2.3......p$ ($= y$ suppose), we shall have a vertical column consisting of y terms; the second term of the horizontal line will, in like manner, give another vertical column consisting of y terms, being all the permutations of the p letters which it contains, one at least of which is different from those in the combinations already treated of. The third combination will, in like manner, give y terms differing from all the others. We shall thus form a table consisting of x columns, each of which contains y terms; and on the whole xy results, which are evidently all the permutations of the n letters, taken p and p together, none being either omitted or repeated; we shall therefore have by formula (1),

$$xy = n(n-1)(n-2)..................(n-p+1)$$

$$\therefore \quad x = \frac{n(n-1)(n-2)..................(n-p+1)}{y}$$

$$= \frac{n(n-1)(n-2)..................(n-p+1)}{1.2.3..........................(p-1)p} \quad ....(4)$$

the expression required.

Hence *we perceive*, that *the number of the combinations of* n *quantities, taken* p *and* p *together, is equal to the number of the permutations of* n *quantities, taken* p *and* p *together, divided by the number of the permutations of* p *quantities taken all together.*

There is a species of notation employed to denote permutations and combinations, which is sometimes used with advantage from its conciseness.

The number of the permutations of n quantities, taken p and p, are represented by .... $(n\,\mathrm{P}\,p)$

The number of the permutations of n quantities, taken *all together*, are represented by .... $(n\,\mathrm{P}\,n)$

The number of the combinations of n quantities, taken p and p, are represented by .... $(n\,\mathrm{C}\,p)$

and so on. It is manifest that the above proposition may be expressed according to this notation by

$$(n\,\mathrm{C}\,p) = \frac{(n\,\mathrm{P}\,p)}{(p\,\mathrm{P}\,p)}$$

METHOD OF UNDETERMINED COEFFICIENTS.

191. The method of undetermined coefficients is a method for the expansion or development of algebraic functions into infinite series, arranged according to the ascending powers of one of the quantities considered as a variable. The principle employed in this method may be stated in the following

Theorem.

If the series $A+Bx+Cx^2+Dx^3+$ &c., whether finite or infinite, be equal to the series $A^1+B^1x+C^1x^2+D^1x^3+$ &c., whatever be the value of x; then the coefficients of the like powers of x must be the same in each series; that is, $A=A^1$, $B=B^1$, $C=C^1$, $D=D^1$, &c.

For since

$$A+Bx+Cx^2+Dx^3+ \text{ \&c. } =A^1+B^1x+C^1x^2+D^1x^3+ \text{ \&c.}$$

by transposition we have

$$(A-A^1)+(B-B^1)x+(C-C^1)x^2+(D-D^1)x^3+ \ldots\ldots =0.$$

Now, if all or any of these coefficients were not $=0$, the equation would determine *particular* values of x, and could only be true for such particular values, which is contrary to the hypothesis. Hence we must have $A-A^1=0$, $B-B^1=0$, $C-C^1=0$, &c., and therefore

$$A=A^1,\ B=B^1,\ C=C^1, \text{ \&c.}$$

Examples.

(1.) Expand the fraction $\dfrac{1}{1-2x+x^2}$ into an infinite series.

Assume $\dfrac{1}{1-2x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+ \ldots\ldots$

then, multiplying by $1-2x+x^2$, we have

$$\begin{aligned} 1=A+ Bx&+ Cx^2+ Dx^3+ Ex^4+ \ldots \\ -2Ax&-2Bx^2-2Cx^3-2Dx^4- \ldots \\ &+ Ax^2+ Bx^3+ Cx^4+ \ldots \end{aligned}$$

hence, by the preceding theorem, we have

$$\begin{array}{rl} A=1 & \therefore A= \ldots =1 \\ B-2A=0 & B=2A \quad =2 \\ C-2B+A=0 & C=2B-A=3 \\ D-2C+B=0 & D=2C-B=4 \\ E-2D+C=0 & E=2D-C=5 \\ \text{\&c.} & \text{\&c.} \end{array}$$

Therefore $\dfrac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+5x^4+6x^5+ \ldots\ldots$

This example has been chosen to illustrate the method of expansion by undetermined coefficients; but the development can be obtained by division in the usual way, or by synthetic division, with more facility than by the principle here employed.

(2.) Extract the square root of $1+x$.

Assume $\sqrt{1+x}=A+Bx+Cx^2+Dx^3+\ldots$, and square both sides;

$$\begin{aligned}\therefore\ 1+x=A^2&+ABx+ACx^2+ADx^3+AEx^4+\ldots\\ &+ABx+B^2x^2+BCx^3+BDx^4+\ldots\\ &\qquad+ACx^2+BCx^3+C^2x^4+\ldots\\ &\qquad\qquad+ADx^3+BDx^4+\ldots\\ &\qquad\qquad\qquad+AEx^4+\ldots\end{aligned}$$

hence, equating the coefficients of the like powers of x, we have

$$\begin{array}{ll} A^2=1 & \therefore A=1 \\ 2AB=1 & B=\frac{1}{2A}=\frac{1}{1\cdot2}=\frac{1}{2} \\ 2AC+B^2=0 & C=-\frac{B^2}{2A}=-\frac{1}{2\cdot4}=-\frac{1}{8} \\ 2AD+2BC=0 & D=-\frac{BC}{A}=\frac{1}{2\cdot8}=\frac{1}{16} \\ 2AE+2BD+C^2=0 & E=-\frac{2BD+C^2}{2A}=-\frac{1}{2}\left\{\frac{1}{16}+\frac{1}{64}\right\}=-\frac{5}{128}. \\ \&c. & \&c. \end{array}$$

Therefore $\sqrt{1+x}=\pm(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\ldots)$

(3.) Decompose $\frac{3x-5}{x^2-13x+40}$ into two fractions having simple binomial denominators.

By quadratics we find $x^2-13x+40=(x-5)(x-8)$; hence we may assume

$$\frac{3x-5}{x^2-13x+40}=\frac{A}{x-5}+\frac{B}{x-8}=\frac{A(x-8)+B(x-5)}{(x-5)(x-8)}=\frac{(A+B)x-8A-5B}{x^2-13x+40}.$$

$$\therefore 3x-5=(A+B)x-(8A+5B);$$

and by the principle of undetermined coefficients we have

$$A+B=3,\text{ and } 8A+5B=5;$$

whence $A=-\frac{10}{3}$ and $B=\frac{19}{3}$, and therefore we get

$$\frac{3x-5}{x^2-13x+40}=\frac{6\frac{1}{3}}{x-8}-\frac{3\frac{1}{3}}{x-5}=\frac{19}{3}\cdot\frac{1}{x-8}-\frac{10}{3}\cdot\frac{1}{x-5}.$$

Note.—The values of A and B might have been determined in the following manner:—

Since $\frac{3x-5}{x^2-13x+40}=\frac{A}{x-5}+\frac{B}{x-8}=\frac{A(x-8)+B(x-5)}{x^2-13x+40}$.

$$\therefore 3x-5=A(x-8)+B(x-5).$$

Now this equation must subsist for every value of x; and, therefore,

if $x=5$, we have $15-5=A(5-8)$. $\therefore A=\frac{15-5}{5-8}=-\frac{10}{3}$;

if $x=8$, we have $24-5=B(8-5)$. $\therefore B=\frac{24-5}{8-5}=\frac{19}{3}$.

This method may frequently be employed with advantage, and will be found useful in the integration of rational fractions, when we come to treat of the Integral Calculus.

EXAMPLES FOR EXERCISE.

(1.) Expand $\frac{1-x}{1+x}$ into an infinite series.

Ans. $1-2x+2x^2-2x^3+2x^4-2x^5+\ldots\ldots$

(2.) Expand $\sqrt{a^2-x^2}$ in a series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-\ldots\ldots$

(3.) Find the development of $\frac{1-x}{1+x+x^2}$.

Ans. $1-2x+x^2+x^3-2x^4+x^5+x^6-2x^7+\ldots\ldots$

(4.) Decompose the fraction $\frac{2x+3}{x^3+x^2-2x}$.

Ans. $-\frac{3}{2x}-\frac{1}{6(x+2)}+\frac{5}{3(x-1)}$.

(5.) Expand the fraction $\frac{1+2x}{1-3x}$ in a series.

Ans. $1+5x+15x^2+45x^3+135x^4+\ldots\ldots$

(6.) Resolve $\frac{x^2}{(x+1)(x+2)(x+3)}$ into partial fractions.

Ans. $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}$

(7.) Resolve $\frac{13+21x+2x^2}{1-5x^2+4x^4}$ into partial fractions.

Ans. $\frac{1}{1+x}-\frac{6}{1-x}+\frac{2}{1+2x}+\frac{16}{1-2x}$.

(8.) Expand $\frac{1+2x}{1-x-x^2}$ to four terms. Ans. $1+3x+4x^2+7x^3+\ldots\ldots$

(9.) Expand $\frac{a-bx}{a+cx}$ to four terms.

Ans. $1-(b+c)\frac{x}{a}+c(b+c)\frac{x^2}{a^2}-c^2(b+c)\frac{x^3}{a^3}+\ldots\ldots$

(10.) Resolve $\frac{x+2}{x^3-x}$ into partial fractions.

Ans. $\frac{1}{2(x+1)}+\frac{3}{2(x-1)}-\frac{2}{x}$.

(11.) Resolve $\frac{6x^2-22x+18}{x^3-6x^2+11x-6}$ into partial fractions.

Ans. $\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}$.

(12.) Expand $\frac{x^2}{x^2+2ax+a^2}$ to four terms.

Ans. $1-\frac{2a}{x}+\frac{3a^2}{x^2}-\frac{4a^3}{x^3}+\ldots\ldots$

(13.) Resolve $\frac{1}{x^4-1}$ into partial fractions.

Ans. $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}$.

PILING OF BALLS AND SHELLS.

(192.) Balls and shells are usually piled in three different forms, called triangular, square, or rectangular, according as the figure on which the pile rests is triangular, square, or rectangular.

(1.) A triangular pile is formed by continued horizontal courses of balls or shells laid one above another, and these courses or rows are usually equilateral triangles whose sides decrease by unity from the bottom to the top row, which is composed simply of *one* shot; and hence the series of balls composing a triangular pile is

$$1+3+6+10+15+\ldots\ldots\frac{n(n+1)}{2},$$

where n denotes the number of courses in the pile.

(2.) A square pile is formed by continued horizontal courses of shot laid one above another, and these courses are squares whose sides decrease by unity from the bottom to the top row, which is also composed simply of *one* shot; and hence the series of balls composing a square pile is

$$1+4+9+16+25+\ldots\ldots n^2,$$

where n denotes the number of courses in the pile.

(3.) The rectangular pile may be conceived to be formed from a square pile, by laying successively on one face of the pyramid a series of triangular strata, each consisting of as many balls as the face itself contains, and the number of these added triangular strata is always one less than the number of shot in the top row; therefore, if n denote the number of courses, and $m+1$ the number of shot in the top row, the series composing a rectangular pile is

$$\begin{aligned}&(m+1)+2(m+2)+3(m+3)+4(m+4)+\ldots\ldots n(m+n)\\&=m+2m+3m+4m+\ldots\ldots nm+1^2+2^2+3^2+4^2+\ldots\ldots n^2\\&=m(1+2+3+4+\ldots\ldots n)+\text{square pile}\\&=\frac{n(n+1)}{2}.m+\text{square pile.}\end{aligned}$$

(4.) The number of balls in a complete triangular or square pile must evidently depend on the number of courses or rows; and the number of balls in a complete rectangular pile depends on the number of courses, and also on the number of shot in the top row, or the amount of shot in the latter pile depends on the length and breadth of the bottom row; for the number of courses is equal to the number of shot in the breadth of the bottom row of the pile. Therefore, the number of shot in a triangular or square pile is a function of n, and the number of shot in a rectangular pile is a function of n and m.

(5.) If the general term of any series of numbers be of the mth degree, the sum of all the terms of such series will be of the $(m+1)$th degree; because

the general term of any progressively increasing series being a function of n of the mth degree, the sum of such series evidently cannot exceed n times the general term, that is, it cannot exceed n times a function of n of the mth degree, and therefore the function itself must of the $(m+1)$th degree.

EXAMPLES.

(1.) Sum $1+2+3+4+5+\ldots\ldots$ to n terms.

Here n, the general term of the series, is of the first degree, and therefore the function expressing the sum of the series is of the second degree; and hence we assume

$$1+2+3+4+\ldots\ldots n=Pn^2+Qn+R.$$

Now this equation must be true for every value of n; hence, when

$$\begin{array}{lll} n=1 \text{ we have } & P+Q+R=1 & =1 \ldots\ldots (1) \\ n=2 \ldots\ldots & 4P+2Q+R=1+2 & =3 \ldots\ldots (2) \\ n=3 \ldots\ldots & 9P+3Q+R=1+2+3=6 & \ldots\ldots (3) \end{array}$$

$$\begin{array}{llll} \text{Hence } (2)-(1) \text{ gives} & 3P+Q=2 & & \\ (3)-(2)\ldots\ldots & 5P+Q=3 & & \\ \therefore & 2P \quad =1 & & \therefore P=\tfrac{1}{2} \\ & \text{Also } Q=2-3P & =2-1\tfrac{1}{2} & \therefore Q=\tfrac{1}{2} \\ & R=1-P-Q=1-\tfrac{1}{2}-\tfrac{1}{2} & & \therefore R=0 \end{array}$$

$$\text{hence } 1+2+3+4+\ldots\ldots n=Pn^2+Qn+R$$
$$=\tfrac{1}{2}n^2+\tfrac{1}{2}n=\frac{n(n+1)}{2}.$$

Formula for a triangular pile.

(2.) Sum n terms of the series $1+3+6+10+15+\ldots\ldots\frac{n(n+1)}{2}$.

$$\text{Assume } 1+3+6+10+15+\ldots\frac{n(n+1)}{2}=Pn^3+Qn^2+Rn+S\ldots\ldots$$

and since there are four coefficients to be determined, we must have a corresponding number of independent equations; hence

$$\begin{array}{llll} \text{when } n=1 \text{ we have} & P+Q+R+S=1 & =1 & (1) \\ n=2\ldots\ldots & 8P+4Q+2R+S=1+3 & =4 & (2) \\ n=3\ldots\ldots & 27P+9Q+3R+S=1+3+6 & =10 & (3) \\ n=4\ldots\ldots & 64P+16Q+4R+S=1+3+6+10 & =20 & (4) \\ \text{Then } (2)-(1) \text{ gives} & 7P+3Q+R & =3\ldots\ldots\ldots\ldots & (5) \\ (3)-(2)\ldots\ldots & 19P+5Q+R & =6\ldots\ldots\ldots\ldots & (6) \\ (4)-(3)\ldots\ldots & 37P+7Q+R & =10\ldots\ldots\ldots\ldots & (7) \\ (6)-(5)\ldots\ldots & 12P+2Q & =3\ldots\ldots\ldots\ldots & (8) \\ (7)-(6)\ldots\ldots & 18P+2Q & =4\ldots\ldots\ldots\ldots & (9) \\ (9)-(8)\ldots\ldots & 6P & =1 & \therefore P=\tfrac{1}{6} \\ & 2Q=3-12P & =1 & \therefore Q=\tfrac{1}{2} \\ & R=3-7P-3Q & =\tfrac{1}{3} & \therefore R=\tfrac{1}{3} \\ & S=1-P-Q-R & =0 & \therefore S=0 \end{array}$$

$$\text{Hence } 1+3+6+10+15+\ldots\frac{n(n+1)}{2}=\frac{1}{6}n^3+\frac{1}{2}n^2+\frac{1}{3}n$$
$$=\frac{n}{6}(n^2+3n+2)$$
$$=\frac{n(n+1)}{2}\cdot\frac{(n+2)}{3}.$$

Formula for a square pile.

(3.) Sum n terms of the series $1+4+9+16+25+\ldots\ldots n^2$.

Assume $1+4+9+16+25+\ldots n^2=Pn^3+Qn^2+Rn+S$;

then, as before,
$$\begin{aligned}
P+\ Q+\ R+S&=1 &&=1\\
8P+\ 4Q+2R+S&=1+4 &&=5\\
27P+\ 9Q+3R+S&=1+4+9 &&=14\\
64P+16Q+4R+S&=1+4+9+16&&=30
\end{aligned}$$

and from these four equations we find, by continued subtraction,

$P=\frac{1}{3}$, $Q=\frac{1}{2}$, $R=\frac{1}{6}$ and $S=0$; therefore

$$\begin{aligned}
+25+\ldots n^2 &=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\\
&=\frac{n}{6}(n^2+3n+1)\\
&=\frac{n(n+1)}{2}\cdot\frac{(2n+1)}{3}.
\end{aligned}$$

Formula for a rectangular pile.

(4.) By Art. 192, we have the number of shot in a rectangular pile

$$\begin{aligned}
&=\frac{n(n+1)}{2}.m+\text{square pile}\\
&=\frac{n(n+1)}{2}m+\frac{n(n+1)}{2}.\frac{(2n+1)}{3}\\
&=\frac{n(n+1)}{2}\cdot\frac{2n+1+3m}{3}.
\end{aligned}$$

Otherwise thus:—

Let $(m+1)+2(m+2)+3(m+3)+\ldots\ldots n(m+n)=Pn^3+Qn^2+Rn+S$.

then
$$\begin{aligned}
P+\ Q+\ R+S&=(m+1) &&=m+1\\
8P+\ 4Q+2R+S&=(m+1)+2(m+2) &&=3m+5\\
27P+\ 9Q+3R+S&=(m+1)+2(m+2)+3(m+3) &&=6m+14\\
64P+16Q+4R+S&=(m+1)+2(m+2)+3(m+3)+4(m+4)&&=10m+30
\end{aligned}$$

and from these four equations we find, as before,

$P=\frac{1}{3}$, $Q=\frac{1}{2}(m+1)$, $R=\frac{1}{6}(3m+1)$, and $S=0$; hence

$$\begin{aligned}
(m+1)+2(m+2)+3(m+3)+\ldots n(m+n)&=\frac{1}{3}n^3+\frac{m+1}{2}n^2+\frac{3m+1}{6}n.\\
&=\frac{n}{6}\left\{2n^2+3(m+1)n+3m+1\right\}\\
&=\frac{n}{6}\left\{2n(n+1)+(n+1)+3m(n+1)\right\}\\
&=\frac{n(n+1)}{2}\cdot\frac{2n+1+3m}{3}.
\end{aligned}$$

Hence we have the subsequent expressions for (S) the number of balls or shells in these three piles, viz.:—

triangular, $S=\frac{n(n+1)}{6}.(n+2) \quad =\frac{1}{3}.\frac{n(n+1)}{2}.(n+1+1)$

square, $S=\frac{n(n+1)}{6}(2n+1) \quad =\frac{1}{3}.\frac{n(n+1)}{2}.(n+1+n)$

rectangular, $S=\frac{n(n+1)}{6}(2n+1+3m)=\frac{1}{3}.\frac{n(n+1)}{2}\left\{(n+m)+(m+1)+(n+m)\right\}$

Now $\frac{n(n+1)}{2}$ is the number of balls in the triangular face of each pile, and the other factor is the sum of the balls in one side of the base, and the two parallel rows at the top and opposite side of the base; hence we have also this general rule, which, as well as the formulæ, should be committed to memory.

Rule.—Add to the number of balls or shells in one side of the base, the numbers in its two parallels at bottom and top (whether row or ball), and the sum multiplied by one-third of the slant end or face gives the number of balls in the pile.

Examples.

(1.) How many balls are in a triangular pile of 15 courses? Ans. 680.

(2.) A complete square pile has 14 courses: how many balls are in the pile, and how many remain after the removal of 5 courses?

Ans. 1015 and 960.

(3.) In an incomplete rectangular pile, the length and breadth at bottom are respectively 46 and 20, and the length and breadth at top are 35 and 9: how many balls does it contain? Ans. 7190.

(4.) The number of balls in an incomplete square pile is equal to 6 times the number removed, and the number of courses left is equal to the number of courses taken away: how many balls were in the complete pile? Ans. 385.

(5.) Let h and k dénote the length and breadth at top of a rectangular truncated pile, and n the number of balls in each of the slanting edges; then, if B be the number of balls in the truncated pile, prove that

$$B=\frac{n}{6}\left\{2n^2+3n(h+k)+6hk-3(h+k+n)+1\right\}.$$

Summation of Series.

193. By a process similar to that we have employed in finding the number of shot in a pile, we may find the general term, as well as the sum of various other series; but we proceed to

The Differential Method.

Let $a, b, c, d, e, \ldots$ be a series of terms, in which each term is less than the succeeding one; and, taking the successive differences, we have

$$\begin{array}{llllll}
 & a & b & c & d & e,\ \&c. \\
(d_1) & b-a & c-b & d-c & e-d, & \&c. \\
 & (d_2) & c-2b+a & d-2c+b & e-2d+c, & \&c. \\
 & & (d_3) & d-3c+3b-a & e-3d+3c-b, & \&c. \\
 & & & (d_4) & e-4d+6c-4b+a, & \&c.
\end{array}$$

Putting $d_1, d_2, d_3, d_4, \ldots$ for the first terms of the first, second, third, fourth, ... differences, we have

$$\begin{array}{ll}
b-a = d_1 & \therefore\ b = a + d_1 \\
c-2b+a = d_2 & \therefore\ c = a + 2d_1 + d_2 \\
d-3c+3b-a = d_3 & \therefore\ d = a + 3d_1 + 3d_2 + d_3 \\
e-4d+6c-4b+a = d_4 & \therefore\ e = a + 4d_1 + 6d_2 + 4d_3 + d_4; \\
\&c. & \&c.
\end{array}$$

Hence the $(n+1)$th term of the proposed series is evidently

$$a+nd_1+\frac{n(n-1)}{1.2}d_2+\frac{n(n-1)(n-2)}{1.2.3}d_3+\ldots$$

and, therefore, the nth term is (by writing $n-1$ for n)

$$a+(n-1)d_1+\frac{(n-1)(n-2)}{1.2}d_2+\frac{(n-1)(n-2)(n-3)}{1.2.3}d_3+\ldots\ldots (1)$$

To find (s) *the sum of* n *terms of a series.*

Let	$a,$	$b,$	$c,$	$d,$	$e,$	&c.
and	$0,$	$a,$	$a+b,$	$a+b+c,$	$a+b+c+d,$	&c.

be two series, of which the $(n+1)$th term of the latter is obviously the sum of n terms of the former; but the first terms of the first, second, third, fourth, ... differences in the latter, are

$$a,\ b-a=d_1,\ c-2b+a=d_2,\ d-3c+3b-a,\ =d_3,\ \&c.;$$

hence the $(n+1)$th term of the latter series, or the sum of n terms of the former is, by eq. (1) above,

$$0+na+\frac{n(n-1)}{1.2}d_1+\frac{n(n-1)(n-2)}{1.2.3}d_2+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d_3+\ldots$$

$$\text{Or } s=na+\frac{n(n-1)}{1.2}d_1+\frac{n(n-1)(n-2)}{1.2.3}d_2+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d_3+\ldots$$

Examples.

(1.) To what is $1.2+2.3+3.4+4.5+\ldots n(n+1)$ equal?

$$\begin{array}{ccccc} 2, & 6, & 12, & 20, & 30, \\ & 4 & 6 & 8 & 10 \\ & & 2 & 2 & 2 \\ & & & 0 & 0 \end{array}$$

Hence, $a=2$, $d_1=4$, $d_2=2$, and $d_3=0$; therefore

$$S=na+\frac{n(n-1)}{2}d_1+\frac{n(n-1)(n-2)}{2.3}d_2;$$

$$=2n+2n(n-1)+\tfrac{1}{3}n(n-1)(n-2)$$

$$=\tfrac{1}{3}n(n+1)(n+2).$$

(2.) Find the sum of n terms of the series $1, 2^3, 3^3, 4^3, 5^3$, &c.

(3.) Find the sum of n terms of the series 1, 4, 10, 20, 35, &c.

(4.) To what is $1.2.3+2.3.4+3.4.5+\ldots\ldots n(n+1)(n+2)$ equal?

(5.) Sum n terms of the series 1, 3, 5, 7, 9, 11, &c. ...

(6.) Find the sum of 15 terms of the series 1, 4, 8, 13, 19, &c.

(7.) Sum 8 terms of the series $1, 2^4, 3^4, 4^4, 5^4, 6^4$, &c.

Answers.

(2.) $\dfrac{n^2(n+1)^2}{4}$

(3.) $\dfrac{n(n+1)(n+2)(n+3)}{1.2.3.4}$

(4.) $\tfrac{1}{4}n(n+1)(n+2)(n+3)$

(5.) n^2

(6.) $\tfrac{1}{6}n(n^2+6n-1) = 785$

(7.) $\dfrac{n^5}{5}+\dfrac{n^4}{2}+\dfrac{n^3}{3}-\dfrac{n}{30} = 8772.$

Summation of Infinite Series.

The summation of series of this kind is the finding of a finite expression equal to the proposed series, and in many cases this finite expression is found by subtraction.

Examples.

(1.) Required the sum of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots$ to infinity.

$$\text{Let } S = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\ldots \text{ ad infinitum.}$$

$$\therefore S-1 = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\ldots\ldots \quad \text{ditto.}$$

Hence, by subtracting the latter from the former, we have

$$1 = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\ldots\ldots \quad \text{ditto.}$$

(2.) Required the sum of the series $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\ldots$ to n terms.

$$\text{Let } S = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \ldots\ldots \frac{1}{n};$$

$$\therefore S-1-\frac{1}{2}+\frac{1}{n+1}+\frac{1}{n+2} = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\ldots\frac{1}{n+2};$$

$$\text{by sub. } 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2} = \frac{2}{1.3}+\frac{2}{2.4}+\frac{2}{3.5}+\frac{2}{4.6}+\ldots\ldots\frac{2}{n(n+2)}$$

$$\therefore \frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\ldots\frac{1}{n(n+2)} = \frac{1}{2}\left\{1+\frac{1}{2}-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right\}$$

$$= \frac{1}{2}\left\{1-\frac{1}{n+1}+\frac{1}{2}-\frac{1}{n+2}\right\}$$

$$= \frac{n}{2n+2}+\frac{n}{4n+8}.$$

When n is infinitely great, then we have

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\ldots \text{ ad inf.} = \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{\infty}-\frac{1}{\infty}\right) = \frac{3}{4}$$

(3.) Sum the series $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\frac{1}{4.6}+\ldots\ldots$ ad infinitum.

Ans. $\frac{1}{4}$.

(4.) Sum the series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\ldots\ldots$ ad infinitum.

Ans. $\frac{1}{18}$.

(5.) Sum the series $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\ldots\ldots$ to n terms.

Ans. $\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}$.

(6.) Sum the series $a+2ar+3ar^2+4ar^3+\ldots$ to n terms.

Ans. $a\left\{\frac{1-r^n}{(1-r)^2}-\frac{n\,r^n}{1-r}\right\}$.

(7.) Sum the series $1+3x+5x^2+7x^3+9x^4\ldots$ ad infinitum.

Ans. $\frac{1+x}{(1-x)^2}$

Investigation of the Binomial Theorem.

194. *Let it be required to expand* $(a+x)^n$ *in a series, whether* n *be integral or fractional, positive or negative.*

Since $(a+x)^n=\left\{a\left(1+\frac{x}{a}\right)\right\}^n=a^n\left(1+\frac{x}{a}\right)^n$, whatever n may be,

if $(1+y)^n=1+Ay+By^2+Cy^3+Dy^4+\ldots\ldots\ldots$

then $(a+x)^n=a^n\left(1+\frac{x}{a}\right)^n$

$$=a^n\left(1+A\frac{x}{a}+B\frac{x^2}{a^2}+C\frac{x^3}{a^3}+\ldots\ldots\right)$$

$$=a^n+Aa^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+\ldots\ldots$$

Case 1. Let n be integral and positive.

Then $(1+y)^n=(1+y).(1+y).(1+y)\ldots\ldots\ldots$ to n factors; and by effecting the multiplication of a few of these equal factors, we shall find that the first two terms of the series are $1+ny$, and that the remaining terms are of the form $By^2+Cy^3+Dy^4+\ldots\ldots$ where B, C, D, ... are undetermined coefficients, entirely independent of y; and therefore we have $A=n$.

Case 2. Let n be integral and negative.

Suppose $n=-m$; then

$$(1+y)^n=(1+y)^{-m}=\frac{1}{(1+y)^m}=\frac{1}{1+my+,\&c.}=1-my+,\ \&c.$$

$$\therefore\ A=-m=n.$$

Case 3. Let n be fractional and positive.

Suppose $n=\frac{p}{q}$; then

$$(1+y)^n=(1+y)^{\frac{p}{q}}=1+Ay+By^2+Cy^3+\ldots$$

$$\therefore\ (1+y)^p=(1+Ay+By^2+Cy^3+\ldots.)^q$$

$$\therefore\ 1+py+,\&c.=\{1+(A+By+Cy^2+\ldots.)y\}^q$$

$$=1+q(A+By+Cy^2+\ldots.)y+,\ \&c.$$

$$=1+qAy+qBy^2+qCy^3+\ldots$$

Hence, equating the coefficients of like powers of y, we have

$$qA=p \text{ or } A=\frac{p}{q}=n.$$

Case 4. If n be fractional and negative.

Let $n=-\frac{p}{q}$; then

$$(1+y)^n=(1+y)^{-\frac{p}{q}}=\frac{1}{(1+y)^{\frac{p}{q}}}=\frac{1}{1+\frac{p}{q}y+\ldots}=1-\frac{p}{q}y+\ldots$$

Hence $A=-\frac{p}{q}=n$, and therefore in all cases $A=n$; and, consequently,

$$(a+x)^n=a^n+na^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+\ldots\ldots$$

Now in order to determine the values of the coefficients B, C, D, &c., we have

$$(a+x+z)^n=\{(a+x)+z\}^n=\{a+(x+z)\}^n,$$

and if we expand according to each of these forms, the two expansions must be identical; hence, by the first form we have

$$\begin{aligned}(a+x+z)^n&=\{(a+x)+z\}^n\\ &=(a+x)^n+n(a+x)^{n-1}z+B(a+x)^{n-2}z^2+C(a+x)^{n-3}z^3+\ldots\\ &=a^n+na^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+Da^{n-4}x^4+\ldots\\ &\quad+n\{a^{n-1}+(n-1)a^{n-2}x+Ba^{n-3}x^2+\ldots\}z\\ &\quad+B\{a^{n-2}+(n-2)a^{n-3}x+\ldots\}z^2\\ &\quad+C\{a^{n-3}+(n-3)a^{n-4}x+\ldots\}z^3\\ &\quad+\text{\&c.}\\ &=a^n+na^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+Da^{n-4}x^4+\ldots\\ &\quad+na^{n-1}z+n(n-1)a^{n-2}xz+Ba^{n-3}x^2z+\ldots\\ &\quad+Ba^{n-2}z^2+B(n-2)a^{n-3}xz^2+\ldots\\ &\quad+Ca^{n-3}z^3+\text{\&c.}\end{aligned}$$

Again: $(a+x+z)^n=\{a+(x+z)\}^n$

$$\begin{aligned}&=a^n+na^{n-1}(x+z)+Ba^{n-2}(x+z)^2+Ca^{n-3}(x+z)^3+\ldots\\ &=a^n+na^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+\ldots\\ &\quad+na^{n-1}z+2Ba^{n-2}xz+3Ca^{n-3}x^2z+\ldots\\ &\quad+Ba^{n-2}z^2+3Ca^{n-3}xz^2+\ldots\\ &\quad+Ca^{n-3}z^3+\ldots\end{aligned}$$

and the coefficients of the same powers of x and z, in these two expansions, must be the same (Art. 191); hence we have

$$\begin{aligned}2\,B&=n(n-1)\ \therefore\ B=\ldots\ldots\ldots\ldots\frac{n(n-1)}{1\cdot 2}\\ 3\,C&=(n-2)B\ \therefore\ C=\frac{(n-2)B}{3}=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\\ 4\,D&=(n-3)C\ \therefore\ D=\frac{(n-3)C}{4}=\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\\ &\text{\&c}\qquad\qquad \text{\&c.}\qquad\qquad \text{\&c.}\end{aligned}$$

Hence we have, generally,

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3+\ldots\ldots\\ +\frac{n(n-1)(n-2)\ldots\ldots(n-p+1)}{1\cdot 2\cdot 3\ldots\ldots\ldots p}a^{n-p}x^p+\text{\&c.}$$

which is the *Binomial Theorem,* and where the last term represents the $(p+1)$th term of the expansion.

Hence $(a-x)^n=a^n-na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3+$ &c.

$$(a+x)^{-n}=a^{-n}+na^{-(n+1)}x+\frac{n(n+1)}{1\cdot 2}a^{-(n+2)}x^2-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}a^{-(n+3)}x^3+\text{\&c.}$$

$$(a-x)^{-n}=a^{-n}+na^{-(n+1)}x+\frac{n(n+1)}{1\cdot 2}a^{-(n+2)}x^2+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}a^{-(n+3)}x^3+\text{\&c.}$$

and in all these formulæ n may be either integral or fractional.

THE EXPONENTIAL THEOREM.

195. *It is required to expand a^x in a series ascending by the powers of x.*

Since $a=1+a-1$; therefore $a^x=\{1+(a-1)\}^x$, and by the Binomial Theorem we have

$$\{1+(a-1)\}^x=1+x(a-1)+\frac{x(x-1)}{1\cdot 2}(a-1)^2+\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}(a-1)^3+\ldots.$$

$$=1+\{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ldots\}x+Bx^2+Cx^3..$$

where B, C, ... denote the coefficients of x^2, x^3, ...; and if we put

$$A=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3+\tfrac{1}{4}(a-1)^4+\ldots..$$

Then $a^x=1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots..$

For x write $x+h$; then we have

$$\begin{aligned}a^{x+h}&=1+A(x+h)+B(x+h)^2+C(x+h)^3+\ldots\\&=1+Ax+Bx^2+Cx^3+Dx^4+\ldots\\&\quad+Ah+2Bxh+3Cx^2h+4Dx^3h+\ldots\\&\quad+Bh^2+3Cxh^2+6Dx^2h^2+\ldots\\&\quad+Ch^3+4Dxh^3+\ldots\\&\quad+Dh^4+\ldots\end{aligned}$$

But $a^{x+h}=a^x\times a^h=(1+Ax+Bx^2+Cx^3+\ldots)(1+Ah+Bh^2+Ch^3+\ldots)$

$$\begin{aligned}&=1+Ax+Bx^2+Cx^3+Dx^4+\ldots\\&\quad+Ah+A^2xh+ABx^2h+ACx^3h+\ldots\\&\quad+Bh^2+ABxh^2+B^2x^2h^2+\ldots\\&\quad+Ch^3+ACxh^3+\ldots\\&\quad+Dh^4+\ldots\end{aligned}$$

Now these two expansions must be identical; and we must, therefore, have the coefficients of like powers of x and h equal; hence

$$2B=A^2 \qquad \therefore\ B=\frac{A^2}{2}$$

$$3C=AB \qquad C=\frac{A\cdot B}{3}=\frac{A^3}{2\cdot 3}$$

$$4D=AC \qquad D=\frac{AC}{4}=\frac{A^4}{2\cdot 3\cdot 4}$$

&c. &c. &c. &c.

$$\text{hence } a^x=1+Ax+\frac{A^2x^2}{1\cdot 2}+\frac{A^3x^3}{1\cdot 2\cdot 3}+\frac{A^4x^4}{1\cdot 2\cdot 3\cdot 4}+\ldots..$$

which is the *Exponential Theorem;* where

$$A=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ldots....$$

Let ϵ be the value of a, which renders $A=1$, then

$$(\epsilon-1)-\tfrac{1}{2}(\epsilon-1)^2+\tfrac{1}{3}(\epsilon-1)^3-\tfrac{1}{4}(\epsilon-1)^4+\ldots=1$$

$$\epsilon^x=1+x+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}+\ldots$$

Now, since this equation is true for every value of x, let $x=1$; then

$$\epsilon=1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\ldots...$$

$$=1+1+\tfrac{1}{2}(1)+\tfrac{1}{3}\left(\frac{1}{1\cdot 2}\right)+\tfrac{1}{4}\left(\frac{1}{1\cdot 2\cdot 3}\right)+\ldots$$

$$=2\cdot 718281828459\ldots...$$

LOGARITHMS.

196. LOGARITHMS are artificial numbers, adapted to natural numbers, in order to facilitate numerical calculations; and we shall now proceed to explain the theory of these numbers, and illustrate the principles upon which their properties depend.

DEFINITION. *In a system of logarithms, all numbers are considered as the powers of some one number, arbitrarily assumed, which is called the* BASE *of the system, and the exponent of that power of the base which is equal to any given number is called the* LOGARITHM *of that number.*

Thus, if a be the base of a system of logarithms, N any number, and x such that

$$N = a^x$$

then x is called the logarithm of N in the system whose base is a.

The base of the common system of logarithms, (called from their inventor "Briggs's Logarithms"), is the number 10. Hence since

$(10)^0 =$	1	, 0	is the logarithm of	1	in this system
$(10)^1 =$	10	, 1	————	10	————
$(10)^2 =$	100	, 2	————	100	————
$(10)^3 =$	1000	, 3	————	1000	————
$(10)^4 =$	10000	, 4	————	10000	————
&c. =	&c.	&c....................... .			

From this it appears, that in the common system the logarithms of every number between 1 and 10 is some number between 0 and 1, *i. e.* is a fraction. The logarithm of every number between 10 and 100 is some number between 1 and 2, *i. e.* is 1 plus a fraction. The logarithm of every number between 100 and 1000 is some number between 2 and 3, *i. e.* is 2 plus a fraction, and so on.

197. In the common tables the fractional part alone of the logarithm is registered and from what has been said above, the rule usually given for finding the *characteristic*, or, *index*, *i. e.* the integral part of the logarithm will be readily understood, viz. *The index of the logarithm of any number greater than unity is equal to one less than the number of integral figures in the given number.* Thus, in searching for the logarithm of such a number as 2970, we find in the tables opposite to 2970 the number 4727564; but since 2970 is a number between 1000 and 10000, its logarithm must be some number between 3 and 4, *i. e.* must be 3 plus a fraction; the fractional part is the number 4727564, which we have found in the tables, affixing to this the index 3, and interposing a decimal point, we have 3.4727564, the logarithm of 2970.

PILING OF BALLS AND SHELLS.

(192.) Balls and shells are usually piled in three different forms, called triangular, square, or rectangular, according as the figure on which the pile rests is triangular, square, or rectangular.

(1.) A triangular pile is formed by continued horizontal courses of balls or shells laid one above another, and these courses or rows are usually equilateral triangles whose sides decrease by unity from the bottom to the top row, which is composed simply of *one* shot; and hence the series of balls composing a triangular pile is

$$1+3+6+10+15+\ldots\ldots\frac{n(n+1)}{2},$$

where n denotes the number of courses in the pile.

(2.) A square pile is formed by continued horizontal courses of shot laid one above another, and these courses are squares whose sides decrease by unity from the bottom to the top row, which is also composed simply of *one* shot; and hence the series of balls composing a square pile is

$$1+4+9+16+25+\ldots\ldots n^2,$$

where n denotes the number of courses in the pile.

(3.) The rectangular pile may be conceived to be formed from a square pile, by laying successively on one face of the pyramid a series of triangular strata, each consisting of as many balls as the face itself contains, and the number of these added triangular strata is always one less than the number of shot in the top row; therefore, if n denote the number of courses, and $m+1$ the number of shot in the top row, the series composing a rectangular pile is

$$(m+1)+2(m+2)+3(m+3)+4(m+4)+\ldots\ldots n(m+n)$$
$$=m+2m+3m+4m+\ldots\ldots nm+1^2+2^2+3^2+4^2+\ldots\ldots n^2$$
$$=m(1+2+3+4+\ldots\ldots n)+\text{ square pile}$$
$$=\frac{n(n+1)}{2}.m+\text{ square pile.}$$

(4.) The number of balls in a complete triangular or square pile must evidently depend on the number of courses or rows; and the number of balls in a complete rectangular pile depends on the number of courses, and also on the number of shot in the top row, or the amount of shot in the latter pile depends on the length and breadth of the bottom row; for the number of courses is equal to the number of shot in the breadth of the bottom row of the pile. Therefore, the number of shot in a triangular or square pile is a function of n, and the number of shot in a rectangular pile is a function of n and m.

(5.) If the general term of any series of numbers be of the mth degree, the sum of all the terms of such series will be of the $(m+1)$th degree; because

the general term of any progressively increasing series being a function of n of the mth degree, the sum of such series evidently cannot exceed n times the general term, that is, it cannot exceed n times a function of n of the mth degree, and therefore the function itself must of the $(m+1)$th degree.

Examples.

(1.) Sum $1+2+3+4+5+\ldots$ to n terms.

Here n, the general term of the series, is of the first degree, and therefore the function expressing the sum of the series is of the second degree; and hence we assume

$$1+2+3+4+\ldots n=Pn^2+Qn+R.$$

Now this equation must be true for every value of n; hence, when

$$\begin{array}{lll}
n=1 \text{ we have } & P+Q+R=1 & =1 \ldots (1)\\
n=2 \ldots\ldots & 4P+2Q+R=1+2 & =3 \ldots (2)\\
n=3 \ldots\ldots & 9P+3Q+R=1+2+3=6 & \ldots (3)
\end{array}$$

$$\begin{array}{llll}
\text{Hence } (2)-(1) \text{ gives} & 3P+Q=2 & & \\
(3)-(2) \ldots & 5P+Q=3 & & \\
\therefore & 2P \quad =1 & . & P=\tfrac{1}{2}\\
& \text{Also } Q=2-3P \quad =2-1\tfrac{1}{2} & . & Q=\tfrac{1}{2}\\
& R=1-P-Q=1-\tfrac{1}{2}-\tfrac{1}{2} & . & R=0
\end{array}$$

$$\text{hence } 1+2+3+4+\ldots n=Pn^2+Qn+R$$
$$=\tfrac{1}{2}n^2+\tfrac{1}{2}n=\frac{n(n+1)}{2}.$$

Formula for a triangular pile.

(2.) Sum n terms of the series $1+3+6+10+15+\ldots \dfrac{n(n+1)}{2}$.

$$\text{Assume } 1+3+6+10+15+\ldots \frac{n(n+1)}{2}=Pn^3+Qn^2+Rn+S \ldots$$

and since there are four coefficients to be determined, we must have a corresponding number of independent equations; hence

$$\begin{array}{llll}
\text{when } n=1 \text{ we have} & P+\ Q+\ R+S = 1 & = 1 & (1)\\
n=2 \ldots\ldots & 8P+\ 4Q+2R+S = 1+3 & = 4 & (2)\\
n=3 \ldots\ldots & 27P+\ 9Q+3R+S = 1+3+6 & =10 & (3)\\
n=4 \ldots\ldots & 64P+16Q+4R+S = 1+3+6+10 & =20 & (4)\\
\text{Then } (2)-(1) \text{ gives} & 7P+\ 3Q+R & = 3 \ldots & (5)\\
(3)-(2) \ldots\ldots & 19P+\ 5Q+R & = 6 \ldots & (6)\\
(4)-(3) \ldots\ldots & 37P+\ 7Q+R & =10 \ldots & (7)\\
(6)-(5) \ldots\ldots & 12P+\ 2Q & = 3 \ldots & (8)\\
(7)-(6) \ldots\ldots & 18P+\ 2Q & = 4 \ldots & (9)\\
(9)-(8) \ldots\ldots & 6P & = 1 & \therefore P=\tfrac{1}{6}\\
& 2Q=3-12P & = 1 & \therefore Q=\tfrac{1}{2}\\
& R=3-7P-3Q & = \tfrac{1}{3} & \therefore R=\tfrac{1}{3}\\
& S=1-P-Q-R & = 0 & \therefore S=0
\end{array}$$

$$\text{Hence } 1+3+6+10+15+\ldots \frac{n(n+1)}{2}=\frac{1}{6}n^3+\frac{1}{2}n^2+\frac{1}{3}n$$
$$=\frac{n}{6}(n^2+3n+2)$$
$$=\frac{n(n+1)}{2}\cdot\frac{(n+2)}{3}.$$

Formula for a square pile.

(3.) Sum n terms of the series $1+4+9+16+25+\ldots\ldots n^2$.

Assume $1+4+9+16+25+\ldots n^2=\mathrm{P}n^3+\mathrm{Q}n^2+\mathrm{R}n+\mathrm{S}$;

then, as before,

$$\begin{aligned}
\mathrm{P}+\mathrm{Q}+\mathrm{R}+\mathrm{S}&=1 &&=1\\
8\mathrm{P}+4\mathrm{Q}+2\mathrm{R}+\mathrm{S}&=1+4 &&=5\\
27\mathrm{P}+9\mathrm{Q}+3\mathrm{R}+\mathrm{S}&=1+4+9 &&=14\\
64\mathrm{P}+16\mathrm{Q}+4\mathrm{R}+\mathrm{S}&=1+4+9+16&&=30
\end{aligned}$$

and from these four equations we find, by continued subtraction,

$$\mathrm{P}=\frac{1}{3},\ \mathrm{Q}=\frac{1}{2},\ \mathrm{R}=\frac{1}{6}\text{ and }\mathrm{S}=0\text{; therefore}$$

$$\begin{aligned}
+25+\ldots n^2 &=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\\
&=\frac{n}{6}(n^2+3n+1)\\
&=\frac{n(n+1)}{2}\cdot\frac{(2n+1)}{3}.
\end{aligned}$$

Formula for a rectangular pile.

(4.) By Art. 192, we have the number of shot in a rectangular pile

$$\begin{aligned}
&=\frac{n(n+1)}{2}.m+\text{square pile}\\
&=\frac{n(n+1)}{2}m+\frac{n(n+1)}{2}.\frac{(2n+1)}{3}\\
&=\frac{n(n+1)}{2}\cdot\frac{2n+1+3m}{3}.
\end{aligned}$$

Otherwise thus:—

Let $(m+1)+2(m+2)+3(m+3)+\ldots\ldots n(m+n)=\mathrm{P}n^3+\mathrm{Q}n^2+\mathrm{R}n+\mathrm{S}$.

then

$$\begin{aligned}
\mathrm{P}+\mathrm{Q}+\mathrm{R}+\mathrm{S}&=(m+1) &&=m+1\\
8\mathrm{P}+4\mathrm{Q}+2\mathrm{R}+\mathrm{S}&=(m+1)+2(m+2) &&=3m+5\\
27\mathrm{P}+9\mathrm{Q}+3\mathrm{R}+\mathrm{S}&=(m+1)+2(m+2)+3(m+3) &&=6m+14\\
64\mathrm{P}+16\mathrm{Q}+4\mathrm{R}+\mathrm{S}&=(m+1)+2(m+2)+3(m+3)+4(m+4)&&=10m+30
\end{aligned}$$

and from these four equations we find, as before,

$$\mathrm{P}=\frac{1}{3},\ \mathrm{Q}=\frac{1}{2}(m+1),\ \mathrm{R}=\frac{1}{6}(3m+1),\text{ and }\mathrm{S}=0\text{; hence}$$

$$\begin{aligned}
(m+1)+2(m+2)+3(m+3)+\ldots n(m+n)&=\frac{1}{3}n^3+\frac{m+1}{2}n^2+\frac{3m+1}{6}n.\\
&=\frac{n}{6}\left\{2n^2+3(m+1)n+3m+1\right\}\\
&=\frac{n}{6}\left\{2n(n+1)+(n+1)+3m(n+1)\right\}\\
&=\frac{n(n+1)}{2}\cdot\frac{2n+1+3m}{3}.
\end{aligned}$$

Hence we have the subsequent expressions for (S) the number of balls or shells in these three piles, viz.:—

triangular, $\mathrm{S}=\dfrac{n(n+1)}{6}.(n+2) \quad =\dfrac{1}{3}.\dfrac{n(n+1)}{2}.(n+1+1)$

square, $\mathrm{S}=\dfrac{n(n+1)}{6}(2n+1) \quad =\dfrac{1}{3}.\dfrac{n(n+1)}{2}.(n+1+n)$

rectangular, $\mathrm{S}=\dfrac{n(n+1)}{6}(2n+1+3m)=\dfrac{1}{3}.\dfrac{n(n+1)}{2}\left\{(n+m)+(m+1)+(n+m)\right\}$

Now $\frac{n(n+1)}{2}$ is the number of balls in the triangular face of each pile, and the other factor is the sum of the balls in one side of the base, and the two parallel rows at the top and opposite side of the base; hence we have also this general rule, which, as well as the formulæ, should be committed to memory.

Rule.—Add to the number of balls or shells in one side of the base, the numbers in its two parallels at bottom and top (whether row or ball), and the sum multiplied by one-third of the slant end or face gives the number of balls in the pile.

Examples.

(1.) How many balls are in a triangular pile of 15 courses? Ans. 680.

(2.) A complete square pile has 14 courses: how many balls are in the pile, and how many remain after the removal of 5 courses?

Ans. 1015 and 960.

(3.) In an incomplete rectangular pile, the length and breadth at bottom are respectively 46 and 20, and the length and breadth at top are 35 and 9: how many balls does it contain? Ans. 7190.

(4.) The number of balls in an incomplete square pile is equal to 6 times the number removed, and the number of courses left is equal to the number of courses taken away: how many balls were in the complete pile? Ans. 385.

(5.) Let h and k denote the length and breadth at top of a rectangular truncated pile, and n the number of balls in each of the slanting edges; then, if B be the number of balls in the truncated pile, prove that

$$\mathrm{B}=\frac{n}{6}\left\{2n^2+3n(h+k)+6hk-3(h+k+n)+1\right\}.$$

Summation of Series.

193. By a process similar to that we have employed in finding the number of shot in a pile, we may find the general term, as well as the sum of various other series; but we proceed to

The Differential Method.

Let $a, b, c, d, e, \ldots.$ be a series of terms, in which each term is less than the succeeding one; and, taking the successive differences, we have

$$\begin{array}{llllll}
a & b & c & d & e, & \&c. \\
(d_1) & b-a & c-b & d-c & e-d, & \&c. \\
 & (d_2) & c-2b+a & d-2c+b & e-2d+c, & \&c. \\
 & & (d_3) & d-3c+3b-a & e-3d+3c-b, & \&c. \\
 & & & (d_4) & e-4d+6c-4b+a, & \&c.
\end{array}$$

Putting $d_1, d_2, d_3, d_4, \ldots.$ for the first terms of the first, second, third, fourth, ... differences, we have

$$\begin{array}{lll}
b-a & =d_1 \therefore & b=a+d_1 \\
c-2b+a & =d_2 \therefore & c=a+2d_1+d_2 \\
d-3c+3b-a & =d_3 \therefore & d=a+3d_1+3d_2+d_3 \\
e-4d+6c-4b+a & =d_4 \therefore & e=a+4d_1+6d_2+4d_3+d_4; \\
\&c. & & \&c.
\end{array}$$

Hence the $(n+1)$th term of the proposed series is evidently

$$a+nd_1+\frac{n(n-1)}{1\,.\,2}d_2+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}d_3+\,.$$

and, therefore, the nth term is (by writing $n-1$ for n)

$$a+(n-1)d_1+\frac{(n-1)(n-2)}{1\,.\,2}d_2+\frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3}d_3+\,\ldots\ldots\,(1)$$

To find (s) *the sum of* n *terms of a series.*

Let a, b, c, d, e, &c.
and 0, a, $a+b$, $a+b+c$, $a+b+c+d$, &c.
be two series, of which the $(n+1)$th term of the latter is obviously the sum of n terms of the former; but the first terms of the first, second, third, fourth, ... differences in the latter, are

$$a,\ b-a=d_1,\ c-2b+a=d_2,\ d-3c+3b-a,\ =d_3,\ \&c.;$$

hence the $(n+1)$th term of the latter series, or the sum of n terms of the former is, by eq. (1) above,

$$0+na+\frac{n(n-1)}{1\,.\,2}d_1+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}d_2+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}d_3+\ldots$$

$$\text{Or } s=na+\frac{n(n-1)}{1\,.\,2}d_1+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}d_2+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}d_3+\ldots$$

EXAMPLES.

(1.) To what is $1.2+2.3+3.4+4.5+\ldots n(n+1)$ equal?

$$\begin{array}{ccccc} 2, & 6, & 12, & 20, & 30, \\ & 4 & 6 & 8 & 10 \\ & & 2 & 2 & 2 \\ & & & 0 & 0 \end{array}$$

Hence, $a=2$, $d_1=4$, $d_2=2$, and $d_3=0$; therefore

$$S=na+\frac{n(n-1)}{2}d_1+\frac{n(n-1)(n-2)}{2\,.\,3}d_2;$$
$$=2n+2n(n-1)+\tfrac{1}{3}n(n-1)(n-2)$$
$$=\tfrac{1}{3}n(n+1)(n+2).$$

(2.) Find the sum of n terms of the series 1, 2^3, 3^3, 4^3, 5^3, &c.
(3.) Find the sum of n terms of the series 1, 4, 10, 20, 35, &c.
(4.) To what is $1.2.3+2.3.4+3.4.5+\ldots\ldots n(n+1)(n+2)$ equal?
(5.) Sum n terms of the series 1, 3, 5, 7, 9, 11, &c. ...
(6.) Find the sum of 15 terms of the series 1, 4, 8, 13, 19, &c.
(7.) Sum 8 terms of the series 1, 2^4, 3^4, 4^4, 5^4, 6^4, &c.

ANSWERS.

(2.) $\dfrac{n^2(n+1)^2}{4}$ (5.) n^2

(3.) $\dfrac{n(n+1)(n+2)(n+3)}{1\,.\,2\,.\,3\,.\,4}$ (6.) $\tfrac{1}{6}n(n^2+6n-1) = 785$

(4.) $\tfrac{1}{4}n(n+1)(n+2)(n+3)$ (7.) $\dfrac{n^5}{5}+\dfrac{n^4}{2}+\dfrac{n^3}{3}-\dfrac{n}{30} = 8772.$

Summation of Infinite Series.

The summation of series of this kind is the finding of a finite expression equal to the proposed series, and in many cases this finite expression is found by subtraction.

Examples.

(1.) Required the sum of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots$ to infinity.

$$\text{Let } S = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\ldots \text{ ad infinitum.}$$

$$\therefore S-1 = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\ldots\ldots \quad \text{ditto.}$$

Hence, by subtracting the latter from the former, we have

$$1 = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\ldots\ldots \quad \text{ditto.}$$

(2.) Required the sum of the series $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\ldots$ to n terms.

$$\text{Let } S = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\frac{1}{n};$$

$$\therefore S-1-\frac{1}{2}+\frac{1}{n+1}+\frac{1}{n+2} = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\ldots\frac{1}{n+2};$$

$$\text{by sub. } 1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2} = \frac{2}{1.3}+\frac{2}{2.4}+\frac{2}{3.5}+\frac{2}{4.6}+\ldots\ldots\frac{2}{n(n+2)}$$

$$\therefore \frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\ldots\frac{1}{n(n+2)}=\frac{1}{2}\left\{1+\frac{1}{2}-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right\}$$

$$=\frac{1}{2}\left\{1-\frac{1}{n+1}+\frac{1}{2}-\frac{1}{n+2}\right\}$$

$$=\frac{n}{2n+2}+\frac{n}{4n+8}.$$

When n is infinitely great, then we have

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\ldots \text{ ad inf.} =\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{\infty}-\frac{1}{\infty}\right)=\frac{3}{4}.$$

(3.) Sum the series $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\frac{1}{4.6}+\ldots$ ad infinitum.

Ans. $\frac{1}{4}$.

(4.) Sum the series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\ldots$ ad infinitum.

Ans. $\frac{1}{18}$.

(5.) Sum the series $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\ldots$ to n terms.

Ans. $\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}$.

(6.) Sum the series $a+2ar+3ar^2+4ar^3+\ldots$ to n terms.

Ans. $a\left\{\frac{1-r^n}{(1-r)^2}-\frac{n\,r^n}{1-r}\right\}$.

(7.) Sum the series $1+3x+5x^2+7x^3+9x^4\ldots$ ad infinitum.

Ans. $\frac{1+x}{(1-x)^2}$

Investigation of the Binomial Theorem.

194. *Let it be required to expand* $(a+x)^n$ *in a series, whether* n *be integral or fractional, positive or negative.*

Since $(a+x)^n=\left\{a\left(1+\frac{x}{a}\right)\right\}^n=a^n\left(1+\frac{x}{a}\right)^n$, whatever n may be,

if $(1+y)^n=1+Ay+By^2+Cy^3+Dy^4+\ldots\ldots\ldots$

then $(a+x)^n=a^n\left(1+\frac{x}{a}\right)^n$

$$=a^n\left(1+A\frac{x}{a}+B\frac{x^2}{a^2}+C\frac{x^3}{a^3}+\ldots\ldots\right)$$

$$=a^n+Aa^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+\ldots\ldots$$

Case 1. Let n be integral and positive.

Then $(1+y)^n=(1+y).(1+y).(1+y)\ldots\ldots\ldots$ to n factors; and by effecting the multiplication of a few of these equal factors, we shall find that the first two terms of the series are $1+ny$, and that the remaining terms are of the form $By^2+Cy^3+Dy^4+\ldots$ where B, C, D, ... are undetermined coefficients, entirely independent of y; and therefore we have $A=n$.

Case 2. Let n be integral and negative.

Suppose $n=-m$; then

$$(1+y)^n=(1+y)^{-m}=\frac{1}{(1+y)^m}=\frac{1}{1+my+,\&c.}=1-my+,\ \&c.$$

$\therefore\ A=-m=n.$

Case 3. Let n be fractional and positive.

Suppose $n=\frac{p}{q}$; then

$$(1+y)^n=(1+y)^{\frac{p}{q}}=1+Ay+By^2+Cy^3+\ldots$$

$$\therefore\ (1+y)^p=(1+Ay+By^2+Cy^3+\ldots.)^q$$

$$\therefore\ 1+py+,\&c.=\{1+(A+By+Cy^2+\ldots.)y\}^q$$

$$=1+q(A+By+Cy^2+\ldots.)y+,\ \&c.$$

$$=1+qAy+qBy^2+qCy^3+\ \ldots$$

Hence, equating the coefficients of like powers of y, we have

$$qA=p \text{ or } A=\frac{p}{q}=n.$$

Case 4. If n be fractional and negative.

Let $n=-\frac{p}{q}$; then

$$(1+y)^n=(1+y)^{-\frac{p}{q}}=\frac{1}{(1+y)^{\frac{p}{q}}}=\frac{1}{1+\frac{p}{q}y+\ldots}=1-\frac{p}{q}y+\ldots$$

Hence $A=-\frac{p}{q}=n$, and therefore in all cases $A=n$; and, consequently,

$$(a+x)^n=a^n+na^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+\ldots\ldots$$

Now in order to determine the values of the coefficients B, C, D, &c., we have

$$(a+x+z)^n=\{(a+x)+z\}^n=\{a+(x+z)\}^n,$$

and if we expand according to each of these forms, the two expansions must be identical; hence, by the first form we have

$$\begin{aligned}(a+x+z)^n&=\{(a+x)+z\}^n\\
&=(a+x)^n+n(a+x)^{n-1}z+B(a+x)^{n-2}z^2+C(a+x)^{n-3}z^3+\dots\\
&=a^n+na^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+Da^{n-4}x^4+\dots\\
&\quad+n\{a^{n-1}+(n-1)a^{n-2}x+Ba^{n-3}x^2+\dots\}z\\
&\quad+B\{a^{n-2}+(n-2)a^{n-3}x+\dots\dots\}z^2\\
&\quad+C\{a^{n-3}+(n-3)a^{n-4}x+\dots\dots\}z^3\\
&\quad+\&c.\end{aligned}$$

$$\begin{aligned}&=a^n+na^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+Da^{n-4}x^4+\dots\\
&\quad+na^{n-1}z+n(n-1)a^{n-2}xz+Ba^{n-3}x^2z+\dots\dots\\
&\quad+Ba^{n-2}z^2+B(n-2)a^{n-3}xz^2+\dots\dots\\
&\quad+Ca^{n-3}z^3+\&c.\end{aligned}$$

Again: $(a+x+z)^n=\{a+(x+z)\}^n$

$$\begin{aligned}&=a^n+na^{n-1}(x+z)+Ba^{n-2}(x+z)^2+Ca^{n-3}(x+z)^3+\dots\\
&=a^n+na^{n-1}x+Ba^{n-2}x^2+Ca^{n-3}x^3+\dots\\
&\quad+na^{n-1}z+2Ba^{n-2}xz+3Ca^{n-3}x^2z+\dots\\
&\quad+Ba^{n-2}z^2+3Ca^{n-3}xz^2+\dots\\
&\quad+Ca^{n-3}z^3+\dots\dots\end{aligned}$$

and the coefficients of the same powers of x and z, in these two expansions, must be the same (Art. 191); hence we have

$$\begin{aligned}2\,B&=n(n-1)\ \therefore\ B=\dots\dots\dots\dots\frac{n(n-1)}{1\cdot 2}\\
3\,C&=(n-2)B\ \therefore\ C=\frac{(n-2)B}{3}=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\\
4\,D&=(n-3)C\ \therefore\ D=\frac{(n-3)C}{4}=\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\\
&\&c \qquad\qquad \&c. \qquad\qquad \&c.\end{aligned}$$

Hence we have, generally,

$$\begin{aligned}(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3+\dots\dots\\
+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\,\frac{\dots\dots(n-p+1)}{\dots\dots\dots p}a^{n-p}x^p+\&c.\end{aligned}$$

which is the *Binomial Theorem*, and where the last term represents the $(p+1)$th term of the expansion.

Hence $(a-x)^n=a^n-na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3+$ &c.

$$(a+x)^{-n}=a^{-n}+na^{-(n+1)}x+\frac{n(n+1)}{1\cdot 2}a^{-(n+2)}x^2-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}a^{-(n+3)}x^3+\&c.$$

$$(a-x)^{-n}=a^{-n}+na^{-(n+1)}x+\frac{n(n+1)}{1\cdot 2}a^{-(n+2)}x^2+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}a^{-(n+3)}x^3+\&c.$$

and in all these formulæ n may be either integral or fractional.

The Exponential Theorem.

195. ***It is required to expand a^x in a series ascending by the powers of x.***
Since $a=1+a-1$; therefore $a^x=\{1+(a-1)\}^x$, and by the Binomial Theorem we have

$$\{1+(a-1)\}^x=1+x(a-1)+\frac{x(x-1)}{1\cdot 2}(a-1)^2+\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}(a-1)^3+\ldots.$$
$$=1+\{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ldots.\}x+Bx^2+Cx^3\ldots$$

where B, C, ... denote the coefficients of x^2, x^3, ...; and if we put
$$A=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3+\tfrac{1}{4}(a-1)^4+\ldots..$$
Then $a^x=1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\ldots..$

For x write $x+h$; then we have

$$\begin{aligned}a^{x+h}&=1+A(x+h)+B(x+h)^2+C(x+h)^3+\ldots.\\&=1+Ax+Bx^2+Cx^3+Dx^4+\ldots.\\&\quad+Ah+2Bxh+3Cx^2h+4Dx^3h+\ldots.\\&\quad+Bh^2+3Cxh^2+6Dx^2h^2+\ldots.\\&\quad+Ch^3+4Dxh^3+\ldots.\\&\quad+Dh^4+\ldots.\end{aligned}$$

But $a^{x+h}=a^x\times a^h=(1+Ax+Bx^2+Cx^3+\ldots.)(1+Ah+Bh^2+Ch^3+\ldots.)$

$$\begin{aligned}&=1+Ax+Bx^2+Cx^3+Dx^4+\ldots.\\&\quad+Ah+A^2xh+ABx^2h+ACx^3h+\ldots.\\&\quad+Bh^2+ABxh^2+B^2x^2h^2+\ldots.\\&\quad+Ch^3+ACxh^3+\ldots.\\&\quad+Dh^4+\ldots.\end{aligned}$$

Now these two expansions must be identical; and we must, therefore, have the coefficients of like powers of x and h equal; hence

$$\begin{array}{lll}2B=A^2 & \therefore\ B=\dfrac{A^2}{2} & \\ 3C=AB & C=\dfrac{A\cdot B}{3}=\dfrac{A^3}{2\cdot 3} & \\ 4D=AC & D=\dfrac{AC}{4}=\dfrac{A^4}{2\cdot 3\cdot 4} & \\ \&c.\ \ \&c. & \&c. & \&c.\end{array}$$

hence $a^x=1+Ax+\dfrac{A^2x^2}{1\cdot 2}+\dfrac{A^3x^3}{1\cdot 2\cdot 3}+\dfrac{A^4x^4}{1\cdot 2\cdot 3\cdot 4}+\ldots..$

which is the *Exponential Theorem;* where
$$A=(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\ldots....$$
Let ϵ be the value of a, which renders $A=1$, then
$$(\epsilon-1)-\tfrac{1}{2}(\epsilon-1)^2+\tfrac{1}{3}(\epsilon-1)^3-\tfrac{1}{4}(\epsilon-1)^4+\ldots=1$$
$$\epsilon^x=1+x+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}+\ldots.$$
Now, since this equation is true for every value of x, let $x=1$; then
$$\epsilon=1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\ldots...$$
$$=1+1+\tfrac{1}{2}(1)+\tfrac{1}{3}\left(\frac{1}{1\cdot 2}\right)+\tfrac{1}{4}\left(\frac{1}{1\cdot 2\cdot 3}\right)+\ldots.$$
$$=2\cdot 718281828459.$$

LOGARITHMS.

196. LOGARITHMS are artificial numbers, adapted to natural numbers, in order to facilitate numerical calculations; and we shall now proceed to explain the theory of these numbers, and illustrate the principles upon which their properties depend.

DEFINITION. *In a system of logarithms, all numbers are considered as the powers of some one number, arbitrarily assumed, which is called the* BASE *of the system, and the exponent of that power of the base which is equal to any given number is called the* LOGARITHM *of that number.*

Thus, if a be the base of a system of logarithms, N any number, and x such that

$$N = a^x$$

then x is called the logarithm of N in the system whose base is a.

The base of the common system of logarithms, (called from their inventor "Briggs's Logarithms"), is the number 10. Hence since

$(10)^0 =$	1	, 0	is the logarithm of	1	in this system
$(10)^1 =$	10	, 1	————	10	———
$(10)^2 =$	100	, 2	————	100	———
$(10)^3 =$	1000	, 3	————	1000	———
$(10)^4 =$	10000	, 4	————	10000	———
&c. =	&c.	&c.	....		

From this it appears, that in the common system the logarithms of every number between 1 and 10 is some number between 0 and 1, *i. e.* is a fraction. The logarithm of every number between 10 and 100 is some number between 1 and 2, *i. e.* is 1 plus a fraction. The logarithm of every number between 100 and 1000 is some number between 2 and 3, *i. e.* is 2 plus a fraction, and so on.

197. In the common tables the fractional part alone of the logarithm is registered and from what has been said above, the rule usually given for finding the *characteristic*, or, *index*, *i. e.* the integral part of the logarithm will be readily understood, viz. *The index of the logarithm of any number greater than unity is equal to one less than the number of integral figures in the given number.* Thus, in searching for the logarithm of such a number as 2970, we find in the tables opposite to 2970 the number 4727564; but since 2970 is a number between 1000 and 10000, its logarithm must be some number between 3 and 4, *i. e.* must be 3 plus a fraction; the fractional part is the number 4727564, which we have found in the tables, affixing to this the index 3, and interposing a decimal point, we have 3.4727564, the logarithm of 2970.

We must not, however, suppose that the number 3.4727564 is the exact logarithm of 2970, or that

$$2970 = (10)^{3.4727564}$$

accurately. The above is only an approximate value of the logarithm of 2970 we can obtain the exact logarithm of very few numbers, but taking a sufficient number of decimals we can approach as nearly as we please to the true logarithm, as will be seen when we come to treat of the construction of tables.

198. It has been shown that in Briggs' system the logarithm of 1 is 0, consequently, if we wish to extend the application of logarithms to fractions, we must establish a convention by which the logarithms of numbers less than 1 may be represented by numbers less than zero, *i. e.* by *negative numbers.*

Extending, therefore, the above principles to negative exponents, since

$$\frac{1}{10} \text{ or } (10)^{-1} = 0.1, \quad -1 \text{ is the logarithm of } .1 \text{ in this system}$$
$$\frac{1}{100} \text{ or } (10)^{-2} = 0.01, \quad -2 \text{ ——— } .01 \text{ ———}$$
$$\frac{1}{1000} \text{ or } (10)^{-3} = 0.001, \quad -3 \text{ ——— } .001 \text{ ———}$$
$$\frac{1}{10000} \text{ or } (10)^{-4} = 0.0001, \quad -4 \text{ ——— } .0001 \text{ ———}$$
$$\&c. = \&c.$$

It appears, then, from this *convention*, that the logarithm of every number between 1 and .1, is some number between 0 and -1; the logarithm of every number between .1 and .01, is some number between -1 and -2; the logarithm of every number between .01 and .001, is some number between -2 and -3; and so on.

From this will be understood the rule given in books of tables, for finding the *characteristic* or *index* of the logarithm of a decimal fraction, viz. *The index of any decimal fraction is a negative number, equal to unity, added to the number of zeros immediately following the decimal point.* Thus, in searching for a logarithm of the number such as .00462, we find in the tables opposite to 462 the number 6646420; but since .00462 is a number between .001 and .0001, its logarithm must be some number between -3 and -4, i. e. must be -3 plus a fraction, the fractional part is the number 6646420, which we have found in the tables, affixing to this the index -3, and interposing a decimal point, we have -3.6646420, the logarithm of .00462.

General Properties of Logarithms.

199. Let N and N′ be any two numbers, x and x' their respective logarithms, a the base of the system. Then, by definition,

$$N = a^x \text{.................................} (1)$$
$$N' = a^{x'} \text{.................................} (2)$$

I. Multiply equations (1) and (2) together,

$$\begin{aligned} N\,N' &= a^x a^{x'} \\ &= a^{x+x'} \end{aligned}$$

$\therefore$ by definition, $x + x'$ is the logarithm of N N′, that is to say,

The logarithm of the product of two or more factors is equal to the sum of the logarithms of those factors.

II. Divide equation (1) by (2),

$$\frac{N}{N'} = \frac{a^x}{a^{x'}}$$
$$= a^{x-x'}$$

$\therefore$ by definition, $x - x'$ is the logarithm of $\frac{N}{N'}$, that is to say,

The logarithm of a fraction, or of the quotient of two numbers, is equal to the logarithm of the numerator minus the logarithm of the denominator.

III. Raise both members of equation (1) to the power of n.

$$N^n = a^{nx}$$

$\therefore$ by definition, nx is the logarithm of N^n, that is to say,

The logarithm of any power of a given number is equal to the logarithm of the number multiplied by the exponent of the power.

IV. Extract the n^{th} root of both members of equation (1).

$$N^{\frac{1}{n}} = a^{\frac{x}{n}}$$

$\therefore$ by definition, $\frac{x}{n}$ is the logarithm of $N^{\frac{1}{n}}$, that is to say,

The logarithm of any root of a given number is equal to the logarithm of the number divided by the index of the root.

Combining the two last cases, we shall find,

$$N^{\frac{m}{n}} = a^{\frac{mx}{n}}$$

whence, $\frac{mx}{n}$ is the logarithm of $N^{\frac{m}{n}}$.

It is of the highest importance to the student to make himself familiar with the application of the above principles to algebraic calculations. The following examples will afford a useful exercise:

Ex. 1. $\log. (a.b.c.d.\ldots\ldots) = \log. a + \log. b + \log. c + \log. d \ldots.$

Ex. 2. $\log. \left(\frac{abc}{de}\right) = \log. a + \log. b + \log. c - \log. d - \log. e.$

Ex. 3. $\log. (a^m b^n c^p \ldots.) = m \log. a + n \log. b + p \log. c \ldots$

Ex. 4. $\log. \left(\frac{a^m b^n}{c^p}\right) = m \log. a + n \log. b - p \log. c$

Ex. 5. $\log. (a^2 - x^2) = \log.(a+x)\times(a-x) = \log.(a+x) + \log.(a-x)$

Ex. 6. $\log. \sqrt{a^2 - x^2} = \frac{1}{2} \log.(a+x) + \frac{1}{2} \log.(a-x)$

Ex. 7. $\log. (a^3 \sqrt[4]{a^3}) \quad = \log. a^3 + \frac{1}{4} \log a^3 = 3 \log. a + \frac{3}{4} \log. a = \frac{15}{4} \log. a$

Ex. 8. $\log. \sqrt[n]{(a^3 - x^3)^m} \quad = \frac{m}{n} \log. (a-x) + \frac{m}{n} \log. (a^2 + ax + x^2)$

$$= \frac{m}{n} \{\log. (a-x) + \log. (a+x+z) + \log. (a+x-z)\}$$

where $z^2 = ax$

Ex. 9. $\log. \sqrt{a^2 + x^2} \quad = \frac{1}{2} \{\log. (a+x+z) + \log. (a+x-z)\}$, where $z^2 = 2ax$

Ex. 10. $\log. \frac{\sqrt{a^2 - x^2}}{(a+x)^2} \quad = \frac{1}{2} \{\log. (a-x) - 3 \log. (a+x)\}$

Let us resume the equation,

$$N = a^x$$

1°. If $a > 1$, making $x = 0$, we have $N = 1$; the hypothesis $x = 1$ gives $N = a$. As x increases from 0 up to 1, and from 1 up to infinity, N will increase from 1 up to a, and from a up to infinity; so that x being supposed to pass through all intermediate values, according to the law of continuity, N increases also, but with much greater rapidity. If we attribute negative values to x, we have $N = a^{-x}$, or $N = \frac{1}{a^x}$. Here, as x increases, N diminishes, so that x being supposed to increase negatively, N will decrease from 1 towards 0, the hypothesis $x = \infty$ gives $N = 0$.

2°. If $a < 1$, put $a = \frac{1}{b}$, where $b > 1$, and we shall then have $N = \frac{1}{b^x}$, or $N = b^x$, according as we attribute positive or negative values to x. We here arrive at the same conclusion as in the former case, with this difference, that when x is positive $N < 1$, and when x is negative $N > 1$.

3°. If $a = 1$, then $N = 1$. whatever may be the value of x.

From this it appears, that,

I. *In every system of logarithms, the logarithm of* 1 *is* 0, *and the logarithm of the base* is 1.

II. *If the base be* $>$ 1, *the logarithms of numbers* $>$ 1 *are positive, and the logarithms of numbers* $<$ 1 *are negative. The contrary takes place if the base be* $<$ 1.

III. *The base being fixed, any number has only one real logarithm; but the same number has manifestly a different logarithm for each value of the base, so that every number has an infinite number of real logarithms.* Thus, since $9^2 = 81$, and $3^4 = 81$, 2 and 4 are the logarithms of the same number 81, according as the base is 9 or 3.

IV. *Negative numbers have no real logarithms, for attributing to* x *all values from* $-\infty$ *up to* $+\infty$, *we find that the corresponding values of* N *are positive numbers only, from* 0 *up to* $+\infty$.

The formation of a table of logarithms consists in determining and registering the values of x which correspond to $N = 1, 2, 3, \ldots$ in the equation,

$$N = a^x$$

200. If we suppose $m = a^{\alpha}$, making

$$\begin{array}{lll} x = 0,\ \alpha,\ 2\alpha,\ 3\alpha,\ 4\alpha,\ 5\alpha, \ldots\ldots\ldots & \textit{logarithms.} \\ y = 1,\ m,\ m^2,\ m^3,\ m^4,\ m^5, \ldots\ldots\ldots & \textit{numbers.} \end{array}$$

the logarithms increase in arithmetical progression, while the numbers increase in geometrical progression; 0 and 1 being the first terms of the corresponding series, and the arbitrary numbers α and m the common difference and the common ratio.

We may, therefore, consider the systems of values of x and y, which satisfy the equation $N = a^x$, as ranged in these two progressions.

201. In order to solve the equation

$$c = a^x$$

where c and a are given, and where x is unknown, we equate the logarithms of the two members, which gives us

$$\log. c = x \log. a$$

Whence,

$$x = \frac{\log. c}{\log. a}$$

To determine the value of x in the equation

$$Aa^x + Ba^{x-b} + Ca^{x-c} + \ldots\ldots\ldots = P$$

we have

$$a^x\left(A + \frac{B}{a^b} + \frac{C}{a^c} + \ldots\ldots\ldots\right) = P$$

Or,

$$Qa^x = P$$

$$\therefore \quad x = \frac{\log. P - \log. Q}{\log. a}$$

If we have an equation $a^z = b$, where z depends upon an unknown quantity x, and we have

$$z = Ax^n + Bx^{n-1} + \ldots\ldots\ldots$$

Since $z = \dfrac{\log. b}{\log. a} = K$ some known number, the problem depends upon the solution of the equation of the n^{th} degree.

$$K = Ax^n + Bx^{n-1} + \ldots\ldots\ldots\ldots$$

For example, let

$$4\left(\frac{2}{3}\right)^{x^2 - 5x + 4} = 9$$

Hence,

$$(x^2 - 5x + 4) \log. \left(\frac{2}{3}\right) = \log. \frac{9}{4}.$$

$$\therefore \quad x^2 - 5x + 4 = -2$$

an equation of the second degree, from which we find $x = 2$, $x = 3$.

To find the value of x from the equation

$$b^{n-\frac{a}{x}} = c^{mx} . f^{x-p}$$

Taking the logarithms of each member,

$$(n - \frac{a}{x})\log. b = mx \log. c + (x-p)\log. f$$

Or,

$$(m \log. c + \log. f)\, x^2 - (n \log. b + p \log. f)\, x + a \log. b = 0$$

a quadratic equation, from which the value of x may be determined.

In like manner, from the equation

$$c^{mx} = a b^{nx-1}$$

we find

$$x = \frac{\log. a - \log. b}{m \log. c - n \log. b}$$

Equations of this nature are called *Exponential Equations.*

202. Let N and N + 1 be two consecutive numbers, the difference of their logarithms, taken in any system, will be

$$\log. (N+1) - \log. N = \log \left(\frac{N+1}{N}\right) = \log. \left(1 + \frac{1}{N}\right)$$

a quantity which approaches to the logarithm of 1, or zero, in proportion as $\frac{1}{N}$ decreases, that is, as N increases. Hence it appears, that

The difference of the logarithms of two consecutive numbers is less in proportion as the numbers themselves are greater.

203. When we have calculated a table of logarithms for any base a, we can easily change the system, and calculate another table for a new base b.

Let $c = b^x$, x is the log. of c in the system whose base is b;

Taking the logs. in the known system, whose base is a, we have

$$x = \frac{\log. c}{\log. b} = \log. c \left(\frac{1}{\log. b}\right) \text{.........(A)} \quad \text{hence}$$

The log. of c *in the system whose base is* b, *is the quotient arising from dividing the log. of* c *by the log. of the new base* b, *both these last logs. being taken in the system whose base is* a.

In order ∴ to have x the log. of c in the new system, we must multiply log. c by $\frac{1}{\log. b}$; this last factor $\frac{1}{\log. b}$ is constant for all numbers, and is called the *Modulus;* that is to say, if we divide the logs. of the same number c taken in two systems, the quotient will be invariable for these systems, whatever may be the value of c, and will be the modulus, the constant multiplier which reduces the first system of logs. to the second.

If we find it inconvenient to make use of a log. calculated to the base 10, we can in this manner, by aid of a set of tables calculated to the base 10, discover the logarithm of the given number in any required system.

For example, let it be required, by aid of Briggs' tables, to find $\frac{2}{3}$, in a system whose base is $\frac{5}{7}$.

Let x be the log. sought, then by (A)

$$x = \frac{\log.\frac{2}{3}}{\log.\frac{5}{7}}$$

$$= \frac{\log. 2 - \log. 3}{\log. 5 - \log. 7}$$

Taking these logs. in Briggs' system, and reducing, we find

$$= \frac{-0.17609125}{-0.14612804}$$

$$= 1.2050476 = \log.\frac{2}{3} \text{ to base } \frac{5}{7}.$$

Similarly, the log. of $\frac{2}{3}$, in the system whose base is $\frac{3}{2}$, is

$$x = \frac{\log. 2 - \log. 3}{\log. 3 - \log. 2}$$

$$= -1$$

which is manifestly the true result; for in this case the general equation $N = a^x$ becomes $\frac{2}{3} = \left(\frac{3}{2}\right)^x = \left(\frac{2}{3}\right)^{-x}$, and x is evidently $= -1$.

In a system whose base is a, we have

$$n = a^{\log. n}$$

for, by the definition of a log. in the equation $n = a^x$, x is the log. n.

In like manner,

$$n^h = a^{\log. (n^h)} = a^{h \log. n}$$

Examples for Exercise.

(1.) Given $2^{2x}+2^x=12$ to find the value of x.

(2.) Given $x+y=a$, and $m^{(x-y)}=n$ to find x and y.

(3.) Given $m^x n^x=a$, and $hx=ky$ to find x and y.

Answers.

(1.) $x=1{\cdot}584962$, or $x=\log.(-4)\div\log. 2$.

(2.) $x=\frac{1}{2}\{a+\log. n\div\log. m\}$ and $y=\frac{1}{2}\{a-\log. n\div\log. m\}$.

(3.) $x=\log. a\div(\log. m+\log. n)$ and $y=\frac{h}{k}\log. a\div(\log. m+\log. n)$.

204. *To find the logarithm of any given number.*

Let N be any given number whose logarithm is x, in a system whose base is a; then

$$a^x = N \text{ and } a^{xz} = N^z;$$

hence, by the exponential theorem, we have from the last equation

$$1 + Axz + A^2\frac{x^2z^2}{1\cdot 2} + \ldots = 1 + A_1 z + A_1^2\frac{z^2}{1\cdot 2} + \ldots.$$

and equating the coefficients of z, we get $Ax = A_1$; hence

$$x = \frac{A_1}{A} = \frac{(N-1) - \frac{1}{2}(N-1)^2 + \frac{1}{3}(N-1)^3 - \ldots}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots}$$

because $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots$ in the expansion of a^{xz}.

and $A_1 = (N-1) - \frac{1}{2}(N-1)^2 + \frac{1}{3}(N-1)^3 - \ldots$ in the expansion of N^z.

205. *To find the logarithm of a number in a converging series.*

We have seen that if $a^x = N^1$, then

$$x = \frac{(N^1-1) - \frac{1}{2}(N^1-1)^2 + \frac{1}{3}(N^1-1)^3 - \frac{1}{4}(N^1-1)^4 + \ldots}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \ldots}$$

Now the reciprocal of the denominator is the *modulus* of the system; and, representing the modulus by M, we have

$$x = \log. N^1 = M\{(N^1-1) - \tfrac{1}{2}(N^1-1)^2 + \tfrac{1}{3}(N^1-1)^3 - \tfrac{1}{4}(N^1-1)^4 + \ldots\}$$

Put $N^1 = 1 + n$; then $N^1 - 1 = n$, and we have

$$\log. (1+n) = M(+n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \tfrac{1}{5}n^5 - \ldots)$$

$$\text{Similarly } \log. (1-n) = M(-n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 - \tfrac{1}{5}n^5 - \ldots)$$

$$\therefore \log. (1+n) - \log. (1-n) = 2M(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \tfrac{1}{7}n^7 + \ldots)$$

$$\text{or } \log. \frac{1+n}{1-n} = 2M(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \tfrac{1}{7}n^7 + \ldots)$$

Put $n = \frac{1}{2P+1}$; then $1+n = \frac{2P+2}{2P+1}$, $1-n = \frac{2P}{2P+1}$, and $\frac{1+n}{1-n} = \frac{P+1}{P}$, consequently

$$\log. (P+1) - \log. P = 2M\left\{\frac{1}{2P+1} + \frac{1}{3(2P+1)^3} + \frac{1}{5(2P+1)^5} + \ldots\right\}$$

$$\therefore \log. (P+1) = \log. P + 2M\left\{\frac{1}{2P+1} + \frac{1}{3(2P+1)^3} + \frac{1}{5(2P+1)^5} + \ldots\right\}$$

Hence, if log. P be known, the log. of the next greater number can be found by this rapidly converging series.

206. *To find the Napierian logarithms of numbers.*

In the preceding series, which we have deduced for log. $(P+1)$, we find a number M, called the *modulus* of the system; and we must assign some value to this number before we can compute the value of the series. Now, as the value of M is arbitrary, we may follow the steps of the celebrated Lord

Napier, the inventor of logarithms, and assign to M the simplest possible value. This value will therefore be unity; and we have

$$\log.(P+1)=\log.P+2\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\ldots\right\}$$

Expounding P successively by by 1, 2, 3, 4, &c., we find

$$\log\ 2= \qquad 2\left(\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\ldots\right)= \cdot 6931472$$

$$\log.\ 3=\log.2+2\left(\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\frac{1}{7\cdot 5^7}+\ldots\right)=1\cdot 0986123$$

$$\log.\ 4=2\log.2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ =1\cdot 3862944$$

$$\log.\ 5=\log.4+2\left(\frac{1}{9}+\frac{1}{3\cdot 9^3}+\frac{1}{5\cdot 9^5}+\frac{1}{7\cdot 9^7}+\ldots\right)=1\cdot 6094379$$

$$\log.\ 6=\log.2+\log.3\ \ldots\ldots\ldots\ldots\ldots\ldots\ =1\cdot 7917595$$

$$\log.\ 7=\log.6+2\left(\frac{1}{13}+\frac{1}{3\cdot 13^3}+\frac{1}{5\cdot 13^5}+\ldots\ldots\right)=1\cdot 9459101$$

$$\log.\ 8=\log.2+\log 4,\text{ or }3\log.2\ \ldots\ldots\ldots\ldots\ =2\cdot 0794415$$

$$\log.\ 9=2\log.3\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ =2\cdot 1972246$$

$$\log.\ 10=\log.2+\log.5\ \ldots\ldots\ldots\ldots\ldots\ldots\ =2\cdot 3025851$$

In this manner the Napierian logarithms of all numbers may be computed.

207. *To find the common logarithms of numbers.*

Let $a^x=N$ and $b^y=N$; then we have

$x=\log.$ N to the base a, or $x=\log._a N$

$y=\log.$ N to the base b, or $y=\log._b N$

hence, $\log._a N=\log._a b^y=y\log._a b$ (Art. 199.)

$$\therefore\ x=y\log._a b$$

$$\text{and } y=\frac{1}{\log._a b}.x$$

and by means of this equation we can pass from one system of logs. to another, by multiplying x, the log. of any number in the system whose base is a, by the reciprocal of log. b in the same system; and thus we shall obtain the log. of the same number in the system whose base is b.

Let the two systems be the Napierian and the common, in which the base of the former is $\epsilon=2\cdot 718281828\ldots$ and the base of the latter is $b=10$, the base of our common system of arithmetic; then we have $b=10$, and $a=\epsilon=2\cdot 718281828\ldots$ and consequently if N denote any number, we shall have

$$\log._{10}N=\frac{1}{\log._\epsilon 10}.\log._\epsilon N;\ \text{that is,}$$

$$\text{com. log. } N=\frac{\text{nap. log. N}}{\text{nap. log. 10}}=\frac{\text{nap. log. N}}{2\cdot 3025851}=\cdot 43429446\times\text{nap. log. N};$$

and the modulus of the common system is, therefore,

$$M=\frac{1}{2\cdot3025851}=\cdot43429448 \quad \therefore 2\,M=\cdot86858896$$

Hence, to construct a table of common logarithms, we have

$$\log.(P+1)=\log.P+\cdot86858896\left\{\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\dots\right\}$$

Expounding P successively by 1, 2, 3, &c., we get

$$\log.\ 2=\cdot86858896\left(\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5\cdot3^5}+\dots\right)$$

$$=\cdot86858896\times\cdot6931472\dots\dots=\cdot3010300$$

$$\log.\ 3=\log.\ 2+\cdot86858896\left(\frac{1}{5}+\frac{1}{3\cdot5^3}+\frac{1}{5^6}+\dots\right)=\cdot4771213$$

$$\log.\ 4=2\log.\ 2\dots\dots=\cdot6020600$$

$$\log.\ 5=\log.\ \tfrac{10}{2}=\log.\ 10-\log.\ 2=1-\log.\ 2\dots\dots=\cdot6989700$$

$$\log.\ 6=\log.\ 2+\log.\ 3\dots\dots=\cdot7781513$$

$$\log.\ 7=\log.\ 6+\cdot86858896\left(\frac{1}{13}+\frac{1}{3\cdot13^3}+\frac{1}{5\cdot13^5}+\dots\right)=\cdot8450980$$

$$\log.\ 8=\log.\ 2^3=3\log.\ 2\dots\dots=\cdot9030900$$

$$\log.\ 9=\log.\ 3^2=2\log.\ 3\dots\dots=\cdot9542426$$

$$\log.\ 10=\dots\dots=1\cdot0000000$$

&c. &c.

208. Since $\log.\ \frac{1+n}{1-n}=2M\,(n+\frac{1}{3}n^3+\frac{1}{5}n^5+\frac{1}{7}n^7+\dots)$

let $\frac{1+n}{1-n}=P$; then $1+n=P\,(1-n)$ or $n=\frac{P-1}{P+1}$

$$\therefore \log.\ P=2M\left\{\frac{P-1}{P+1}+\frac{1}{3}\cdot\left(\frac{P-1}{P+1}\right)^3+\frac{1}{5}\cdot\left(\frac{P-1}{P+1}\right)^5+\dots\right\}$$

and thus we have a series for computing the logs. of all numbers, without knowing the log. of the previous number.

Examples in Logarithms.

(1.) Given the log. of 2=0·3010300, to find the logs. of 25 and ·0125.

Here $25=\frac{100}{4}=\frac{10^2}{2^2}$; therefore log. 25=2 log. 10—2 log. 2=1·3979406

Again $\cdot0125=\frac{125}{10000}=\frac{1}{80}=\frac{1}{10\times2^3}$

$\therefore$ log. ·0125=log. 1—log. 10—3 log. 2=—1—3 log. 2=$\bar{2}\cdot0969100$

(2.) Calculate the common logarithm of 17. Ans. 1·2304489.

(3.) Given the logs. of 2 and 3 to find the logarithm of 22·5.

Ans. 1+2 log. 3—2 log. 2.

(4.) Having given the logs. of 3 and ·21, to find the logarithm of 83349.

Ans. 6+2 log. 3+3 log. ·21.

On Exponential Equations.

209. An *exponential equation* is an equation in which the unknown appears in the form of an exponent or index; thus, the following are exponential equations:

$$a^x = b,\ x^x = a,\ a^{b^x} = c,\ x^{x^x} = a, \text{ \&c.}$$

When the equation is of the form $a^x = b$, or $a^{b^x} = c$, the value of x is readily obtained by logarithms, as we have already seen in Art. 201. But if the equation be of the form $x^x = a$, the value of x may be obtained by the rule of *double position*, as in the following example:

Ex. Given $x^x = 100$, to find an approximate value of x.

The value of x is evidently between 3 and 4, since $3^3 = 27$ and $4^4 = 256$; hence, taking the logs. of both sides of the equation, we have

$$x \log. x = \log. 100 = 2^*$$

$$\begin{aligned} \text{First, let } x_1 &= 3{\cdot}5;\ \text{then} \\ 3{\cdot}5 \log. 3{\cdot}5 &= 1{\cdot}9042380 \\ \text{true no.} &= 2{\cdot}0000000 \\ \hline \text{error} &= -{\cdot}0957620 \end{aligned}$$

$$\begin{aligned} \text{Second, let } x_2 &= 3{\cdot}6;\ \text{then} \\ 3{\cdot}6 \log. 3{\cdot}6 &= 2{\cdot}0026890 \\ \text{true no.} &= 2{\cdot}0000000 \\ \hline \text{error} &= +{\cdot}0026890 \end{aligned}$$

Then, as the difference of the results is to the difference of the assumed numbers, so is the least error to a correction of the assumed number corresponding to the least error; that is,

$$\cdot 098451 : \cdot 1 :: \cdot 002689 : \cdot 00273;$$

hence $x = 3{\cdot}6 - {\cdot}00273 = 3{\cdot}59727$, nearly

Again, by forming the value of x^x for $x = 3{\cdot}5972$, we find the error to be $-{\cdot}0000841$, and for $x = 3{\cdot}5973$, the error is $+{\cdot}0000149$;

$$\text{hence, as } \cdot 000099 : \cdot 0001 :: \cdot 0000149 : \cdot 0000151;$$

therefore $x = 3{\cdot}5973 - {\cdot}0000151 = 3{\cdot}5972849$, the value nearly.

Examples for Practice.

(1.) Find x from the equation $x^x = 5$. Ans. 2·129372.

(2.) Solve the equation $x^x = 123456789$. Ans. 8·6400268.

(3.) Find x from the equation $x^x = 2000$. Ans. 4·8278226.

* In equations of this kind, the following method may be adopted:—Let $x^x = a$; then x log. $x =$ log. a; put log. $x = y$, and log. $a = b$; then $xy = b$, and log. $x +$ log. $y =$ log. b; hence $y +$ log. $y =$ log. b. Now, y may be found by double position, as above, and then x becomes known. When a is less than unity, put $x = \frac{1}{y}$ and $a = \frac{1}{b}$; then we have $b^y = y$ $\therefore$ y log. $b =$ log. y, and if log. $b = c$, and log. $y = x$; then $cy = z$, and log. $c +$ log. $y =$ log. z, or log. $c + z =$ log. x. Hence z may be found by the preceding method, and then y and x become known.

INTEREST AND ANNUITIES.

The solution of all questions connected with interest and annuities may be greatly facilitated by the employment of algebraical formulæ.

In treating of this subject we may employ the following notation:

Let p pounds denote the principal.

r	..	...	interest of £1 for one year.
i	...	...	interest of p pounds for t years.
s	...	...	amount of p pounds for t years at the rate of interest denoted by r.
t	...	...	the number of years that p is put out to interest.

SIMPLE INTEREST.

Problem I.—*To find the interest of a sum* p *for* t *years at the rate* r.

Since the interest of one pound for one year is r, the interest of p pounds for one year must be p times as much, or pr; and for t years t times as much as for one year, consequently,

$$i = p\,t\,r \qquad \text{...}(1)$$

Problem II.—*To find the amount of a sum* p *laid out for* t *years at simple interest at the rate* r.

The amount must evidently be equal to the principal together with the interest upon that principal for the given time,

Hence,

$$\begin{aligned} s &= p + p\,t\,r \\ &= p\,(1 + t\,r) \qquad \text{..................}(2) \end{aligned}$$

Example 1.

Required the interest of £873. 15*s.* for $2\frac{1}{2}$ years at $4\frac{3}{4}$ per cent. per annum.

It will be found convenient to reduce broken sums of money and periods of time to decimals of a pound and of a year, respectively.

By the formula (1) we have

$$i = p\,t\,r$$

In the example before us

$$\begin{array}{lllcl} p & = & £873.\ 15s. & \ldots\ldots\ldots\ldots\ldots\ldots & = £873.75 \\ r & & & \ldots\ldots\ldots\ldots\ldots\ldots & = £.0475^{*} \\ t & = & 2\tfrac{1}{2} \text{ years} & \ldots\ldots\ldots\ldots\ldots\ldots & = 2.5 \text{ years.} \end{array}$$

$$\begin{aligned} \therefore\ i &= 873.75 \times 2.5 \times .0475 \text{ pounds.} \\ &= £103.7578125 \\ &= £103.\ 15s.\ 1\tfrac{3}{4}d. \end{aligned}$$

The *amount* of the above sum at the end of the given time will be

$$\begin{aligned} s &= p + p\,t\,r \\ &= £873.\ 15s. + £103.\ 15s.\ 1\tfrac{1}{4}d. \\ &= £977.\ 10s.\ 1\tfrac{3}{4}d. \end{aligned}$$

PRESENT VALUE AND DISCOUNT AT SIMPLE INTEREST.

The present value of any sum s *due* t *years hence is the principal which in the time* t *will amount to* s.

The discount upon any sum due t *years hence is the difference between that sum and its present value.*

PROBLEM III.—*To find the present value of* s *pounds due* t *years hence, simple interest being calculated at the rate* r.

By formula (2) we find the amount of a sum p at the end of t years to be

$$s = p + p\,t\,r$$

Consequently p will represent the present value of the sum s due t years hence, and we shall have

$$p = \frac{s}{1 + t\,r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

for the expression required.

PROBLEM IV.—*To find the discount on* s *pounds due* t *years hence, at the rate* r, *simple interest.*

* r is the interest of $L1$ for one year. To find the value of r when interest is calculated at the rate of $L4\tfrac{3}{4}$, or $L4.75$ *per cent.* per annum, we have the following proportion.

$$L100 : L1 :: L4.75 : r$$

$$\begin{aligned} \therefore\ r &= L\frac{4.75}{100} \\ &= L0.0475 \end{aligned}$$

In like manner,

When the rate of interest *per cent.* is				$L5$	then r	=	$L0.05$
—	—	—	—	$4\tfrac{3}{4}$	—	=	0.0475
—	—	—	—	$4\tfrac{1}{2}$	—	=	0.045
—	—	—	—	$4\tfrac{1}{4}$	—	=	0.0425
—	—	—	—	4	—	=	0.04
—	—	—	—	$3\tfrac{3}{4}$	—	=	0.0375
&c.			&c.			&c.	

Since the discount on s is the difference between s and its present value, we shall have

$$d = s - \frac{s}{1 + t r}$$
$$= \frac{s t r}{1 + t r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (4)$$

Example.

Required the discount on £100, due 3 months hence, interest being calculated at the rate of 5 per cent. per annum.

Here,

$$s = £100 = £100$$
$$t = 3 \text{ months} = .25 \text{ years}$$
$$r = \quad = £ .05$$

Here the present value of p is

$$p = \frac{s}{1 + t r}$$
$$= \frac{100}{1 + .25 \times .05}$$
$$= \frac{100}{1.0125}$$
$$= 98.76543 \text{ pounds.}$$

But,

$$s = £100$$
$$p = £98.76543 = £98.\ 15s.\ 3\tfrac{1}{2}d.$$
$$\therefore\ s - p \text{ or } d = £1.\ 4s.\ 8\tfrac{1}{2}d.$$

ANNUITIES AT SIMPLE INTEREST.

PROBLEM V.—*To find the amount of an annuity* a *continued for* t *years, simple interest being allowed at the rate* r *upon the successive payments.*

At the end of the first year the annuity a will be due, at the end of the second year a second payment a will become due, together with $a\,r$, the interest for one year upon the first payment, at the end of the third year a third payment a becomes due, together with $2\,a\,r$, the interest for one year upon the two former payments, and so on, the sum of all these will be the amount required,

Thus,

At the end of the first year the sum due is a.

..	..	second	..	..	$a + a\,r$.
..	..	third	..	..	$a + 2\,a\,r$.
..	..	fourth	..	..	$a + 3\,a\,r$.
&c.		&c.	&c.		
..	..	t^{th}	..	..	$a + (t - 1)\,a\,r$.

Hence, adding these all together for the whole amount,

$$s = t\,a + a\,r\,(1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots\ldots (t - 1)\,)$$

Or, taking the expression for the sum of the arithmetical series, $1 + 2 + 3 \ldots\ldots\ldots (t - 1)$

$$s = t\,a + r\,a\,.\,\frac{t\,(t-1)}{1\,.\,2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Problem VI.—*To find the present value of an annuity* a *payable for* t *years, simple interest being allowed at the rate* r.

It is manifest that the present value of the annuity must be a sum such, that, if put out to interest for t years at the rate r, its amount at the end of that period will be the same with the amount of the annuity.

Hence, if we call this present value p, we shall have by Probs. I and V.

$$\begin{aligned} p + p\,t\,r &= \text{amount of annuity.} \\ &= t\,a + r\,a\,.\,\frac{t\,(t-1)}{1\,.\,2} \\ \therefore p &= \frac{t\,a + r\,a\,.\,\frac{t\,(t-1)}{1\,.\,2}}{1 + t\,r} \\ &= \frac{t\,a}{2}\,.\,\frac{2 + (t-1)\,r}{1 + t\,r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)^* \end{aligned}$$

COMPOUND INTEREST.

Problem VII.—*To find the amount of a sum* p *laid out for* t *years, compound interest being allowed at the rate* r.

At the end of the first year the amount will be, by Problem II.

$$p + p\,r, \text{ or } p\,(1 + r)$$

Since compound interest is allowed, this sum $p\,(1 + r)$ now becomes the principal, and hence, at the end of the second year, the amount will be $p\,(1 + r)$, together with the interest on $p\,(1 + r)$ for one year; that is, it will be

$$p\,(1 + r) + p\,r\,(1 + r), \text{ or } p\,(1 + r)^2$$

The sum $p\,(1 + r)^2$ must now be considered as the principal, and hence the whole amount at the end of the third year will be

$$p\,(1 + r)^2 + p\,r\,(1 + r)^2, \text{ or } p\,(1 + r)^3$$

And, in like manner, at the end of the t^{th} year we shall have

$$s = p\,(1 + r)^t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Any three of the four quantities, s, p, r, t, being given, the fourth may always be found from the above equations.

Example I.

Find the amount of £15. 10*s*. for 9 years, compound interest being allowed

* It is unnecessary to give any examples on this rule, as the purchase of annuities at simple interest can never be of practical utility. Thus, if we wished to ascertain by this formula the present value of an annuity of *L*50, to continue for 40 years, calculating interest at 5 per cent., we should find it to be *L*1316 13*s*. 4*d*. But the interest of *L*1316 13*s*. 4*d*. at the same rate is upwards of £65 per annum continued for ever.

at the rate of £3½ per cent. per annum. The interest payable at the end of each year.

By equation (7),

$$s = p(1+r)^t$$
$$\therefore \log. s = \log. p + t \log. (1+r)$$

Hence,

$$p = £15.\ 10s. \ldots\ldots\ldots = £15.5.$$
$$t = \quad = 9 \text{ years.}$$
$$r = \quad = £.035$$
$$\therefore \log. p = 1.1903317$$
$$t \log. (1+r) = 0.1344627$$
$$\therefore \log. s = 1.3247944 = \log. 21.12481$$
$$\therefore s = £21.12481$$
$$= £21.\ 2s.\ 5\tfrac{3}{4}d.$$

Example 2.

Find the amount of £182. 12s. 6d. for 18 years, 6 months, and 10 days, at the rate of 3½ per cent. per annum, compound interest; the interest being payable at the end of each year.

In this case, it will be convenient, first, to find the amount at compound interest of the above sum for 18 years, and then calculate the interest on the result for the remaining period.

By formula (7),

$$s = p(1+r)^t$$
$$\log. s = \log. p + t \log (1+r)$$

Here,

$$p = £182.\ 12s.\ 6d. = £182.625$$
$$r = \quad = £.035$$
$$t = \quad = 18 \text{ years,}$$
$$\therefore \log. p = 2.2615602$$
$$t \log. (1+r) = 0.2689254$$
$$\therefore \log. s = 2.5304856 = \log. 339.224.$$

Again, to find the interest on this sum for the short period, we have

$$i = s t' r$$
$$\therefore \log. i = \log. s + \log. t' + \log. r.$$

Here,

$$s = £339.224$$
$$r = £.035$$
$$t' = 6 \text{ months, } 10 \text{ days} = .527402 \text{ years.}$$
$$\therefore \log. s = 2.5304856$$
$$\log. r = \bar{2}.5440680$$
$$\log. t' = \bar{1}.7221401$$
$$\therefore \log. s t' r = .07966937 = \log. 6.2617200$$
$$\therefore s t' r = £6.26172$$

The whole amount required will therefore be

$$s + s t' r = £339.224 + £6.26172$$
$$= £345.\ 9s.\ 8\tfrac{1}{4}d.$$

Example 3.

Required the compound interest upon £410 for $2\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent. per annum, the interest being payable half yearly.

In this case, the time t must be calculated in *half years*; and since we have supposed r to be the interest of £1 for one year, we must substitute $\frac{r}{2}$, which will be the interest of £1 for half a year; the formula (7) will thus become

$$s = p\left(1+\frac{r}{2}\right)^{2t}$$

$$\therefore \log. s = \log. p + 2t \log. \left(1+\frac{r}{2}\right)$$

Here,

$$\begin{aligned} p &= £410 \\ r &= £.045 \\ 2t &= 5 \text{ half years} \end{aligned}$$

$$\begin{array}{rcl} \therefore \log. p &=& 2.6127839 \\ 5 \log. 1.0225 &=& 0.0483165 \\ \hline \therefore \log. s &=& 2.6611004 = \log. 458.2471 \\ \therefore s &=& £458.2471. \end{array}$$

The *interest* must be the difference between this amount and the original principal,

$$\begin{aligned} \therefore i &= s - p \\ &= £458.247 - £410 \\ &= £48\ 4s.\ 11\tfrac{1}{4}d. \end{aligned}$$

Example 4.

£400 was put out at compound interest, and at the end of 9 years amounted to £569 6*s*. 8*d*; required the rate of interest per cent.

Here s, p, t are given, and r is sought.

From formula

$$s = p\,(1+r)^t$$

We have

$$\log. (1+r) = \frac{1}{t}(\log. s - \log. p)$$

Here,

$$\begin{array}{rcl} s = £569\ 6s.\ 8d &=& £569.3333 \\ p = &=& £400 \\ t = &=& 9 \text{ years} \\ \because \log. s &=& 2.7553666 \\ \log. p &=& 2.6020600 \\ \hline \therefore \log. s - \log. p &=& .1533066 \\ \log. (1+r) &=& \dfrac{.1533066}{9} \\ &=& .0170340 \\ &=& \log. 1.04 \\ \therefore r &=& .04 = 4 \text{ per cent.} \end{array}$$

Example 5.

In what time will a sum of money double itself, allowing 4 per cent. compound interest?

Here s, p, r are given, and t is sought.

From the formula (7) we have

$$s = p(1+r)^t$$

But here,

$$\begin{aligned} s &= 2p \\ \therefore 2p &= p(1+r)^t \\ \therefore 2 &= (1+r)^t \\ t &= \frac{\text{log. } 2.}{\text{log. } (1+r)} \\ &= \frac{.3010300}{.0170333} \\ &= 17.673 \text{ years} \\ &= 17 \text{ years, } 8 \text{ months, } 2 \text{ days.} \end{aligned}$$

In like manner, if it be required to find in what time a sum will triple itself at the same rate, we have

$$\begin{aligned} t &= \frac{\text{log. } 3.}{\text{log. } 1.04} \\ &= \frac{.4771213}{.0170333} \\ &= 28.011 \text{ years.} \\ &= 28 \text{ years, } 0 \text{ months, } 3 \text{ days.} \end{aligned}$$

PRESENT VALUE AND DISCOUNT AT COMPOUND INTEREST.

If we call p the present value of a sum s due t years hence, and d its discount, reasoning precisely in the same manner as in the case of simple interest, we shall find

$$p = \frac{s}{(1+r)^t} \quad \dots\dots\dots\dots\dots\dots \quad (8)$$

$$d = s\left(1 - \frac{1}{(1+r)^t}\right) \quad \dots\dots\dots\dots\dots\dots \quad (9)$$

ANNUITIES AT COMPOUND INTEREST.

Problem VIII. *To find the amount of an annuity* a *continued for* t *years, compound interest being allowed at the rate* r.

At the end of the first year the annuity a will become due, at the end of the second year a second payment a will become due, together with the interest of the first payment a for one year, that is, $a\,r$; the whole sum upon which interest must now be computed is thus $2\,a + a\,r$.

At the end of the third year a further payment a becomes due, together with the interest on $2\,a + a\,r$, *i. e.* $2\,a\,r + a\,r^2$; the whole sum upon which interest must now be computed is $3\,a + 3\,a\,r + a\,r^2$. The result will appear evident when exhibited under the following form:

$$
\begin{array}{lll}
\text{Whole amount at the end of 1st year} & = & a \\
\text{.................................. 2nd ...} & = & a + a + ar \\
 & = & a + a(1+r) \\
\text{.................................. 3d ...} & = & a + a + a(1+r) + ar + ar(1+r) \\
 & = & a + a(1+r) + a(1+r)^2 \\
\text{.................................. 4th ...} & = & a + a + a(1+r) + a(1+r)^2 + ar \\
 & & + ar(1+r) + ar(1+r)^2 \\
 & = & a + a(1+r) + a(1+r)^2 + a(1+r)^3 \\
\text{.................................. \&c. ...} & = & \text{\&c.} \\
\text{.................................. } t^{\text{h}} \text{ ...} & = & a + a(1+r) + a(1+r)^2 + a(1+r)^3 \\
 & & + \text{...........} a(1+r)^{t-1}.
\end{array}
$$

Hence the whole amount is

$$
\begin{aligned}
s &= a\left\{1 + (1+r) + (1+r)^2 + \text{...........} + (1+r)^{t-1}\right\} \\
&= a \cdot \frac{(1+r)^t - 1}{r} \text{ .... } (10).
\end{aligned}
$$

PROBLEM IX. *To find the present value of an annuity* a *payable for* t *years, compound interest being allowed at the rate* r.

It is manifest that the present value of this annuity must be a sum such, that if put out to interest for t years at the rate r, its amount at the end of that period will be the same as the amount of the annuity.

Hence, if we call this present value p, we shall have, by Probs. VII. and VIII.

$$
\begin{aligned}
p(1+r)^t &= \text{amount of annuity} \\
&= a \cdot \frac{(1+r)^t - 1}{r} \\
\therefore p &= \frac{(1+r)^t - 1}{r(1+r)^t} \cdot a \\
&= \frac{a}{r} \cdot \frac{(1+r)^t - 1}{(1+r)^t} \text{ .... } (11)
\end{aligned}
$$

Example.

What is the present value of an annuity of £500, to last for 40 years, compound interest being allowed at the rate of $2\frac{1}{2}$ per cent. per annum.

By formula (11),

$$p = \frac{a}{r} \cdot \frac{(1+r)^t - 1}{(1+r)^t}$$

Here,

$$
\begin{aligned}
a &= £500 \\
r &= £.025 \\
t &= 40 \text{ years,} \\
\therefore (1+r)^t &= (1.025)^{40}
\end{aligned}
$$

Now,

$$\begin{aligned} \log. (1.025)^{40} &= 40 \log. 1.025 \\ &= 40 \times .0107239 \\ &= .4289560 \\ &= \log. 2.685072 \\ \therefore (1.025)^{40} &= 2.685072 = (1+r)^{t} \end{aligned}$$

Also,

$$\frac{a}{r} = \frac{500}{.025} = 20000$$

$$\begin{aligned} \therefore p &= 20000 \times \frac{1.685072}{2.685072} \\ &= 20000 \times .62757\ldots \\ &= 12551.4 \text{ pounds.} \end{aligned}$$

REVERSION OF ANNUITIES.

PROBLEM X. *To find the present value* (P) *of an annuity* a *which is to commence after* T *years, and to continue for* t *years.*

The present value required is manifestly the present value of a for T + t years, minus the present value of a for T years.

By Problem IX. the present value of a for T + t years $= \frac{a}{r} \cdot \frac{(1+r)^{T+t} - 1}{(1+r)^{T+t}}$

.... a for T .... $= \frac{a}{r} \cdot \frac{(1+r)^{T} - 1}{(1+r)^{T}}$

$$P = \frac{a}{r} \cdot \{(1+r)^{-T} - (1+r)^{-(T+t)}\} \ldots\ldots\ldots\ldots (12)$$

PURCHASE OF ESTATES.

PROBLEM XI. *To find the present value* p *of an estate or perpetuity, whose annual rental is* a, *compound interest being calculated at the rate* r.

The present value of an annuity a, to continue for t years, by Prob. IX. is

$$p = \frac{a}{r} \{1 - (1+r)^{-t}\}$$

but if the annuity last *for ever*, as in the case of an estate, then $t = \infty$, and $\therefore \frac{1}{(1+r)^{t}} = \frac{1}{\infty} = 0$; hence, in the present case,

$$p = \frac{a}{r} \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Example.

What is the value of an estate, whose rental is £1000, allowing the purchaser 5 per cent. for his money?

Here,

$$a = £1000$$
$$r = £.05$$

$$1000$$

$$= £20000, \text{ or } 20 \text{ years purchase.}$$

REVERSION OF PERPETUITIES.

PROBLEM XII. *To find the present value of an estate or perpetuity, whose annual rental is* a *pounds, to a person to whom it will revert after* T *years, compound interest being allowed at the rate* **r.**

By Problem X., the present value of an annuity, to commence after T years and to continue for t years, is

$$= \frac{a}{r}\left\{(1+r)^{-T} - (1+r)^{-(T+t)}\right\}$$

In the present case, $t = \infty$, and $\therefore (1+r)^{-(T+t)} = 0$; hence we shall have

$$p = \frac{a}{r}(1+r)^{-T} \quad \text{..............................} (14).$$

EXAMPLES FOR PRACTICE

1. Find the interest of £555 for $2\frac{1}{4}$ years at $4\frac{3}{4}$ per cent. simple interest. Ans. £65 18*s.* $1\frac{1}{2}$*d.*

2. In what time will the interest of £1 amount to 15*s.*, allowing $4\frac{1}{2}$ per cent. simple interest? Ans. 16 years, 8 months.

3. What is the amount of £120 10*s.* for $2\frac{1}{2}$ years, at $4\frac{3}{4}$ per cent. simple interest? Ans. £134 16*s.* $2\frac{1}{4}$*d.*

4. The interest of £25 for $3\frac{1}{2}$ years, at simple interest, was found to be £3 18*s.* 9*d.*; required the rate per cent. per annum. Ans. $4\frac{1}{2}$.

5. Find the discount on £100 due at the end of 3 months, interest being calculated at the rate of 5 per cent. per annum. Ans. £1 4*s.* $8\frac{1}{4}$*d.*

6. What is the present value of the compound interest of £100 to be received five years hence, at 5 per cent. per annum. Ans. £78 7*s.* $0\frac{1}{2}$*d.*

7. What is the amount of £721, for 21 years, at 4 per cent. per annum, compound interest? Ans. £1642 19*s.* $9\frac{1}{4}$*d.*

8. The rate of interest being 5 per cent., in what number of years, at compound interest, will £1 amount to £100? Ans. 94 years, 141.4 days.

9. Find the present value of £430, due nine months hence, discount being allowed at $4\frac{1}{2}$ per cent. per annum. Ans. £415 19*s.* $2\frac{1}{2}$*d.*

10. Find the amount of £1000, for 1 year, at 5 per cent. per annum, compound interest, the interest being payable daily. Ans. £1051. 5*s.* 9*d.* nearly.

11. What sum ought to be given for the lease of an estate for 20 years, of the clear annual rental of £100, in order that the purchaser may make 8 per cent. of his money? Ans. £981 16*s.* $3\frac{1}{4}$*d.*

12. Find the present value of £20, to be paid at the end of every five years, for ever, interest being calculated at 5 per cent. Ans. £72 7*s.* $9\frac{1}{2}$*d.*

13. What is the present value of an annuity of £20, to continue for ever, and to commence after two years, interest being calculated at 5 per cent.? Ans. £362 16*s.* $2\frac{3}{4}$*d.*

14. The present value of a freehold estate of £100 per annum, subject to the payment of a certain sum (A) at the end of every two years, is £1000, allowing 5 per cent. compound interest. Find the sum (A). Ans. A = £102 10*s.*

15. What is the present value of an annuity of £79 4*s.* to commence 7 years hence and continue for ever, interest being calculated at the rate of $4\frac{1}{2}$ per cent.? Ans. £1293 5*s.* $11\frac{1}{4}$*d.*

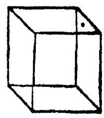
GEOMETRY.

DEFINITIONS.

1. A POINT is that which has position, but no magnitude, nor dimensions; neither length, breadth, nor thickness.

2. A line is length without breadth or thickness.

3. A Surface or Superficies, is an extension or a figure of two dimensions, length and breadth; but without thickness.



4. A Body or Solid, is a figure of three dimensions, namely, length, breadth, and depth, or thickness.

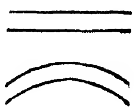
5. Lines are either Right, or Curved, or Mixed of these two.

6. A Right Line, or Straight Line, lies all in the same direction, between its extremities; and is the shortest distance between two points.

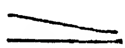
When a Line is mentioned simply, it means a Right Line.

7. A Curve continually changes its direction between its extreme points.

8. Lines are either Parallel, Oblique, Perpendicular, or Tangential.

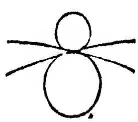


9. Parallel Lines are always at the same perpendicular distance; and they never meet, though ever so far produced.

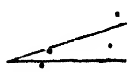


10. Oblique Lines change their distance, and would meet, if produced on the side of the least distance.

11. One line is Perpendicular to another, when it inclines not more on the one side than the other, or when the angles on both sides of it are equal.



12. A line or circle is Tangential, or is a Tangent to a circle, or other curve, when it touches it, without cutting, although both are produced.



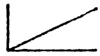
13. An Angle is the inclination or opening of two lines, having different directions, and meeting in a point.

14. Angles are Right or Oblique, Acute or Obtuse

15. A Right Angle is that which is made by one line perpendicular to another. Or when the angles on each side are equal to one another, they are right angles.

16. An Oblique Angle is that which is made by two oblique lines; and is either less or greater than a right angle.

17. An Acute Angle is less than a right angle.



18. An Obtuse Angle is greater than a right angle.

19. Superficies are either Plane or Curved.

20. A Plane Superficies, or a Plane, is that with which a right line may, every way, coincide. Or, if the line touch the plane in two points, it will touch it in every point. But, if not, it is curved.

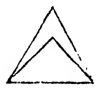
21. Plane Figures are bounded either by right lines or curves.

22. Plane figures that are bounded by right lines have names according to the number of their sides, or of their angles; for they have as many sides as angles; the least number being three.

23. A figure of three sides and angles is called a Triangle. And it receives particular denominations from the relations of its sides and angles.

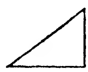
24. An Equilateral Triangle is that whose three sides are all equal.

25. An Isosceles Triangle is that which has two sides equal.



26. A Scalene Triangle is that whose three sides are all unequal.

27. A Right-angled Triangle is that which has one right angle.



28. Other triangles are Oblique-angled, and are either obtuse or acute.

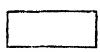
29. An Obtuse-angled Triangle has one obtuse angle.

30. An Acute-angled Triangle has all its three angles acute.

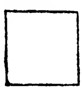
31. A figure of Four sides and angles is called a Quadrangle, or a Quadrilateral.

32. A Parallelogram is a quadrilateral which has both its pairs of opposite sides parallel. And it takes the following particular names, viz. Rectangle, Square, Rhombus, Rhomboid.

33. A Rectangle is a parallelogram, having a right angle.



34. A Square is an equilateral rectangle; having its length and breadth equal, or all its sides equal, and all its angles equal.



35. A Rhomboid is an oblique-angled parallelogram.

36. A Rhombus is an equilateral rhomboid; having all its sides equal, but its angles oblique.

37. A Trapezium is a quadrilateral which has not its opposite sides parallel.

38. A Trapezoid has only one pair of opposite sides parallel.

39. A Diagonal is a line joining any two opposite angles of a quadrilateral.

40. Plane figures that have more than four sides are, in general, called Polygons: and they receive other particular names, according to the number of their sides or angles. Thus,

41. A Pentagon is a polygon of five sides; a Hexagon, of six sides; a Heptagon, seven; an Octagon, eight; a Nonagon, nine; a Decagon, ten; an Undecagon, eleven; and a Dodecagon, twelve sides.

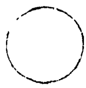
42. A Regular Polygon has all its sides and all its angles equal.—If they are not both equal, the polygon is Irregular.

43. An Equilateral Triangle is also a Regular Figure of three sides, and the Square is one of four: the former being also called a Trigon, and the latter a Tetragon.

44. Any figure is equilateral, when all its sides are equal: and it is equiangular when all its angles are equal. When both these are equal, it is a regular figure.

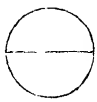
45. A Circle is a plane figure bounded by a curve line, called the Circumference, which is everywhere equidistant from a certain point within, called its Centre.

The circumference itself is often called a circle, and also the Periphery.

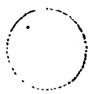


46. The Radius of a circle is a line drawn from the centre to the circumference.

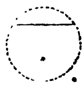
47. The Diameter of a circle is a line drawn through the centre, and terminating at the circumference on both sides.



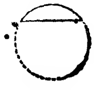
48. An Arc of a circle is any part of the circumference.

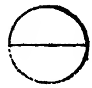


49. A Chord is a right line joining the extremities of an arc.



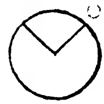
50. A Segment is any part of a circle bounded by an arc and its chord.



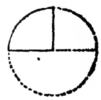


51. A Semicircle is half the circle, or a segment cut off by a diameter.

The half circumference is sometimes called the Semicircle.



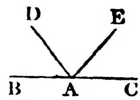
52. A Sector is any part of a circle which is bounded by an arc, and two radii drawn to its extremities.



53. A Quadrant, or Quarter of a circle, is a sector having a quarter of the circumference for its arc, and its two radii are perpendicular to each other. A quarter of the circumference is sometimes called a Quadrant.

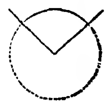
54. The Height or Altitude of a figure is a perpendicular let fall from an angle, or its vertex, to the opposite side, called the base.

55. In a right-angled triangle, the side opposite the right angle is called the Hypothenuse; and the other two sides are called the Legs, and sometimes the *Base* and *Perpendicular*.

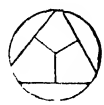


56. When an angle is denoted by three letters, of which one stands at the angular point, and the other two on the two sides, that which stands at the angular point is read in the middle.

57. The circumference of every circle is supposed to be divided into 360 equal parts called degrees; and each degree into 60 Minutes, each Minute into 60 Seconds, and so on. Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

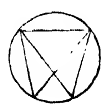


58. The Measure of an angle is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc.



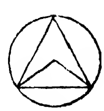
59. Lines, or chords, are said to be Equidistant from the centre of a circle, when perpendiculars drawn to them from the centre are equal.

60. And the right line on which the Greater Perpendicular falls, is said to be farther from the centre.

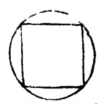


61. An Angle in a Segment is that which is contained by two lines, drawn from any point in the arc of the segment, to the two extremities of that arc.

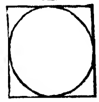
62. An Angle on a segment, or an arc, is that which is contained by two lines, drawn from any point in the opposite or supplementary part of the circumference, to the extremities of the arc, and containing the arc between them.



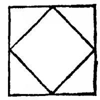
63. An Angle at the circumference, is that whose angular point or summit is any where in the circumference And an angle at the centre, is that whose angular point is at the centre.



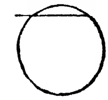
64. A right-lined figure is Inscribed in a circle, or the circle Circumscribes it, when all the angular points of the figure are in the circumference of the circle.



65. A right-lined figure Circumscribes a circle, or the circle is Inscribed in it, when all the sides of the figure touch the circumference of the circle.



66. One right-lined figure is inscribed in another, or the latter circumscribes the former, when all the angular points of the former are placed in the sides of the latter.



67. A Secant is a line that cuts a circle, lying partly within, and partly without it.

68. Two triangles, or other right-lined figures, are said to be mutually equilateral, when all the sides of the one are equal to the corresponding sides of the other, each to each: and they are said to be mutually equiangular, when the angles of the one are respectively equal to those of the other.

69. Identical figures, are such as are both mutually equilateral and equiangular; or that have all the sides and all the angles of the one, respectively equal to all the sides and all the angles of the other, each to each; so that if the one figure were applied to, or laid upon the other, all the sides of the one would exactly fall upon and cover all the sides of the other; the two becoming as it were but one and the same figure.

70. Similar figures, are those that have all the angles of the one equal to all the angles of the other, each to each, and the sides about the equal angles proportional.

71. The Perimeter of a figure, is the sum of all its sides taken together.

72. A Proposition, is something which is either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

73. A Problem, is something proposed to be done.

74. A Theorem, is something proposed to be demonstrated.

75. A Lemma, is something which is premised, or demonstrated, in order to render what follows more easy.

76. A Corollary, is a consequent truth, gained immediately from some preceding truth, or demonstration.

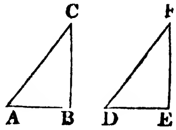
77. A Scholium, is a remark or observation made upon something going before it.

AXIOMS.

1. THINGS which are equal to the same thing are equal to each other.
2. When equals are added to equals, the wholes are equal.
3. When equals are taken from equals, the remainders are equal.
4. When equals are added to unequals, the wholes are unequal.
5. When equals are taken from unequals, the remainders are unequal.
6. Things which are double of the same thing, or equal things, are equal to each other.
7. Things which are halves of the same thing, are equal.
8. Every whole is equal to all its parts taken together.
9. Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.
10. All right angles are equal to one another.
11. Angles that have equal measures, or arcs, are equal.

THEOREM I.

If two triangles have two sides and the included angle in the one, equal to two sides and the included angle in the other, the triangles will be identical, or equal in all respects.



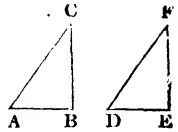
In the two triangles ABC, DEF, if the side AC be equal to the side DF, and the side BC equal to the side EF, and the angle C equal to the angle F; then will the two triangles be identical, or equal in all respects.

For conceive the triangle ABC to be applied to, or placed on, the triangle DEF, in such a manner that the point C may coincide with the point F, and the side AC with the side DF, which is equal to it.

Then, since the angle F is equal to the angle C (by hyp.), the side BC will fall on the side EF. Also, because AC is equal to DF, and BC equal to EF (by hyp.), the point A will coincide with the point D, and the point B with the point E; consequently the side AB will coincide with the side DE. Therefore the two triangles are identical, and have all their other corresponding parts equal (ax. 9), namely, the side AB equal to the side DE, the angle A to the angle D, and the angle B to the angle E. Q. E. D.

THEOREM II.

When two triangles have two angles and the included side in the one, equal to two angles and the included side in the other, the triangles are identical, or have their other sides and angles equal.

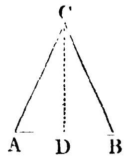


Let the two triangles ABC, DEF, have the angle A equal to the angle D, the angle B equal to the angle E, and the side AB equal to the side DE; then these two triangles will be identical.

For, conceive the triangle ABC to be placed on the triangle DEF, in such manner that the side AB may fall exactly on the equal side DE. Then, since the angle A is equal to the angle D (by hyp.), the side AC must fall on the side DF; and, in like manner, because the angle B is equal to the angle E, the side BC must fall on the side EF. Thus the three sides of the triangle ABC will be exactly placed on the three sides of the triangle DEF: consequently the two triangles are identical (ax. 9), having the other two sides AC, BC, equal to the two DF, EF, and the remaining angle C equal to the remaining angle F. Q. E. D.

THEOREM III.

In an isosceles triangle, the angles at the base are equal. Or, if a triangle have two sides equal, their opposite angles will also be equal.



If the triangle ABC have the side AC equal to the side BC: then will the angle B be equal to the angle A.

For, conceive the angle C to be bisected, or divided into two equal parts, by the line CD, making the angle ACD equal to the angle BCD.

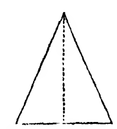
Then, the two triangles ACD, BCD, have two sides and the contained angle of the one, equal to two sides and the contained angle of the other, *viz.* the side AC equal to BC, the angle ACD equal to BCD, and the side CD common; therefore these two triangles are identical, or equal in all respects (th. 1); and consequently the angle A equal to the angle B. Q. E. D.

Corol. 1. Hence the line which bisects the vertical angle of an isosceles triangle, bisects the base, and is also perpendicular to it.

Corol. 2. Hence too it appears, that every equilateral triangle, is also equiangular, or has all its angles equal.

THEOREM IV.

When a triangle has two of its angles equal, the sides opposite to them are also equal.



If the triangle ABC, have the angle A equal to the angle B, it will also have the side AC equal to the side BC.

For, conceive the side AB to be bisected in the point D, making AD equal to DB; and join DC, dividing the whole triangle into the two triangles ACD, BCD. Also conceive the triangle ACD to be turned over upon the triangle BCD, so that AD may fall on BD. (th. 3, Cor. 1)

Then, because the line AD is equal to the line DB (by hyp.), the point A coincides with the point B, and the point D with the point D. Also, because the angle A is equal to the angle B (by hyp.), the line AC will fall on the line BC, and the extremity C of the side AC will coincide with the extremity C of

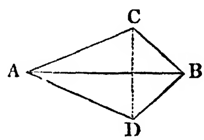
the side BC, because DC is common to both; consequently the side AC is equal to BC. Q. E. D.

Corol. Hence every equiangular triangle is also equilateral.

THEOREM V.

When two triangles have all the three sides in the one, equal to all the three sides in the other, the triangles are identical, or have also their three angles equal, each to each.

Let the two triangles ABC, ABD, have their three sides respectively equal, viz. the side AB equal to AB, AC to AD, and BC to BD; then shall the two triangles be identical, or have their angles equal, *viz.* those angles that are opposite to the equal sides; namely, the angle BAC to the angle BAD, the angle ABC to the angle ABD, and the angle C to the angle D.



For, conceive the two triangles to be joined together by their longest equal sides, and draw the line CD.

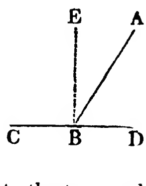
Then, in the triangle ACD, because the side AC is equal to AD (by hyp.), the angle ACD is equal to the angle ADC (th. 3). In like manner, in the triangle BCD, the angle BCD is equal to the angle BDC, because the side BC is equal to BD. Hence then, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, by equal additions the sum of the two angles, ACD, BCD, is equal to the sum of the two ADC, BDC, (ax. 2), that is, the whole angle ACB equal to the whole angle ADB.

Since, then, the two sides AC, CB, are equal to the two sides AD, DB, each to each, (by hyp.), and their contained angles ACB, ADB, also equal, the two triangles ABC, ABD, are identical (th. 1), and have the other angles equal, *viz.* the angle BAC to the angle BAD, and the angle ABC to the angle ABD. Q. E. D.

THEOREM VI.

When one line meets another, the angles which it makes on the same side of the other, are together equal to two right angles.

Let the line AB meet the line CD: then will the two angles ABC, ABD, taken together, be equal to two right angles.



For, first, when the two angles ABC, ABD, are equal to each other, they are both of them right angles (def. 15).

But when the angles are unequal, suppose BE drawn perpendicular to CD. Then, since the two angles EBC, EBD, are right angles (def. 15), and the angle EBD is equal to the two angles EBA, ABD, together (ax. 8), the three angles, EBC, EBA, and ABD, are equal to two right angles.

But the two angles EBC, EBA, are together equal to the angle ABC (ax. 8). Consequently, the two angles ABC, ABD, are also equal to two right angles. Q. E. D.

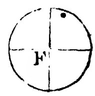
Corol. 1. Hence also, conversely, if the two angles ABC, ABD, on both

sides of the line AB, make up together two right angles, then CB and BD form one continued right line CD.

Corol. 2. Hence, all the angles which can be made, at any point B, by any number of lines, on the same side of the right line CD, are, when taken all together, equal to two right angles.

Corol. 3. And, as all the angles that can be made on the other side of the line CD are also equal to two right angles; therefore, all the angles that can be made quite round a point B, by any number of lines, are equal to four right angles.

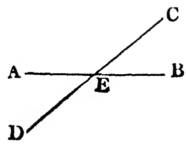
Corol. 4. Hence, also, the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the centre F (def. 57), is the measure of four right angles. Consequently, a semicircle, or 180 degrees, is the measure of two right angles; and a quadrant, or 90 degrees, the measure of one right angle.



THEOREM VII.

When two lines intersect each other, the opposite angles are equal.

Let the two lines AB, CD, intersect in the point E; then will the angle AEC be equal to the angle BED, and the angle AED be equal to the angle CEB.



For, since the line CE meets the line AB, the two angles AEC, BEC, taken together, are equal to two right angles (th. 6).

In like manner, the line BE, meeting the line CD, makes the two angles BEC, BED, equal to two right angles.

Therefore, the sum of the two angles AEC, BEC, is equal to the sum of the two BEC, BED (ax. 1).

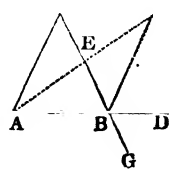
And if the angle BEC, which is common, be taken away from both these, the remaining angle AEC will be equal to the remaining angle BED (ax. 3).

And in like manner it may be shown, that the angle AED is equal to the opposite angle BEC.

THEOREM VIII.

When one side of a triangle is produced, the outward angle is greater than either of the two inward opposite angles.

Let ABC be a triangle, having the side AB produced to D; then will the outward angle CBD be greater than either of the inward opposite angles A or C.



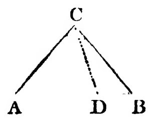
For, conceive the side BC to be bisected in the point E, and draw the line AE, producing it till EF be equal to AE; and join BF.

Then, since the two triangles AEC, BEF, have the side AE = the side EF, and the side CE = the side BE (by suppos.), and the included or opposite angles at E also equal (th. 7), therefore, those two triangles are equal in all respects (th. 1), and have the angle C = the corresponding angle EBF. But the angle CBD is greater than the angle EBF; consequently, the said outward angle CBD is also greater than the angle C.

In like manner, if CB be produced to G, and AB be bisected, it may be shown that the outward angle ABG, or its equal CBD, is greater than the other angle A.

THEOREM IX.

The greater side, of every triangle, is opposite to the greater angle; and the greater angle opposite to the greater side.



Let ABC be a triangle, having the side AB greater than the side AC; then will the angle ACB, opposite the greater side AB, be greater than the angle B, opposite the less side AC.

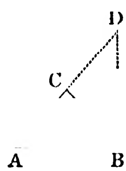
For, on the greater side AB, take the part AD equal to the less side AC, and join CD. Then, since BCD is a triangle, the outward angle ADC is greater than the inward opposite angle B (th. 8). But the angle ACD is equal to the said outward angle ADC, because AD is equal to AC (th. 3). Consequently, the angle ACD also is greater than the angle B. And since the angle ACD is only a part of ACB, much more must the whole angle ACB be greater than the angle B. Q. E. D.

Again, conversely, if the angle C be greater than the angle B, then will the side AB, opposite the former, be greater than the side AC, opposite the latter.

For, if AB be not greater than AC, it must be either equal to it, or less than it. But it cannot be equal, for then the angle C would be equal to the angle B (th. 3), which it is not, by the supposition. Neither can it be less, for then the angle C would be less than the angle B, by the former part of this; which is also contrary to the supposition. The side AB, then, being neither equal to AC, nor less than it, must necessarily be greater. Q. E. D.

THEOREM X.

The sum of any two sides of a triangle is greater than the third side.



Let ABC be a triangle; then will the sum of any two of its sides be greater than the third side, as for instance, AC + CB greater than AB.

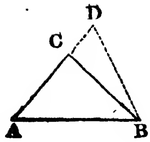
For, produce AC till CD be equal to CB, or AD equal to the sum of the two AC + CB; and join BD:—Then, because CD is equal to CB (by constr.), the angle D is equal to the angle CBD (th. 3). But the angle ABD is greater than the angle CBD, consequently, it must also be greater than the angle D. And, since the greater side of any triangle is opposite to the greater angle (th. 9), the side AD (of the triangle ABD) is greater than the side AB. But AD is equal to AC and CD, or AC and CB, taken together (by constr.); therefore, AC + CB is also greater than AB. Q. E. D.

Corol. The shortest distance between two points, is a single right line drawn from the one point to the other.

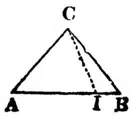
THEOREM XI.

The difference of any two sides of a triangle, is less than the third side.

Let ABC be a triangle; then will the difference of any two sides, as AB — AC, be less than the third side BC.



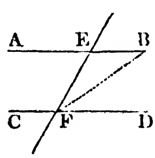
For, produce the less side AC to D, till AD be equal to the greater side AB, so that CD may be the difference of the two sides AB — AC; and join BD. Then, because AD is equal to AB (by constr.), the opposite angles D and ABD are equal (th. 3). But the angle CBD is less than the angle ABD, and consequently also less than the equal angle D. And since the greater side of any triangle is opposite to the greater angle (th. 9), the side CD (of the triangle BCD) is less than the side BC. Q. E. D.



Otherwise. Set off upon AB a distance AI equal to AC. Then (th. 20) AC + CB is greater than AB, that is, greater than AI + IB. From these, take away the equal parts, AC, AI, respectively; and there remains CB greater than IB. Consequently, IB is less than CB. Q. E. D.

THEOREM XII.

When a line intersects two parallel lines, it makes the alternate angles equal to each other.



Let the line EF cut the two parallel lines AB, CD; then will the angle AEF be equal to the alternate angle EFD.

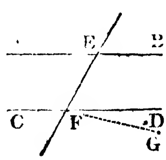
For if they are not equal, one of them must be greater than the other; let it be EFD for instance which is the greater, if possible; and conceive the line FB to be drawn, cutting off the part or angle EFB equal to the angle AEF, and meeting the line AB in the point B.

Then, since the outward angle AEF, of the triangle BEF, is greater than the inward opposite angle EFB (th. 8); and since these two angles also are equal (by the constr.) it follows, that these angles are both equal and unequal at the same time: which is impossible. Therefore the angle EFD is not unequal to the alternate angle AEF, that is, they are equal to each other. Q. E. D.

Corol. Right lines which are perpendicular to one, of two parallel lines, are also perpendicular to the other.

THEOREM XIII.

When a line, cutting two other lines, makes the alternate angles equal to each other, those two lines are parallel.



Let the line EF, cutting the two lines AB, CD, make the alternate angles AEF, DFE, equal to each other; then will AB be parallel to CD.

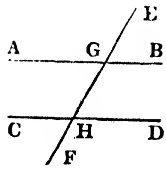
For if they be not parallel, let some other line, as FG, be parallel to AB. Then, because of these parallels, the angle AEF is equal to the alternate angle EFG (th. 12). But the angle AEF is equal to the angle EFD (by hyp.) Therefore the angle EFD is equal to the angle EFG (ax. 1); that is, a part is equal to the whole, which is impossible. Therefore no line but CD can be parallel to AB. Q. E. D.

Corol. Those lines which are perpendicular to the same line, are parallel to each other.

THEOREM XIV.

When a line cuts two parallel lines, the outward angle is equal to the inward opposite one, on the same side; and the two inward angles, on the same side, are together equal to two right angles.

Let the line EF cut the two parallel lines AB, CD; then will the outward angle EGB be equal to the inward opposite angle GHD, on the same side of the line EF; and the two inward angles BGH, GHD, taken together, will be equal to two right angles.



For since the two lines AB, CD, are parallel, the angle AGH is equal to the alternate angle GHD, (th. 12). But the angle AGH is equal to the opposite angle EGB (th. 7). Therefore the angle EGB is also equal to the angle GHD (ax. 1). Q. E. D.

Again, because the two adjacent angles EGB, BGH, are together equal to two right angles (th. 6); of which the angle EGB has been shown to be equal to the angle GHD; therefore the two angles BGH, GHD, taken together, are also equal to two right angles.

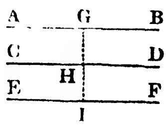
Corol. 1. And, conversely, if one line meeting two other lines, make the angles on the same side of it equal, those two lines are parallels.

Corol. 2. If a line, cutting two other lines, make the sum of the two inward angles on the same side, less than two right angles, those two lines will not be parallel, but will meet each other when produced.

THEOREM XV.

Those lines which are parallel to the same line, are parallel to each other.

Let the lines AB, CD, be each of them parallel to the line EF; then shall the lines AB, CD, be parallel to each other.

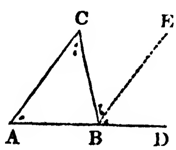


For, let the line GI be perpendicular to EF. Then will this line be also perpendicular to both the lines AB, CD (corol. th. 12), and consequently the two lines AB, CD, are parallels (corol. th. 13). Q. E. D

THEOREM XVI.

When one side of a triangle is produced, the outward angle is equal to both the inward opposite angles taken together.

Let the side AB, of the triangle ABC, be produced to D; then will the outward angle CBD be equal to the sum of the two inward opposite angles A and C.

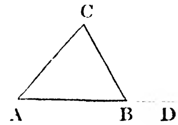


For, conceive BE to be drawn parallel to the side AC of the triangle. Then BC, meeting the two parallels AC, BE, makes the alternate angles C and CBE equal (th. 12). And AD, cutting the same two parallels AC, BE, makes the inward and outward angles on the same side, A and EBD, equal

to each other (th. 14). Therefore, by equal additions, the sum of the two angles A and C, is equal to the sum of the two CBE and EBD, that is, to the whole angle CBD (by ax. 2). Q. E. D.

THEOREM XVII.

In any triangle, the sum of all the three angles is equal to two right angles.



Let ABC be any plane triangle; then the sum of the three angles A + B + C is equal to two right angles.

For, let the side AB be produced to D. Then the outward angle CBD is equal to the sum of the two inward opposite angles A + C (th. 16). To each of these equals add the inward angle B, then will the sum of the three inward angles A + B + C be equal to the sum of the two adjacent angles ABC + CBD (ax. 2). But the sum of these two last adjacent angles is equal to two right angles (th. 6). Therefore also the sum of the three angles of the triangle A + B + C is equal to two right angles (ax. 1). Q. E. D.

Corol. 1. If two angles in one triangle, be equal to two angles in another triangle, the third angles will also be equal (ax. 3), and the two triangles equiangular.

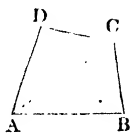
Corol. 2. If one angle in one triangle, be equal to one angle in another, the sums of the remaining angles will also be equal (ax. 3).

Corol. 3. If one angle of a triangle be right, the sum of the other two will also be equal to a right angle, and each of them singly will be acute, or less than a right angle.

Corol. 4. The two least angles of every triangle are acute, or each less than a right angle.

THEOREM XVIII.

In any quadrangle, the sum of all the four inward angles, is equal to four right angles.



Let ABCD be a quadrangle; then the sum of the four inward angles, A + B + C + D is equal to four right angles.

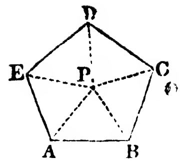
Let the diagonal AC be drawn, dividing the quadrangle into two triangles, ABC, ADC. Then, because the sum of the three angles of each of these triangles is equal to two right angles (th. 17); it follows, that the sum of all the angles of both triangles, which make up the four angles of the quadrangle, must be equal to four right angles (ax. 2). Q. E. D.

Corol. 1. Hence, if three of the angles be right ones, the fourth will also be a right angle.

Corol. 2. And if the sum of two of the four angles be equal to two right angles, the sum of the remaining two will also be equal to two right angles.

THEOREM XIX.

In any figure whatever, the sum of all the inward angles, taken together, is equal to twice as many right angles, wanting four, as the figure has sides.

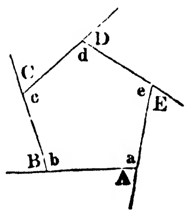


Let ABCDE be any figure; then the sum of all its inward angles, A + B + C + D + E, is equal to twice as many right angles, wanting four, as the figure has sides.

For, from any point P, within it, draw lines, PA, PB, PC, &c. to all the angles, dividing the polygon into as many triangles as it has sides. Now the sum of the three angles of each of these triangles, is equal to two right angles (th. 17); therefore the sum of the angles of all the triangles is equal to twice as many right angles as the figure has sides. But the sum of all the angles about the point P, which are so many of the angles of the triangles, but no part of the inward angles of the polygon, is equal to four right angles (corol. 3, th. 6), and must be deducted out of the former sum. Hence it follows, that the sum of all the inward angles of the polygon alone, A + B + C + D + E, is equal to twice as many right angles as the figure has sides, wanting the said four right angles. Q. E. D.

THEOREM XX.

When every side of any figure is produced out, the sum of all the outward angles thereby made, is equal to four right angles.

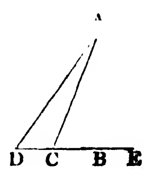


Let A, B, C, &c. be the outward angles of any polygon, made by producing all the sides; then will the sum A + B + C + D + E, of all those outward angles, be equal to four right angles.

For every one of these outward angles, together with its adjacent inward angle, make up two right angles, as A + *a* equal to two right angles, being the two angles made by one line meeting another (th. 6). And there being as many outward, or inward angles, as the figure has sides; therefore the sum of all the inward and outward angles, is equal to twice as many right angles as the figure has sides; therefore the sum of all the inward and outward angles, is equal to twice as many right angles as the figure has sides. But the sum of all the inward angles, with four right angles, is equal to twice as many right angles as the figure has sides (th. 19). Therefore the sum of all the inward and all the outward angles, is equal to the sum of all the inward angles and four right angles (by ax. 1). From each of these take away all the inward angles, and there remain all the outward angles equal to four right angles (by ax. 3). Q. E. D.

THEOREM XXI.

A perpendicular is the shortest line that can be drawn from a given point to an indefinite line. And, of any other lines drawn from the same point, those that are nearest the perpendicular are less than those more remote.



If AB, AC, AD, &c. be lines drawn from the given point A, to the indefinite line DE, of which AB is perpendicular; then shall the perpendicular AB be less than AC, and AC less than AD, &c.

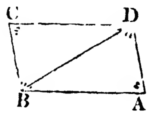
For, the angle B being a right one, the angle C is acute, (by cor. 3, th. 17), and therefore less than the angle B. But the less angle of a triangle is subtended by the less side (th. 9). Therefore the side AB is less than the side AC.

Again, the angle ACB being acute, as before, the adjacent angle ACD will be obtuse (by th. 6); consequently the angle D is acute (corol. 3, th. 17), and therefore is less than the angle C. And since the less side is opposite to the less angle, therefore the side AC is less than the side AD. Q. E. D.

Corol. A perpendicular is the least distance of a given point from a line.

THEOREM XXII.

The opposite sides and angles of any parallelogram are equal to each other; and the diagonal divides it into two equal triangles.



Let ABCD be a parallelogram, of which the diagonal is BD; then will its opposite sides and angles be equal to each other, and the diagonal BD will divide it into two equal parts, or triangles.

For, since the sides AB and DC are parallel, as also the sides AD and BC (defin. 32), and the line BD meets them; therefore the alternate angles are equal (th. 12), namely, the angle ABD to the angle CDB, and the angle ADB to the angle CBD. Hence the two triangles, having two angles in the one equal to two angles in the other, have also their third angles equal (cor. 1, th. 17), namely, the angle A equal to the angle C, which are two of the opposite angles of the parallelogram.

Also, if to the equal angles ABD, CDB, be added the equal angles CBD, ADB, the wholes will be equal (ax. 2), namely, the whole angle ABC to the whole ADC, which are the other two opposite angles of the parallelogram. Q. E. D.

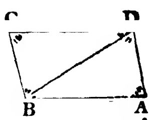
Again, since the two triangles are mutually equiangular, and have a side in each equal, viz. the common side BD; therefore the two triangles are identical (th. 2), or equal in all respects, namely, the side AB equal to the opposite side DC, and AD equal to the opposite side BC, and the whole triangle ABD equal to the whole triangle BCD. Q. E. D.

Corol. 1. Hence, if one angle of a parallelogram be a right angle, all the other three will also be right angles, and the parallelogram a rectangle.

Corol. 2. Hence also, the sum of any two adjacent angles of a parallelogram is equal to two right angles.

THEOREM XXIII.

Every quadrilateral, whose opposite sides are equal, is a parallelogram, or has its opposite sides parallel.



Let ABCD be a quadrangle, having the opposite sides equal, namely, the side AB equal to DC, and AD equal to BC; then shall these equal sides be also parallel, and the figure a parallelogram.

For, let the diagonal BD be drawn. Then, the triangles, ABD, CBD, being mutually equilateral (by hyp.), they are also mutually equiangular (th. 5), or have their corresponding angles equal; consequently the opposite sides are parallel (th. 13); viz. the side AB parallel to DC, and AD parallel to BC, and the figure is a parallelogram. Q. E. D.

THEOREM XXIV.

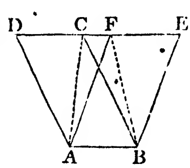
Those lines which join the corresponding extremes of two equal and parallel lines, are themselves equal and parallel.

Let AB, DC, be two equal and parallel lines; then will the lines AD, BC, which join their extremes, be also equal and parallel. [See the fig. above.]

For, draw the diagonal BD. Then, because AB and DC are parallel (by hyp.), the angle ABD is equal to the alternate angle BDC (th. 12). Hence then, the two triangles having two sides and the contained angles equal, viz. the side AB equal to the side DC, and the side BD common, and the contained angle ABD equal to the contained angle BDC, they have the remaining sides and angles also respectively equal (th. 1); consequently AD is equal to BC, and also parallel to it (th. 12). Q. E. D.

THEOREM XXV.

Parallelograms, as also triangles, standing on the same base, and between the same parallels, are equal to each other.



Let ABCD, ABEF, be two parallelograms, and ABC, ABF, two triangles, standing on the same base, AB, and between the same parallels AB, DE; then will the parallelogram ABCD be equal to the parallelogram ABEF, and the triangle ABC equal to the triangle ABF.

For, since the line DE cuts the two parallels AF, BE, and the two AD, BC, it makes the angle E equal to the angle AFD, and the angle D equal to the angle BCE (th. 14); the two triangles ADF, BCE, are therefore equiangular (cor. 1, th. 17); and having the two corresponding sides AD, BC, equal (th. 22), being opposite sides of a parallelogram, these two triangles are identical, or equal in all respects (th. 2). If each of these equal triangles then be taken from the whole space ABED, there will remain the parallelogram ABEF in the one case, equal to the parallelogram ABCD in the other (by ax. 3).

Also the triangles ABC, ABF, on the same base AB, and between the same parallels, are equal, being the halves of the said equal parallelograms (th. 22). Q. E. D.

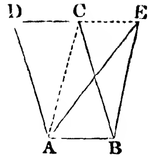
Corol. 1. Parallelograms, or triangles, having the same base and altitude, are equal. For the altitude is the same as the perpendicular or distance between the two parallels, which is every where equal, by the definition of parallels.

Corol. 2. Parallelograms, or triangles, having equal bases and altitudes, are equal. For, if the one figure be applied with its base on the other, the bases will coincide or be the same, because they are equal: and so the two figures, having the same base and altitude, are equal.

THEOREM XXVI.

If a parallelogram and a triangle, stand on the same base, and between the same parallels, the parallelogram will be double the triangle, or the triangle half the parallelogram.

Let ABCD be a parallelogram, and ABE a triangle, on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be double the triangle ABE, or the triangle half the parallelogram.



For, draw the diagonal AC of the parallelogram, dividing it into two equal parts (th. 22). Then because the triangles ABC, ABE, on the same base, and between the same parallels, are equal (th. 25); and because the one triangle ABC is half the parallelogram ABCD (th. 22), the other equal triangle ABE is also equal to half the same parallelogram ABCD. Q. E. D.

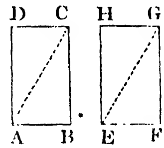
Corol. 1. A triangle is equal to half a parallelogram of the same base and altitude, because the altitude is the perpendicular distance between the parallels, which is everywhere equal, by the definition of parallels.

Corol. 2. If the base of a parallelogram be half that of a triangle, of the same altitude, or the base of the triangle be double that of the parallelogram the two figures will be equal to each other.

THEOREM XXVII.

Rectangles that are contained by equal lines, are equal to each other.

Let BD, FH, be two rectangles, having the sides AB, BC, equal to the sides EF, FG, each to each; then will the rectangle BD be equal to the rectangle FH.



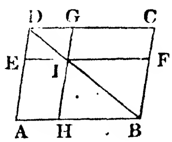
For, draw the two diagonals AC, EG, dividing the two parallelograms each into two equal parts. Then the two triangles ABC, EFG, are equal to each other (th. 1), because they have the two sides AB, BC, and the contained angle B, equal to the two sides EF, FG, and the contained angle F (by hyp). But these equal triangles are the halves of the respective rectangles. And because the halves, or the triangles, are equal, the wholes, or the rectangles DB, HF, are also equal (by ax. 6). Q. E. D.

Corol. The squares on equal lines are also equal; for every square is a species of rectangle.

THEOREM XXVIII.

The complements of the parallelograms, which are about the diagonal of any parallelogram, are equal to each other.

Let AC be a parallelogram, BD a diagonal, EIF parallel to AB or DC, and GIH parallel to AD or BC, making AI, IC, complements to the parallelograms EG, HF, which are about the diagonal DB: then will the complement AI be equal to the complement IC.

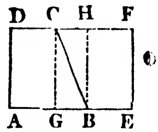


For, since the diagonal DB bisects the three parallelograms AC, EG, HF (th. 22); therefore, the whole triangle DAB being equal to the whole triangle DCB, and the parts DEI, IHB respectively equal to the parts DGI, IFB, the remaining parts AI, IC, must also be equal (by ax. 3). Q. E. D.

THEOREM XXIX.

A trapezoid, or trapezium having two sides parallel, is equal to half a para-

lelogram, whose base is the sum of these two sides, and its altitude the perpendicular distance between them.

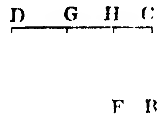


Let ABCD be the trapezoid, having its two sides AB, DC, parallel; and in AB produced take BE equal to DC, so that AE may be the sum of the two parallel sides; produce DC also, and let EF, GC, BH, be all three parallel to AD. Then is AF a parallelogram of the same altitude with the trapezoid ABCD, having its base AE equal to the sum of the parallel sides of the trapezoid; and it is to be proved that the trapezoid ABCD is equal to half the parallelogram AF.

Now, since triangles, or parallelograms, of equal bases and altitude, are equal (corol. 2, th. 25), the parallelogram DG is equal to the parallelogram HE, and the triangle CGB equal to the triangle CHB; consequently, the line BC bisects, or equally divides, the parallelogram AF, and ABCD is the half of it. Q.E.D.

THEOREM XXX.

The sum of all the rectangles contained under one whole line, and the several parts of another line, any way divided, is equal to the rectangle contained under the two whole lines.



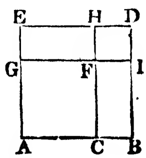
Let AD be the one line, and AB the other, divided into the parts AE, EF, FB; then will the rectangle contained by AD and AB, be equal to the sum of the rectangles of AD and AE, and AD and EF, and AD and FB: thus expressed, $AD \cdot AB = AD \cdot AE + AD \cdot EF + AD \cdot FB$.

For, make the rectangle AC of the two whole lines AD, AB; and draw EG, FH, perpendicular to AB, or parallel to AD, to which they are equal (th. 22). Then the whole rectangle AC is made up of all the other rectangles AG, EH, FC. But these rectangles are contained by AD and AE, EG and EF, FH and FB; which are equal to the rectangles of AD and AE, AD and EF, AD and FB, because AD is equal to each of the two EG, FH. Therefore, the rectangle $AD \cdot AB$ is equal to the sum of all the other rectangles $AD \cdot AE$, $AD \cdot EF$, $AD \cdot FB$. Q. E. D.

Corol. If a right line be divided into any two parts, the square on the whole line, is equal to both the rectangles of the whole line and each of the parts.

THEOREM XXXI.

The square of the sum of two lines, is greater than the sum of their squares, by twice the rectangle of the said lines. Or, the square of a whole line is equal to the squares of its two parts, together with twice the rectangle of those parts.



Let the line AB be the sum of any two lines AC, CB; then will the square of AB be equal to the squares of AC, CB, together with twice the rectangle of $AC \cdot CB$. That is, $AB^2 = AC^2 + CB^2 + 2\,AC \cdot CB$.

For, let ABDE be the square on the sum or whole line AB, and ACFG the square on the part AC. Produce CF and GF to the other sides at H and I.

From the lines CH, GI, which are equal, being each equal to the sides of the

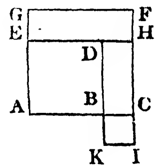
square AB or BD (th. 22), take the parts CF, GF, which are also equal, being the sides of the square AF, and there remains FH equal to FI, which are also equal to DH, DI, being the opposite sides of the parallelogram. Hence, the figure HI is equilateral: and it has all its angles right ones (corol. 1. th. 22); it is therefore a square on the line FI, or the square of its equal CB. Also the figures EF, FB, are equal to two rectangles under AC and CB; because GF is equal to AC, and FH or FI equal to CB. But the whole square AD is made up of the four figures, viz. the two squares AF, FD, and the two equal rectangles EF, FB. That is, the square of AB is equal to the squares of AC, CB, together with twice the rectangle of AC, CB. Q. E. D.

Corol. Hence, if a line be divided into two equal parts; the square of the whole line will be equal to four times the square of half the line.

THEOREM XXXII.

The square of the difference of two lines, is less than the sum of their squares by twice the rectangle of the said lines.

Let AC, BC, be any two lines, and AB their difference: then will the square of AB be less than the squares of AC, BC, by twice the rectangle of AC and BC. Or, $AB^2 = AC^2 + BC^2 - 2\,AC \cdot BC$.



For, let ABDE be the square on the difference AB, and ACFG the square on the line AC. Produce ED to H; also produce DB and HC, and draw KI, making BI the square of the other line BC.

Now, it is visible that the square AD is less than the two squares AF, BI, by the two rectangles EF, DI. But GF is equal to the one line AC, and GE or FH is equal to the other line BC; consequently, the rectangle EF, contained under EG and GF, is equal to the rectangle of AC and BC.

Again, FH being equal to CI or BC or DH, by adding the common part HC, the whole HI will be equal to the whole FC, or equal to AC; and consequently, the figure DI is equal to the rectangle contained by AC and BC.

Hence, the two figures EF, DI, are two rectangles of the two lines AC, BC; and consequently the square of AB is less than the squares of AC, BC, by twice the rectangle AC . BC. Q. E. D.

THEOREM XXXIII.

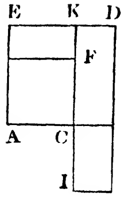
The rectangle under the sum and difference of two lines, is equal to the difference of the squares of those lines. *

* This and the two preceding theorems, are evinced algebraically, by the three expressions,

$$(a+b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab$$
$$(a-b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2ab$$
$$(a+b)(a-b) = a^2 - b^2$$

Let AB, AC, be any two unequal lines; then will the difference of the squares of AB, AC, be equal to a rectangle under their sum and difference. That is,

$$AB^2 - AC^2 = \overline{AB + AC} \cdot \overline{AB - AC}.$$



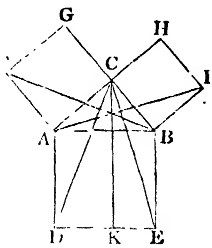
For, let ABDE be the square of AB, and ACFG the square of AC, Produce DB till BH be equal to AC; draw HI parallel to AB or ED, and produce FC both ways to I and K.

Then the difference of the two squares AD, AF, is evidently the two rectangles EF, KB. But the rectangles EF, BI, are equal, being contained under equal lines; for EK and BH are each equal to AC, and GE is equal to CB, being each equal to the difference between AB and AC, or their equals AE and AG. Therefore, the two EF, KB, are equal to the two KB, BI, or to the whole KH; and consequently KH is equal to the difference of the squares AD, AF. But KH is a rectangle contained by DH, or the sum of AB and AC, and by KD, or the difference of AB and AC. Therefore, the difference of the squares of AB, AC, is equal to the rectangle under their sum and difference. Q. E. D.

THEOREM XXXIV.

In any right-angled triangle, the square of the hypothenuse is equal to the sum of the squares of the other two sides.

Let ABC be a right-angled triangle, having the right angle C; then will the square of the hypothenuse AB, be equal to the sum of the squares of the other two sides AC, CB. Or $AB^2 = AC^2 + BC^2$.



For, on AB describe the square AE, and on AC, CB, the squares AG, BH; then draw CK parallel to AD or BE; and join AI, BF, CD, CE.

Now, because the line AC meets the two CG, CB, so as to make two right angles, these two form one straight line GB (corol. 1, th. 6). And because the angle FAC is equal to the angle DAB, being each a right angle, or the angle of a square; to each of these equals add the common angle BAC, so will the whole angle or sum FAB, be equal to the whole angle or sum CAD. But the line FA is equal to the line AC, and the line AB to the line AD, being sides of the same square; so that the two sides FA, AB, and their included angle FAB, are equal to the two sides CA, AD, and the contained angle CAD, each to each: therefore, the whole triangle AFB is equal to the whole triangle ACD (th. 1).

But the square AG is double the triangle AFB, on the same base FA, and between the same parallels FA, GB (th. 26); in like manner, the parallelogram AK is double the triangle ACD, on the same base AD, and between the same parallel AD, CK. And since the doubles of equal things are equal (by ax. 6); therefore, the square AG is equal to the parallelogram AK.

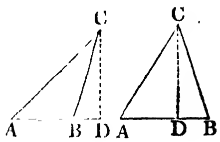
In like manner, the other square BH is proved equal to the other parallelogram BK. Consequently, the two squares AG and BH together, are equal to the two parallelograms AK and BK together, or to the whole square AE. That is, the sum of the two squares on the two less sides, is equal to the square on the greatest side. Q. E. D

Corol. 1. Hence, the square of either of the two less sides, is equal to the difference of the squares of the hypothenuse and the other side (ax. 3); or, equal to the rectangle contained by the sum and difference of the said hypothenuse and other side (th. 33).

Corol. 2. Hence, also, if two right-angled triangles have two sides of the one equal to two corresponding sides of the other; their third sides will also be equal, and the triangles identical.

THEOREM XXXV.

In any triangle, the difference of the squares of the two sides, is equal to the difference of the squares of the segments of the base, or of the two lines, or distances, included between the extremes of the base and the perpendicular.



Let ABC be any triangle, having CD perpendicular to AB; then will the difference of the squares of AC, BC, be equal to the difference of the squares of AD, BD; that is, $AC^2 - BC^2 = AD^2 - BD^2$.

For, since AC^2 is equal to $AD^2 + CD^2$ and BC^2 is equal to $BD^2 + CD^2$ } (by th. 34);

Therefore the difference between AC^2 and BC^2, is equal to the difference between $AD^2 + CD^2$ and $BD^2 + CD^2$, or equal to the difference between AD^2 and BD^2, by taking away the common square CD^2. Q. E. D.

Corol. The rectangle of the sum and difference of the two sides of any triangle, is equal to the rectangle of the sum and difference of the distances between the perpendicular and the two extremes of the base, or equal to the rectangle of the base and the difference or sum of the segments, according as the perpendicular falls within or without the triangle.

That is, $(AC + BC) . (AC - BC) = (AD + BD) . (AD - BD)$
Or, $(AC + BC) . (AC - BC) = AB . (AD - BD)$ in the 2d fig.
And, $(AC + BC) . (AC - BC) = AB . (AD + BD)$ in the 1st fig.

THEOREM XXXVI.

In any obtuse-angled triangle, the square of the side subtending the obtuse angle, is greater than the sum of the squares of the other two sides, by twice the rectangle of the base and the distance of the perpendicular from the obtuse angle.

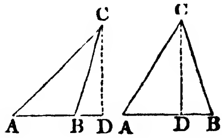
Let ABC be a triangle, obtuse angled at B, and CD perpendicular to AB; then will the square of AC be greater than the squares of AB, BC, by twice the rectangle of AB, BD. That is, $AC^2 = AB^2 + BC^2 + 2 AB . BD$. See the 1st fig. above.

For, $AD^2 = AB^2 + BD^2 + 2 AB . BD$ (th. 31).
And $AD^2 + CD^2 = AB^2 + BD^2 + CD^2 + 2 AB . BD$ (ax. 2.)
But $AD^2 + CD^2 = AC^2$, and $BD^2 + CD^2 = BC^2$ (th. 34).
Therefore $AC^2 = AB^2 + BC^2 + 2 AB . BD$. Q. E. D.

THEOREM XXXVII.

In any triangle, the square of the side subtending an acute angle, is less than the squares of the base and the other side, by twice the rectangle of the base and the distance of the perpendicular from the acute angle.

Let ABC be a triangle, having the angle A acute, and CD perpendicular to AB; then will the square of BC be less than the squares of AB, AC, by twice the rectangle of AB, AD. That is, $BC^2 = AB^2 + AC^2 - 2\,AD \,.\, AB$.

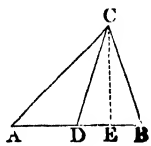


For, $BD^2 = AD^2 + AB^2 - 2\,AD \,.\, AB$ (th. 32).
And $BD^2 + DC^2 = AD^2 + DC^2 + AB^2 - 2\,AD \,.\, AB$ (ax. 2).
Therefore $BC^2 = AC^2 + AB^2 - 2\,AD \,.\, AB$ (th. 34). Q. E. D.

THEOREM XXXVIII.

In any triangle, the double of the square of a line drawn from the vertex to the middle of the base, together with double the square of the half base, is equal to the sum of the squares of the other two sides.

Let ABC be a triangle, and CD the line drawn from the vertex to the middle of the base AB, bisecting it into the two equal parts AD, DB; then will the sum of the squares of AC, CB, be equal to twice the sum of the squares of CD, AD; or $AC^2 + CB^2 = 2\,CD^2 + 2\,AD^2$.

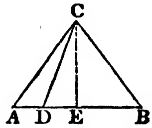


For, $AC^2 = CD^2 + AD^2 + 2\,AD \,.\, DE$ (th. 36).
And, $BC^2 = CD^2 + BD^2 - 2\,AD \,.\, DE$ (th. 37).
Therefore, $AC^2 + BC^2 = 2CD^2 + AD^2 + BD^2$
$= 2CD^2 + 2AD^2$ (ax. 2). Q. E. D.

THEOREM XXXIX.

In an isosceles triangle, the square of a line drawn from the vertex to any point in the base, together with the rectangle of the segments of the base, is equal to the square of one of the equal sides of the triangle.

Let ABC be the isosceles triangle, and CD a line drawn from the vertex to any point D in the base: then will the square of AC be equal to the square of CD, together with the rectangle of AD and DB. That is, $AC^2 = CD^2 + AD \,.\, DB$.

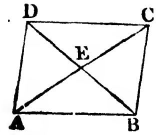


For $AC^2 - CD^2 = AE^2 - DE^2$ (th. 35).
$= AD \,.\, DB$ (th. 33).
Therefore $AC^2 = CD^2 + AD \,.\, DB$ (ax. 2). Q. E. D.

THEOREM XL.

In any parallelogram, the two diagonals bisect each other; and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.

Let ABCD be a parallelogram, whose diagonals intersect each other in E: then will AE be equal to EC, and BE to ED; and the sum of the squares of AC, BD, will be equal to the sum of the squares of AB, BC, CD, DA. That is,



$$AE = EC, \text{ and } BE = ED,$$
$$\text{and } AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2,$$

For, the triangles AEB, DEC, are equiangular, because they have tne opposite angles at E equal (th. 7), and the two lines AC, BD, meeting the parallels AB, DC, make the angle BAE equal to the angle DCE, and the angle ABE equal to the angle CDE, and the side AB equal to the side DC (th. 22); therefore, these two triangles are identical, and have their corresponding sides equal (th. 2), viz. $AE = EC$, and $BE = ED$.

Again, since AC is bisected in E, the sum of the squares $AD^2 + DC^2 = 2AE^2 + 2DE^2$ (th. 38).

In like manner, $AB^2 + BC^2 = 2AE^2 + 2BE^2$ or $2DE^2$.

Therefore, $AB^2 + BC^2 + CD^2 + DA^2 = 4AE^2 + 4DE^2$ (ax. 2).

But, because the square of a whole line is equal to 4 times the square of half the line (cor. th. 31), that is, $AC^2 = 4AE^2$, and $BD^2 = 4DE^2$:

Therefore, $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$ (ax. 1). Q. E. D.

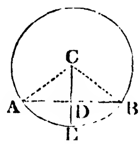
Cor. 1. If $AD = DC$, or the parallelogram be a rhombus; then $AD^2 = AE^2 + ED^2$, $CD^2 = DE^2 + CE^2$, &c.

Cor. 2. Hence, and by th. 34, the diagonals of a rhombus intersect at right angles.

THEOREM XLI.

If a line, drawn through or from the centre of a circle, bisect a chord, it will be perpendicular to it; or, if it be perpendicular to the chord, it will bisect both the chord and the arc of the chord.

Let AB be any chord in a circle, and CD a line drawn from the centre C to the chord. Then, if the chord be bisected in the point D, CD will be perpendicular to AB.



Draw the two radii CA, CB. Then the two triangles ACD, BCD, having CA equal to CB (def. 44), and CD common, also AD equal to DB (by hyp.); they have all the three sides of the one, equal to all the three sides of the other, and so have their angles also equal (th. 5). Hence, then, the angle ADC being equal to the angle BDC, these angles are right angles, and the line CD is perpendicular to AB (def. 11).

Again, if CD be perpendicular to AB, then will the chord AB be bisected at the point D, or have AD equal to DB; and the arc AEB bisected in the point E, or have AE equal EB.

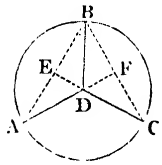
For, having drawn CA, CB, as before; then, in the triangle ABC, because the side CA is equal to the side CB, their opposite angles A and B are also equal (th. 3). Hence, then, in the two triangles ACD, BCD, the angle A is equal to the angle B, and the angles at D are equal (def. 11); therefore, their third angles are also equal (corol. 1, th. 17). And having the side CD common, they have also the side AD equal to the side DB (th. 2).

Also, since the angle ACE is equal to the angle BCE, the arc AE, which measures the former (def. 57), is equal to the arc BE, which measures the latter, since equal angles must have equal measures.

Corol. Hence, a line bisecting any chord at right angles, passes through the centre of the circle.

THEOREM XLII.

If more than two equal lines can be drawn from any point within a circle to the circumference, that point will be the centre.



Let ABC be a circle, and D a point within it: then, if any three lines, DA, DB, DC, drawn from the point D to the circumference, be equal to each other, the point D will be the centre.

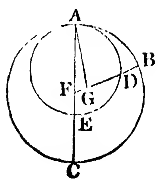
Draw the chords AB, BC, which let be bisected in the points E, F, and join DE, DF.

Then, the two triangles DAE, DBE, have the side DA equal to the side DB by supposition, and the side AE equal to the side EB by hypothesis, also the side DE common: therefore, these two triangles are identical, and have the angles at E equal to each other (th. 5); consequently, DE is perpendicular to the middle of the chord AB (def. 11), and therefore passes through the centre of the circle (corol. th. 41).

In like manner, it may be shown that DF passes through the centre. Consequently, the point D is the centre of the circle, and the three equal lines DA, DB, DC, are radii. Q. E. D.

THEOREM XLIII.

If two circles, placed one within another, touch, the centres of the circles and the point of contact will be all in the same right line.



Let the two circles ABC, ADE, touch one another internally in the point A; then will the point A and the centres of those circles be all in the same right line.

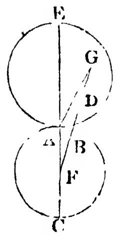
Let F be the centre of the circle ABC, through which draw the diameter AFC. Then, if the centre of the other circle can be out of this line AC, let it be supposed in some other point as G; through which draw the line FG, cutting the two circles in B and D.

Now, in the triangle AFG, the sum of the two sides FG, GA, is greater than the third side AF (th. 10), or greater than its equal radius FB. From each of these take away the common part FG, and the remainder GA will be greater than the remainder GB. But the point G being supposed the centre of the inner circle, its two radii, GA, GD, are equal to each other; consequently, GD will also be greater than GB. But ADE being the inner circle, GD is necessarily less than GB. So that GD is both greater and less than GB; which is absurd. To get quit of this absurdity we must abandon the supposition that produced it, which was that G might be out of the line AFC. Consequently, the centre G *cannot* be out of the line AFC. Q. E. D.

THEOREM XLIV.

If two circles touch one another externally, the centres of the circles and point of contact will be all in the same right line.

Let the two circles ABC, ADE, touch one another externally at the point A; then will the point of contact A and the centres of the two circles be all in the same right line.



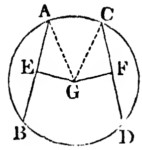
Let F be the centre of the circle ABC, through which draw the diameter AFC, and produce it to the other circle at E. Then, if the centre of the other circle ADE can be out of the line FE, let it, if possible, be supposed in some other point as G; and draw the lines AG, FBDG, cutting the two circles in B and D.

Then, in the triangle AFG, the sum of the two sides AF, AG, is greater than the third side FG (th. 10). But, F and G being the centres of the two circles, the two radii GA, GD, are equal, as are also the two radii AF, FB. Hence the sum of GA, AF, is equal to the sum of GD, BF; and, therefore, this latter sum also, GD, BF, is greater than GF, which is absurd. Consequently, the centre G cannot be out of the line EF. Q. E. D.

THEOREM XLV.

Any chords in a circle, which are equally distant from the centre, are equal to each other; or if they be equal to each other, they will be equally distant from the centre.

Let AB, CD, be any two chords at equal distances from the centre G; then will these two chords AB, CD, be equal to each other.



Draw the two radii GA, GC, and the two perpendiculars GE, GF, which are the equal distances from the centre G. Then, the two right-angled triangles, GAE, GCF, having the side GA equal the side GC, and the side GE equal the side GF, and the angle at E equal to the angle at F, therefore those two triangles are identical (cor. 2, th. 34), and have the line AE equal to the line CF. But AB is the double of AE, and CD is the double of CF (th. 41); therefore AB is equal to CD (by ax. 6). Q. E. D.

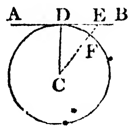
Again, if the chord AB be equal to the chord CD; then will their distances from the centre, GE, GF, also be equal to each other.

For, since AB is equal CD by supposition, the half AE is equal the half CF. Also, the radii GA, GC, being equal, as well as the right angles E and F, therefore the third sides are equal (cor. 2, th. 34), or the distance GE equal the distance GF. Q. E. D.

THEOREM XLVI.

A line perpendicular to the extremity of a radius, is a tangent to the circle.

Let the line ADB be perpendicular to the radius CD of a circle; then shall AB touch the circle in the point D only.



From any other point E in the line AB draw CFE to the centre, cutting the circle in F.

Then, because the angle D, of the triangle CDE, is a right angle, the angle at E is acute (cor. 3, th. 17), and consequently less than the angle D. But the greater side is always opposite to the greater angle (th. 9); therefore the side CE is greater

than the side CD, or greater than its equal CF. Hence the point B is without the circle; and the same for every other point in the line AB. Consequently the whole line is without the circle, and meets it in the point D only.

THEOREM XLVII.

When a line is a tangent to a circle, a radius drawn to the point of contact is perpendicular to the tangent.

Let the line AB touch the circumference of a circle at the point D; then will the radius CD be perpendicular to the tangent AB. [See the last figure.]

For, the line AB being wholly without the circumference except at the point D, every other line, as CE, drawn from the centre C to the line AB, must pass out of the circle to arrive at this line. The line CD is therefore the shortest that can be drawn from the point C to the line AB, and consequently (th. 21,) it is perpendicular to that line.

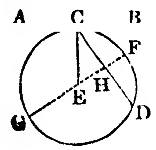
Corol. Hence, conversely, a line drawn perpendicular to a tangent, at the point of contact, passes through the centre of the circle.

THEOREM XLVIII.

The angle formed by a tangent and chord is measured by half the arc of that chord.

Let AB be a tangent to a circle, and CD a chord drawn from the point of contact C; then is the angle BCD measured by half the arc CDF, and the angle ACD measured by half the arc CGD.

Draw the radius EC to the point of contact, and the radius EF perpendicular to the chord at H.



Then the radius EF, being perpendicular to the chord CD, bisects the arc CFD (th. 41). Therefore CF is half the arc CFD.

In the triangle CEH, the angle H being a right one, the sum of the two remaining angles E and C is equal to a right angle (cor. 3, th. 17), which is equal to the angle BCE, because the radius CE is perpendicular to the tangent. From each of these equals take away the common part or angle C, and there remains the angle E equal to the angle BCD. But the angle E is measured by the arc CF (def. 57), which is the half of CFD; therefore the equal angle BCD must also have the same measure, namely, half the arc CFD of the chord CD.

Again, the line GEF, being perpendicular to the chord CD, bisects the arc CGD (th. 41). Therefore CG is half the arc CGD. Now, since the line CE, meeting FG, makes the sum of the two angles at E equal to two right angles (th. 6), and the line CD makes with AB the sum of the two angles at C equal to two right angles; if from these two equal sums there be taken away the parts or angles CEH and BCH, which have been proved equal, there remains the angle CEG equal to the angle ACH. But the former of these, CEG, being an angle at the centre, is measured by the arc CG (def. 57); consequently the equal angle ACD must also have the same measure CG, which is half the arc CGD of the chord CD. Q. E. D.

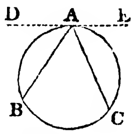
Corol. 1. The sum of the two right angles is measured by half the circumference. For the two angles BCD, ACD, which make up two right angles, are

measured by the arcs CF, CG, which make up half the circumference, FG being a diameter.

Corol. 2. Hence also one right angle must have for its measure a quarter of the circumference, or 90 degrees.

THEOREM XLIX.

An angle at the circumference of a circle is measured by half the arc that subtends it.

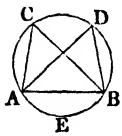


Let BAC be an angle at the circumference; it has for its measure, half the arc BC which subtends it.

For, suppose the tangent DE to pass through the point of contact A: then, the angle DAC being measured by half the arc ABC, and the angle DAB by half the arc AB (th. 48); it follows, by equal subtraction, that the difference, or angle BAC, must be measured by half the arc BC, which it stands upon. Q. E. D.

THEOREM L.

All angles in the same segment of a circle, or standing on the same arc, are equal to each other.

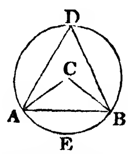


Let C and D be two angles in the same segment ACDB, or, which is the same thing, standing on the supplemental arc AEB; then will the angle C be equal to the angle D.

For, each of these angles is measured by half the arc AEB; and thus, having equal measures, they are equal to each other (ax. 11).

THEOREM LI.

An angle at the centre of a circle is double the angle at the circumference, when both stand on the same arc.

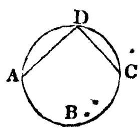


Let C be an angle at the centre C, and D an angle at the circumference, both standing on the same arc or same chord AB; then will the angle C be double of the angle D, or the angle D equal to half the angle C.

For, the angle at the centre C is measured by the whole arc AEB (def. 57), and the angle at the circumference D is measured by half the same arc AEB (th. 49); therefore the angle D is only half the angle C, or the angle C double the angle D.

THEOREM LII.

An angle in a semicircle, is a right angle.

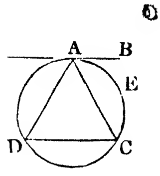


If ABC or ADC be a semicircle; then any angle D in that semicircle, is a right angle.

For, the angle D, at the circumference, is measured by half the arc ABC (th. 49), that is, by a quadrant of the circumference. But a quadrant is the measure of a right angle (cor. 4, th. 6; or cor. 2, th. 48). Therefore the angle D is a right angle.

THEOREM LIII.

The angle formed by a tangent to a circle, and a chord drawn from the point of contact, is equal to the angle in the alternate segment.



If AB be a tangent, and AC a chord, and D any angle in the alternate segment ADC; then will the angle D be equal to the angle BAC made by the tangent and chord of the arc AEC.

For, the angle D, at the circumference, is measured by half the arc AEC (th. 49); and the angle BAC, made by the tangent and chord, is also measured by the same half arc AEC (th. 48); therefore these two angles are equal (ax. 11).

THEOREM LIV.

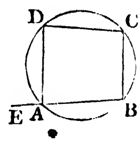
The sum of any two opposite angles of a quadrangle inscribed in a circle, is equal to two right angles.

Let ABCD be any quadrilateral inscribed in a circle; then shall the sum of the two opposite angles A and C, or B and D, be equal to two right angles.

For the angle A is measured by half the arc DCB, which it stands upon, and the angle C by half the arc DAB (th. 49); therefore the sum of the two angles A and C is measured by half the sum of these two arcs, that is, by half the circumference. But half the circumference is the measure of two right angles (cor. 4, th. 6); therefore the sum of the two opposite angles A and C is equal to two right angles. In like manner it is shown, that the sum of the other two opposite angles, D and B, is equal to two right angles. Q. E. D.

THEOREM LV.

If any side of a quadrangle, inscribed in a circle, be produced out, the outward angle will be equal to the inward opposite angle.

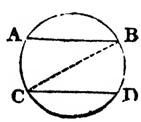


If the side AB, of the quadrilateral ABCD, inscribed in a circle, be produced to E; the outward angle DAE will be equal to the inward opposite angle C.

For, the sum of the two adjacent angles DAE and DAB is equal to two right angles (th. 6); and the sum of the two opposite angles C and DAB is also equal to two right angles (th. 54); therefore the former sum, of the two angles DAE and DAB, is equal to the latter sum, of the two C and DAB (ax. 1). From each of these equals taking away the common angle DAB, there remains the angle DAE equal the angle C. Q. E. D.

THEOREM LVI.

Any two parallel chords intercept equal arcs.



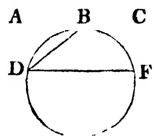
Let the two chords AB, CD, be parallel: then will the arcs AC, BD, be equal; or AC = BD.

Draw the line BC. Then, because the lines AB, CD, are parallel, the alternate angles B and C are equal (th. 12). But the angle at the circumference B, is measured by half the arc AC (th. 49); and the other equal angle

at the circumference C is measured by half the arc BD: therefore the halves of the arcs AC, BD, and consequently the arcs themselves, are also equal. Q. E. D.

THEOREM LVII.

When a tangent and chord are parallel to each other, they intercept equal arcs.

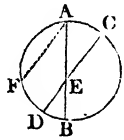


Let the tangent ABC be parallel to the chord DF; then are the arcs BD, BF, equal; that is, BD = BF.

Draw the chord BD. Then, because the lines AB, DF, are parallel, the alternate angles D and B are equal (th. 12). But the angle B, formed by a tangent and chord, is measured by half the arc BD (th. 48); and the other angle at the circumference D is measured by half the arc BF (th. 49); therefore the arcs BD, BF, are equal. Q. E. D.

THEOREM LVIII.

The angle formed within a circle, by the intersection of two chords, is measured by half the sum of the two intercepted arcs.



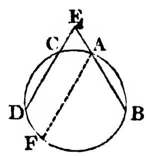
Let the two chords AB, CD, intersect at the point E: then the angle AEC, or DEB, is measured by half the sum of the two arcs AC, DB.

Draw the chord AF parallel to CD. Then, because the lines AF, CD, are parallel, and AB cuts them, the angles on the same side A and DEB are equal (th. 14). But the angle at the circumference A is measured by half the arc BF, or of the sum of FD and DB (th. 49); therefore, the angle E is also measured by half the sum of FD and DB.

Again, because the chords AF, CD, are parallel, the arcs AC, FD, are equal (th. 56); therefore, the sum of the two arcs AC, DB, is equal to the sum of the two FD, DB; and consequently the angle E, which is measured by half the latter sum, is also measured by half the former. Q. E. D.

THEOREM LIX.

The angle formed out of a circle, by two secants, is measured by half the difference of the intercepted arcs.



Let the angle E be formed by two secants EAB and ECD; this angle is measured by half the difference of the two arcs AC, DB, intercepted by the two secants.

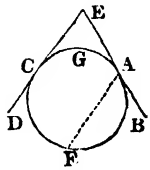
Draw the chord AF parallel to CD. Then, because the lines AF, CD, are parallel, and AB cuts them, the angles on the same side A and BED are equal (th. 14). But the angle A, at the circumference, is measured by half the arc BF (th. 49), or of the difference of DF and DB: therefore, the equal angle E is also measured by half the difference of DF, DB.

Again, because the chords AF, CD are parallel, the arcs AC, FD, are equal (th. 56); therefore, the difference of the two arcs AC, DB, is equal to the difference of the two DF, DB. Consequently, the angle E, which is measured by half the latter difference, is also measured by half the former. Q. E. D.

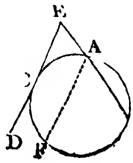
THEOREM LX.

The angle formed by two tangents, is measured by half the difference of the two intercepted arcs.

Let EB, ED, be two tangents to a circle at the points A, C; then the angle E is measured by half the difference of the two arcs CFA, CGA.



Draw the chord AF parallel to ED. Then, because the lines AF, ED, are parallel, and EB meets them, the angles on the same side A and E are equal (th. 14). But the angle A, formed by the chord AF and tangent AB, is measured by half the arc AF (th. 48); therefore, the equal angle E is also measured by half the same arc AF, or half the difference of the arcs CFA and CF, or CGA (th. 57.)

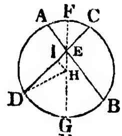


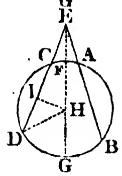
Corol. In like manner it is proved, that the angle E, formed by a tangent ECD, and a secant EAB, is measured by half the difference of the two intercepted arcs CA and CFB.

THEOREM LXI.

When two lines, meeting a circle each in two points, cut one another, either within it or without it; the rectangle of the parts of the one, is equal to the rectangle of the parts of the other; the parts of each being measured from the point of meeting to the two intersections with the circumference.

Let the two lines AB, CD, meet each other in E; then the rectangle of AE, EB, will be equal to the rectangle of CE, ED. Or, AE . EB = CE . ED.

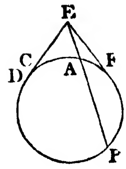




For, through the point E draw the diameter FG; also, from the centre H draw the radius DH, and draw HI perpendicular to CD.

Then, since DEH is a triangle, and the perp. HI bisects the chord CD (th. 41), the line CE is equal to the difference of the segments DI, EI, the sum of them being DE. Also, because H is the centre of the circle, and the radii DH, FH, GH, are all equal, the line EG is equal to the sum of the sides DH, HE; and EF is equal to their difference.

But the rectangle of the sum and difference of the two sides of a triangle is equal to the rectangle of the sum and difference of the segments of the base (th. 35); therefore the rectangle of FE, EG, is equal to the rectangle of CE, ED. In like manner it is proved, that the same rectangle of FE, EG, is equal to the rectangle of AE, EB. Consequently, the rectangle of AE, EB, is also equal to the rectangle of CE, ED (ax 1). Q. E. D.

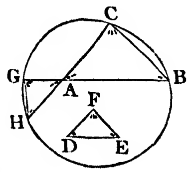


Corol. 1. When one of the lines in the second case, as DE, by revolving about the point E, comes into the position of the tangent EC or ED, the two points C and D running into one; then the rectangle of CE, ED, becomes the square of CE, because CE and DE are then equal. Consequently, the rectangle of the parts of the secant, AE . EB, is equal to the square of the tangent, CE^2.

Corol. 2. Hence, both the tangents EC, EF, drawn from the same point E, are equal; since the square of each is equal to the same rectangle or quantity AE . EB.

THEOREM LXII.

In equiangular triangles, the rectangles of the corresponding or like sides, ta en alternately, are equal.



Let ABC, DEF, be two equiangular triangles, having the angle A = the angle D, the angle B = the angle E, and the angle C = the angle F; also the like sides AB, DE, and AC, DF, being those opposite the equal angles; then will the rectangle of AB, DF, be equal to the rectangle of AC, DE.

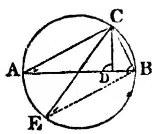
In BA, produced take AG equal to DF; and through the three points B, C, G, conceive a circle BCGH to be described, meeting CA produced at H, and join GH.

Then the angle G is equal to the angle C on the same arc BH, and the angle H equal to the angle B on the same arc CG (th. 50); also the opposite angles at A are equal (th. 7): therefore the triangle AGH is equiangular to the triangle ACB, and consequently to the triangle DFE also. But the two like sides AG, DF, are also equal by supposition, consequently the two triangles AGH, DFE, are identical (th. 2), having the two sides AG, AH, equal to the two DF, DE, each to each.

But the rectangle GA . AB is equal to the rectangle HA . AC (th. 61): consequently the rectangle DF . AB is equal to the rectangle DE . AC. Q. E. D.

THEOREM LXIII.

The rectangle of the two sides of any triangle, is equal to the rectangle of the perpendicular on the third side and the diameter of the circumscribing circle.



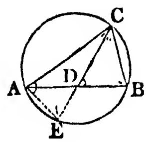
Let CD be the perpendicular, and CE the diameter of the circle about the triangle ABC; then the rectangle CA . CB is = the rectangle CD . CE.

For, join BE: then in the two triangles ACD, ECB, the angles A and E are equal, standing on the same arc BC (th. 50); also the right angle D is equal to the angle B, which is also a right angle, being in a semicircle (th. 52): therefore these two triangles have also their third angles equal, and are equiangular. Hence, AC, CE, and CD, CB, being like sides, subtending the equal angles, the rectangle AC . CB, of the first and last of them, is equal to the rectangle CE . CD, of the other two (th. 62).

THEOREM LXIV.

The square of a line bisecting any angle of a triangle, together with the rectangle of the two segments of the opposite side, is equal to the rectangle of the two other sides including the bisected angle.

Let CD bisect the angle C of the triangle ABC; then the square $CD^2 +$ the rectangle AD . DB is $=$ the rectangle AC . CB.



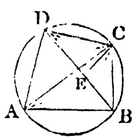
For, let CD be produced to meet the circumscribing circle at E, and join AE.

Then the two triangles ACE, BCD, are equiangular: for the angles at C are equal by supposition, and the angles B and E are equal, standing on the same arc AC (th. 50); consequently the third angles at A and D are equal (cor. 1, th. 17): also AC, CD, and CE, CB, are like or corresponding sides, being opposite to equal angles: therefore the rectangle AC . CB is $=$ the rectangle CD . CE (th. 62). But the latter rectangle CD . CE is $= CD^2 +$ the rectangle CD . DE (th. 30); therefore the former rectangle AC . CB is also $= CD^2 +$ CD . DE, or equal to $CD^2 +$ AD . DB, since CD . DE is $=$ AD . DB (th. 61). Q. E. D.

THEOREM LXV.

The rectangle of the two diagonals of any quadrangle inscribed in a circle, is equal to the sum of the two rectangles of the opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and AC, BD, its two diagonals: then the rectangle AC . BD, is $=$ the rectangle AB . DC $+$ the rectangle AD . BC.



For, let CE be drawn, making the angle BCE equal to the angle DCA. Then the two triangles ACD, BCE, are equiangular; for the angles A and B are equal, standing on the same arc DC; and the angles DCA, BCE, are equal by supposition; consequently the third angles ADC, BEC, are also equal: also AC, BC, and AD, BE, are like or corresponding sides, being opposite to the equal angles: therefore the rectangle AC . BE is $=$ the rectangle AD . BC (th. 62).

Again, the two triangles ABC, DEC, are equiangular: for the angles BAC, BDC, are equal, standing on the same arc BC; and the angle DCE is equal to the angle BCA, by adding the common angle ACE to the two equal angles DCA, BCE; therefore the third angles E and ABC are also equal: but AC, DC, and AB, DE, are the like sides: therefore the rectangle AC . DE is $=$ the rectangle AB . DC (th. 62).

Hence, by equal additions, the sum of the rectangles AC . BE $+$ AC . DE is $=$ AD . BC $+$ AB . DC. But the former sum of the rectangles AC . BE $+$ AC DE is $=$ the rectangle AC . BD (th. 30): therefore the same rectangle AC . BD is equal to the latter sum, the rect. AD . BC $+$ the rect. AB . DC (ax. 1). Q. E. D.

Corol. Hence, if ABD be an equilateral triangle, and C any point in the arc BCD of the circumscribing circle, we have AC $=$ BC $+$ DC. For AC . BD being $=$ AD . BC $+$ AB . DC; dividing by BD $=$ AB $=$ AD, there results AC $=$ BC $+$ DC.

OF RATIOS AND PROPORTIONS.

DEFINITIONS.

DEF. 76. RATIO is the proportion or relation which one magnitude bears to another magnitude of the same kind, with respect to quantity.

Note. The measure, or quantity, of a ratio, is conceived, by considering what part or parts the leading quantity, called the Antecedent, is of the other, called the consequent; or what part or parts the number expressing the quantity of the former, is of the number denoting in like manner the latter. So, the ratio of a quantity expressed by the number 2 to a like quantity expressed by the number 6, is denoted by 2 divided by 6, or $\frac{2}{6}$ or $\frac{1}{3}$: the number 2 being 3 times contained in 6, or the third part of it. In like manner, the ratio of the quantity 3 to 6, is measured by $\frac{3}{6}$ or $\frac{1}{2}$; the ratio of 4 to 6 is $\frac{4}{6}$ or $\frac{2}{3}$; that of 6 to 4 is $\frac{6}{4}$ or $\frac{3}{2}$; &c.

77. Proportion is an equality of ratios. Thus,

78. Three quantities are said to be proportional, when the ratio of the first to the second is equal to the ratio of the second to the third. As of the three quantities A (2), B (4), C (8), where $\frac{2}{4} = \frac{4}{8} = \frac{1}{2}$, both the same ratio.

79. Four quantities are said to be proportional, when the ratio of the first to the second, is the same as the ratio of the third to the fourth. As of the four A (4), B (2), C (10), D (5), where $\frac{4}{2} = \frac{10}{5} = 2$, both the same ratio.

Note. To denote that four quantities, A, B, C, D, are proportional, they are usually stated or placed thus, A : B :: C : D; and read thus, A is to B as C is to D. But when three quantities are proportional, the middle one is repeated, and they are written thus, A : B :: B : C.

The proportionality of quantities may also be expressed very generally by the equality of fractions, as at pa. 121. Thus, if $\frac{A}{B} = \frac{C}{D}$, then A : B :: C : D, also B : A :: D : C, A : C :: B : D. and C : A :: D : B.

80. Of three proportional quantities, the middle one is said to be a Mean Proportional between the other two; and the last, a Third Proportional to the first and second.

81. Of four proportional quantities, the last is said to be a Fourth Proportional to the other three. taken in order.

82. Quantities are said to be Continually Proportional, or in Continued Proportion, when the ratio is the same between every two adjacent terms, viz. when the first is to the second, as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on, all in the same common ratio.

As in the quantities 1, 2, 4, 8, 16, &c.; where the common ratio is equal to 2.

83. Of any number of quantities, A, B, C, D, the ratio of the first A, to the last D, is said to be Compounded of the ratios of the first to the second, of the second to the third, and so on to the last.

84. Inverse ratio is, when the antecedent is made the consequent, and the consequent the antecedent. Thus, if 1 : 2 :: 3 · 6; then inversely, 2 : 1 :: 6 : 3.

85. Alternate proportion is, when antecedent is compared with antecedent, and consequent with consequent.—As, if $1 : 2 :: 3 : 6$; then, by alternation, or permutation, it will be $1 : 3 :: 2 : 6$.

86. Compound ratio is, when the sum of the antecedent and consequent is compared, either with the consequent, or with the antecedent.—Thus, if $1 : 2 :: 3 : 6$, then, by composition, $1 + 2 : 1 :: 3 + 6 : 3$, and $1 + 2 : 2 :: 3 + 6 : 6$.

87. Divided ratio is, when the difference of the antecedent and consequent is compared, either with the antecedent or with the consequent.—Thus, if $1 : 2 :: 3 : 6$, then, by division, $2 - 1 : 1 :: 6 - 3 : 3$, and $2 - 1 : 2 :: 6 - 3 : 6$.

Note. The term Divided, or Division, here means subtracting, or parting; being used in the sense opposed to compounding, or adding, in def. 86.

THEOREM LXVI.

Equimultiples of any two quantities have the same ratio as the quantity themselves.

Let A and B be any two quantities, and mA, mB, any equimultiples of them, m being any number whatever: then will mA and mB have the same ratio as A and B, or $A : B :: mA : mB$.

For $\frac{mB}{mA} = \frac{B}{A}$, the same ratio.

Corol. Hence, like parts of quantities have the same ratio as the wholes; because the wholes are equimultiples of the like parts, or A and B are like parts of mA and mB.

THEOREM LXVII.

If four quantities, of the same kind, be proportionals; they will be in proportion by alternation or permutation, or the antecedents will have the same ratio as the consequents.

Let $A : B :: mA : mB$; then will $A : mA :: B : mB$.

For $\frac{mA}{A} = \frac{m}{1}$, and $\frac{mB}{B} = \frac{m}{1}$, both the same ratio.

Otherwise. Let $A : B :: C : D$; then shall $B : A :: C : D$.

For, let $\frac{A}{B} = \frac{C}{D} = r$; then $A = Br$, and $C = Dr$: therefore $B = \frac{A}{r}$, and $D = \frac{C}{r}$. Hence $\frac{B}{A} = \frac{1}{r}$, and $\frac{D}{C} = \frac{1}{r}$. Whence it is evident that $\frac{B}{A} = \frac{D}{C}$ (ax. 1), or $B : A :: D : C$.

In a similar manner may most of the other theorems be demonstrated.

THEOREM LXVIII.

If four quantities be proportional; they will be in proportion by inversion, or inversely.

Let $A : B :: mA : mB$; then will $B : A :: mB : mA$.

For $\frac{mA}{mB} = \frac{A}{B}$, both the same ratio.

THEOREM LXIX.

If four quantities be proportional; they will be in proportion by composition and division.

Let $A : B :: mA : mB$;

Then will $B \pm A : A :: mB \pm mA : mA$,

and $B \pm A : B :: mB \pm mA : mB$.

For $\dfrac{mA}{mB \pm mA} = \dfrac{A}{B \pm A}$; and $\dfrac{mB}{mB \pm mA} = \dfrac{B}{B \pm A}$.

Corol. It appears from hence, that the sum of the greatest and least of four proportional quantities, of the same kind, exceeds the sum of the other two. For since $A : A+B :: mA :: mAemB$ --- $A : A+B :: mA+mB$, where A is the least, and $mA+mB$ the greatest; then $m+1.A+mB$, the sum of the greatest and least, exceeds $\overline{m+1}.A+B$, the sum of the two other quantities.

THEOREM LXX.

If, of four proportional quantities, there be taken any equimultiples whatever of the two antecedents, and any equimultiples whatever of the two consequents; the quantities resulting will still be proportional.

Let $A : B :: mA : mB$; also, let pA and pmA be any equimultiples of the two antecedents, and qB and qmB any equimultiples of the two consequents; then will - - - - $pA : qB :: pmA : qmB$.

For $\dfrac{qmB}{pmA} = \dfrac{qB}{pA}$, both the same ratio.

THEOREM LXXI.

If there be four proportional quantities, and the two consequents be either augmented or diminished by quantities that have the same ratio as the respective antecedents; the results and the antecedents will still be proportionals.

Let $A : B :: mA : mB$, and nA and nmA any two quantities having the same ratio as the two antecedents; then will $A : B \pm nA :: mA : mB \pm nmA$.

For $\dfrac{mB \pm nmA}{mA} = \dfrac{B \pm nA}{A}$, both the same ratio.

THEOREM LXXII.

If any number of quantities be proportional, then any one of the antecedents will be to its consequent, as the sum of all the antecedents, is to the sum of all the

Let $A : B :: mA : mB :: nA : nB$, &c.; then will $A : B :: A + mA + nA : B + mB + nB$, &c.

For $\dfrac{B + mB + nB}{A + mA + nA} = \dfrac{(1 + m + n)B}{(1 + m + n)A} = \dfrac{B}{A}$, the same ratio.

THEOREM LXXIII.

If a whole magnitude be to a whole, as a part taken from the first, is to a part taken from the other; then the remainder will be to the remainder, as the whole to the whole.

Let $A : B :: \frac{m}{n} A : \frac{m}{n} B$;

then will $A : B :: A - \frac{m}{n} A : B - \frac{m}{n} B$.

For $\dfrac{B - \frac{m}{n} B}{A - \frac{m}{n} A} = \dfrac{B}{A}$, both the same ratio.

THEOREM LXXIV.

If any quantities be proportional; their squares, or cubes, or any like powers, or roots, of them, will also be proportional.

Let $A : B :: mA : mB$; then will $A^n : B^n :: m^n A^n : m^n B^n$.

For $\dfrac{m^n B^n}{m^n A^n} = \dfrac{B^n}{A^n}$, both the same ratio.

See also th. VIII.

THEOREM LXXV.

If there be two sets of proportionals; then the products or rectangles of the corresponding terms will also be proportional.

Let $A : B :: mA : mB$,
and $C : D :: nC : nD$;
then will $AC : BD :: mnAC : mnBD$.

For, $\dfrac{mnBD}{mnAC} = \dfrac{BD}{AC}$, both the same ratio.

THEOREM LXXVI.

If four quantities be proportional; the rectangle or product of the two extremes, will be equal to the rectangle or product of the two means. And the converse.

Let $A : B :: mA : mB$;
then is $A \times mB = B \times mA = mAB$, as is evident.

THEOREM LXXVII.

If three quantities be continued proportionals; the rectangle or product of the two extremes, will be equal to the square of the mean. And the converse.

Let A, mA, m^2A, be three proportionals,
or $A : mA :: mA : m^2A$;
then is $A \times m^2A = m^2A^2$, as is evident.

THEOREM LXXVIII.

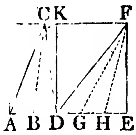
If any number of quantities be continued proportionals; the ratio of the first to the third, will be duplicate or the square of the ratio of the first and second; and the ratio of the first and fourth will be triplicate or the cube of that of the first and second; and so on.

Let A, mA, m^2A, m^3A, &c. be proportionals;

then is $\frac{A}{mA} = \frac{1}{m}$; but $\frac{A}{m^2A} = \frac{1}{m^2}$; and $\frac{A}{m^3A} = \frac{1}{m^3}$. &c

THEOREM LXXIX.

Triangles, and also parallelograms, having equal altitudes, are to each other as their bases.



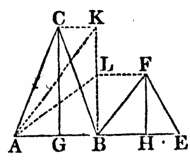
Let the two triangles ADC, DEF, have the same altitude, or be between the same parallels AE, IF; then is the surface of the triangle ADC, to the surface of the triangle DEF, as the base AD is to the base DE. Or AD : DE : : the triangle ADC : the triangle DEF.

For, let the base AD be to the base DE, as any one number m (2), to any other number n (3); and divide the respective bases into those parts, AB, BD, DG, GH, HE, all equal to one another; and from the points of division draw the lines BC, GF, HF, to the vertices C and F. Then will these lines divide the triangles ADC, DEF, into the same number of parts as their bases, each equal to the triangle ABC, because those triangular parts have equal bases and altitude (cor. 2, th. 25); namely, the triangle ABC equal to each of the triangles BDC, DFG, GFH, HFE. So that the triangle ADC, is to the triangle DFE, as the number of parts m (2) of the former, to the number n (3) of the latter, that is, as the base AD to the base DE (def. 79).

In like manner, the parallelogram ADKI is to the parallelogram DEFK, as the base AD is to the base DE; each of these having the same ratio as the number of their parts, m to n. Q. E. D.

THEOREM LXXX.

Triangles, and also parallelograms, having equal bases, are to each other as their altitudes.



Let ABC, BEF, be two triangles having the equal bases AB, BE, and whose altitudes are the perpendiculars CG, FH; then will the triangle ABC : the triangle BEF : : CG : FH.

For, let BK be perpendicular to AB, and equal to CG; in which let there be taken BL = FH; drawing AK and AL.

Then triangles of equal bases and heights being equal (cor. 2, th. 25), the triangle ABK is = ABC, and the triangle ABL = BEF. But, considering now ABK, ABL, as two triangles on the bases BK, BL, and having the same altitude AB, these will be as their bases (th. 79), namely, the triangle ABK : the triangle ABL : : BK : BL.

But the triangle ABK = ABC, and the triangle ABL = BEF, also BK = CG and BL = FH.

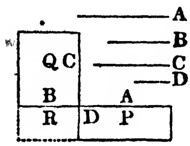
Therefore, the triangle ABC : triangle BEF : : CG : FH.

And since parallelograms are the doubles of these triangles, having the same bases and altitudes, they will likewise have to each other the same ratio as their altitudes. Q. E. D.

Corol. Since, by this theorem, triangles and parallelograms, when their bases are equal, are to each other as their altitudes; and by the foregoing one, when their altitudes are equal, they are to each other as their bases; therefore, universally, when neither are equal, they are to each other in the compound ratio, or as the rectangle or product of their bases and altitudes.

THEOREM LXXXI.

If four lines be proportional; the rectangle of the extremes will be equal to the rectangle of the means. And, conversely, if the rectangle of the extremes, of four lines, be equal to the rectangle of the means, the four lines, taken alternately, will be proportional.



Let the four lines A, B, C, D, be proportionals, or A : B :: C : D; then will the rectangle of A and D be equal to the rectangle of B and C; or the rectangle A . D = B . C.

For, let the four lines be placed with their four extremities meeting in a common point, forming at that point four right angles; and draw lines parallel to them to complete the rectangles P, Q, R, where P is the rectangle of A and D, Q the rectangle of B and C, and R the rectangle of B and D.

Then the rectangles P and R, being between the same parallels, are to each other as their bases A and B (th. 79); and the rectangles Q and R, being between the same parallels, are to each other as their bases C and D. But the ratio of A to B, is the same as the ratio of C to D, by hypothesis; therefore the ratio of P to R, is the same as the ratio of Q to R; and consequently the rectangles P and Q are equal. Q. E. D.

Again, if the rectangle of A and D, be equal to the rectangle of B and C; these lines will be proportional, or A : B :: C : D.

For, the rectangles being placed the same as before: then, because parallelograms between the same parallels, are to one another as their bases, the rectangle P : R :: A : B, and Q : R :: C : D. But as P and Q are equal, by supposition, they have the same ratio to R, that is, the ratio of A to B is equal to the ratio of C to D, or A : B :: C : D. Q. E. D.

Corol. 1. When the two means, namely, the second and third terms, are equal, their rectangle becomes a square of the second term, which supplies the place of both the second and third. And hence it follows, that when three lines are proportionals, the rectangle of the two extremes is equal to the square of the mean; and, conversely, if the rectangle of the extremes be equal to the square of the mean, the three lines are proportionals.

Corol. 2. Since it appears, by the rules of proportion in arithmetic and algebra, that when four quantities are proportional, the product of the extremes is equal to the product of the two means; and, by this theorem, the rectangle o the extremes is equal to the rectangle of the two means; it follows, that the area or space of a rectangle is represented or expressed by the product of its length

and breadth multiplied together. And, in general, a rectangle in geometry is similar to the product of the measures of its two dimensions of length and breadth, or base and height. Also, a square is similar to, or represented by, the measure of its side multiplied by itself. So that, what is shown of such products is to be understood of the squares and rectangles.

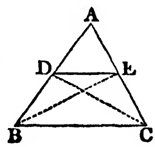
Corol. 3. Since the same reasoning, as in this theorem, holds for any parallelograms whatever, as well as for the rectangles, the same property belongs to all kinds of parallelograms, having equal angles, and also to triangles, which are the halves of parallelograms; namely, that if the sides about the equal angles of parallelograms, or triangles, be reciprocally proportional, the parallelograms or triangles will be equal; and, conversely, if the parallelograms or triangles be equal, their sides about the equal angles will be reciprocally proportional.

Corol. 4. Parallelograms, or triangles, having an angle in each equal, are in proportion to each other as the rectangles of the sides which are about these equal angles.

THEOREM LXXXII.

If a line be drawn in a triangle parallel to one of its sides, it will cut the other two sides proportionally.

Let DE be parallel to the side BC of the triangle ABC; then will AD : DB : : AE : EC.



For, draw BE and CD. Then the triangles DBE, DCE are equal to each other, because they have the same base DE, and are between the same parallels DE, BC (th. 25). But the two triangles ADE, BDE, on the bases AD, DB, have the same altitude; and the two triangles ADE, CDE, on the bases AE, EC, have also the same altitude; and because triangles of the same altitude are to each other as their bases, therefore

the triangle ADE : BDE : : AD : DB,
and triangle ADE : CDE : : AE : EC.

But BDE is = CDE; and equals must have to equals the same ratio; therefore AD : DB : : AE : EC. Q. E. D.

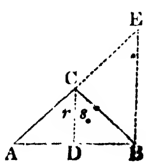
Corol. Hence, also, the whole lines, AB, AC, are proportional to their corresponding proportional segments (corol. th. 66),

viz. AB : AC : : AD : AE,
and AB : AC : : BD : CE.

THEOREM LXXXIII.

A line which bisects any angle of a triangle, divides the opposite side into two segments, which are proportional to the two other adjacent sides.

Let the angle ACB, of the triangle ABC, be bisected by the line CD, making the angle r equal to the angle s: then will the segment AD be to the segment DB, as the side AC is to the side CB. Or, AD : DB : : AC : CB.

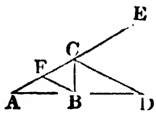


For, let BE be parallel to CD, meeting AC produced at E. Then, because the line BC cuts the two parallels CD, BE, it makes the angle CBE equal to the alternate angle s (th. 12), and therefore also equal to the angle r, which is

equal to s by the supposition. Again, because the line AE cuts the two parallels DC, BE, it makes the angle E equal to the angle r on the same side of it (th. 14). Hence, in the triangle BCE, the angles B and E, being each equal to the angle r, are equal to each other, and consequently their opposite sides CB, CE, are also equal (th. 3).

But now, in the triangle ABE, the line CD, being drawn parallel to the side BE, cuts the two other sides AB, AE, proportionally (th. 82), making AD to DB, as is AC to CE or to its equal CB. Q. E. D.

Case 2. The proposition is also applicable when an external angle of a triangle is bisected.



Let AC, one of the sides of the triangle ABC, be produced to E, and let the angle BCE be bisected by the straight CD, cutting AB produced in D; then

$$AD : DB :: AC : CB.$$

Let BF be parallel to CD.

Then, because the line BC cuts the parallel lines CD, FB, it makes the angle CBF equal to the alternate angle BCD; and, therefore, also equal to the angle DCE, which is equal to BCD by supposition. Again, because the line EA cuts the two parallel lines CD, FB, it makes the angle DCE equal to the angle CFB, on the same side of the line. Hence, in the triangle BCF, the angle BFC, and FBC, being each equal to the angle DCE, are equal to each other; and, consequently, their opposite sides BC, CF, are also equal.

Now, in the triangle ADC, the line BF being drawn parallel to the side CD cuts the two sides AD, AC, proportionally; making

$$AD : AC :: DB : CF \text{ (theor. 72)};$$

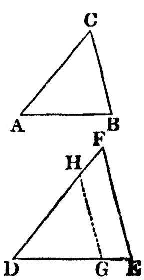
$$\text{Or,} \quad AD : DB :: AC : CF.$$

But, BC is equal to CF; therefore,

$$AD : DB :: AC : CB.$$

THEOREM LXXXIV.

Equiangular triangles are similar, or have their like sides proportional.



Let ABC, DEF, be two equiangular triangles, having the angle A equal to the angle D, the angle B to the angle E, and consequently the angle C to the angle F; then will AB : AC :: DE : DF.

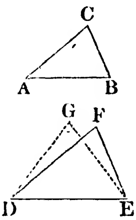
For, make DG = AB, and DH = AC, and join GH. Then the two triangles ABC, DGH, having the two sides AB, AC, equal to the two DG, DH, and the contained angles A and D also equal, are identical, or equal in all respects (th. 1), namely, the angles B and C are equal to the angles G and H. But the angles B and C are equal to the angles E and F by the hypothesis; therefore also the angles G and H are equal to the angles E and F (ax. 1), and consequently the line GH is parallel to the side EF (cor. 1, th. 14).

Hence then, in the triangle DEF, the line GH, being parallel to the side EF, divides the two other sides proportionally, making DG : DH :: DE : DF (cor. th. 82). But DG and DH are equal to AB and AC ; therefore also AB : AC :: DE : DF. Q. E. D.

THEOREM LXXXV.

Triangles which have their sides proportional, are also equiangular.

In the two triangles ABC, DEF, if AB : DE :: AC : DF :: BC : EF; the two triangles will have their corresponding angles equal.



For, if the triangle ABC be not equiangular with the triangle DEF, suppose some other triangle, as DEG, to be equiangular with ABC. But this is impossible: for if the two triangles ABC, DEG, were equiangular, their sides would be proportional (th. 84). So that, AB being to DE as AC to DG, and AB to DE as BC to EG, it follows that DG and EG, being fourth proportionals to the same three quantities, as well as the two DF, EF, the former, DG, EG, would be equal to the latter, DF, EF. Thus, then, the two triangles, DEF, DEG, having their three sides equal, would be identical (th. 5); which is absurd, since their angles are unequal.

THEOREM LXXXVI.

Triangles, which have an angle in the one equal to an angle in the other, and the sides about these angles proportional, are equiangular.

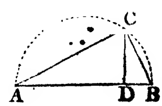
Let ABC, DEF, be two triangles, having the angle A = the angle D, and the sides AB, AC, proportional to the sides DE, DF: then will the triangle ABC be equiangular with the triangle DEF.

For, make DG = AB, and DH = AC, and join GH.

Then, the two triangles ABC, DGH, having two sides equal, and the contained angles A and D equal, are identical and equiangular (th. 1), having the angles G and H equal to the angles B and C. But, since the sides DG, DH, are proportional to the sides DE, DF, the line GH is parallel to EF (th. 82); hence the angles E and F are equal to the angles G and H (th. 14), and consequently to their equals B and C. Q. E. D. [See fig. th. 84.]

THEOREM LXXXVII.

In a right-angled triangle, a perpendicular from the right angle, is a mean proportional between the segments of the hypothenuse; and each of the sides, about the right angle, is a mean proportional between the hypothenuse and the adjacent segment.



Let ABC be a right-angled triangle, and CD a perpendicular from the right angle C to the hypothenuse AB; then will

CD be a mean proportional between AD and DB;
AC a mean proportional between AB and AD;
BC a mean proportional between AB and BD.

Or, AD : CD :: CD : DB; and AB : BC :: BC : BD; and AB : AC :: AC : AD.

For, the two triangles ABC, ADC, having the right angles at C and D equal, and the angle A common, have their third angles equal, and are equiangular (cor. 1, th. 17). In like manner, the two triangles ABC, BDC, having the right angles at C and D equal, and the angle B common, have their third angles equal, and are equiangular.

Hence then, all the three triangles, ABC, ADC, BDC, being equiangular, will have their like sides proportional (th. 84)

viz. AD : CD :: CD : DB;
and AB : AC :: AC : AD;
and AB : BC :: BC : BD. Q. E. D.

Corol. 1. Because the angle in a semicircle is a right angle (th. 52); it follows, that if, from any point C in the periphery of the semicircle, a perpendicular be drawn to the diameter AB; and the two chords CA, CB, be drawn to the extremities of the diameter: then are AC, BC, CD, the mean proportionals as in this theorem, or (by th. 77), $CD^2 = AD . DB$; $AC^2 = AB . AD$; and $BC^2 = AB . BD$.

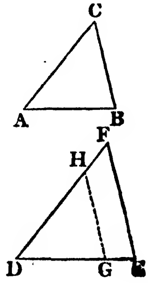
Corol. 2. Hence $AC^2 : BC^2 :: AD : BD$.

Corol. 3. Hence we have another demonstration of th. 34.

For since $AC^2 = AB . AD$, and $BC^2 = AB . BD$
By addition $AC^2 + BC^2 = AB (AD + BD) = AB^2$.

THEOREM LXXXVIII.

Equiangular or similar triangles, are to each other as the squares of their like sides.



Let ABC, DEF, be two equiangular triangles, AB and DE being two like sides: then will the triangle ABC be to the triangle DEF, as the square of AB is to the square of DE, or as AB^2 to DE^2.

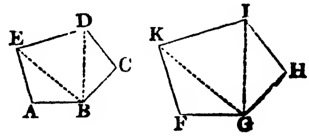
For, the triangles being similar, they have their like sides proportional (th. 84), and are to each other as the rectangles of the like pairs of their sides (cor. 4, th. 81);

therefore AB : DE :: AC : DF (th. 84),
and AB : DE :: AB : DE of equality:
therefore $AB^2 : DE^2 :: AB . AC : DE . DF$ (th. 75).
But $\triangle ABC : \triangle DEF :: AB . AC : DE . DF$ (cor. 4, th. 81),
therefore $\triangle ABC : \triangle DEF :: AB^2 : DE$ Q. E. D.

THEOREM LXXXIX.

All similar figures are to each other, as the squares of their like sides.

Let ABCDE, FGHIK, be any two similar figures, the like sides being AB, FG, and BC, GH, and so on in the same order: then will the figure ABCDE be to the figure FGHIK, as the square of AB to the square of FG, or as AB^2 to FG^2.



For, draw BE, BD, GK, GI, dividing the figures into an equal number of triangles, by lines from two equal angles B and G.

The two figures being similar (by suppos.), they are equiangular, and have their like sides proportional (def. 67).

Then, since the angle A is = the angle F, and the sides AB, AE, proportional to the sides FG, FK, the triangles ABE, FGK, are equiangular (th. 86). In like manner, the two triangles BCD, GHI, having the angle C = the angle H, and the sides BC, CD, proportional to the sides GH, HI, are also equiangular. Also, if from the equal angles AED, FKI, there be taken the equal angles AEB, FKG, there will remain the equals BED, GKI; and if from the equal angles CDE, HIK, be taken away the equals CDB, HIG, there will remain the equals BDE, GIK; so that the two triangles BDE, GIK, having two angles equal, are also equiangular. Hence each triangle of the one figure, is equiangular with each corresponding triangle of the other.

But equiangular triangles are similar, and are proportional to the squares of their like sides (th. 88).

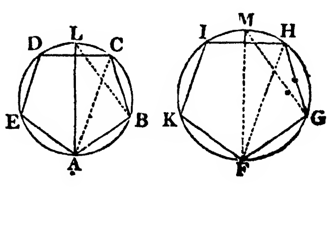
$$\text{Therefore the } \triangle ABE : \triangle FGK :: AB^2 : FG^2,$$
$$\text{and } \triangle BCD : \triangle GHI :: BC^2 : GH^2,$$
$$\text{and } \triangle BDE : \triangle GIK :: DE^2 : IK^2.$$

But as the two polygons are similar, their like sides are proportional, and consequently their squares also proportional; so that all the ratios AB^2 to FG^2, and BC^2 to GH^2, and DE^2 to IK^2, are equal among themselves, and consequently the corresponding triangles also, ABE to FGK, and BCD to GHI, and BDE to GIK, have all the same ratio, viz. that of AB^2 to FG^2: and hence all the antecedents, or the figure ABCDE, have to all the consequents, or the figure FGHIK, still the same ratio, viz. that of AB^2 to FG^2 (th. 72). Q. E. D.

THEOREM XC.

Similar figures inscribed in circles, have their like sides, and also their whole perimeters, in the same ratio as the diameters of the circles in which they are inscribed.

Let ABCDE, FGHIK, be two similar figures, inscribed in the circles whose diameters are AL and FM; then will each side AB, BC, &c. of the one figure be to the like side FG, GH, &c. of the other figure, or the whole perimeter AB + BC + &c. of the one figure, to the



whole perimeter FG + GH + &c. of the other figure, as the diameter AL to the diameter FM.

For, draw the two corresponding diagonals, AC, FH, as also the lines BL, GM. Then, since the polygons are similar, they are equiangular, and their like sides have the same ratio (def. 67); therefore the two triangles ABC, FGH, have the angle B = the angle G, and the sides AB, BC, proportional to the two sides FG, GH; consequently these two triangles are equiangular (th. 86), and have the angle ACB = FHG. But the angle ACB = ALB, standing on the same arc AB; and the angle FHG = FMG, standing on the same arc FG; therefore the angle ALB = FMG (ax. 1). And since the angle ABL = FGM, being both right angles, because in a semicircle; therefore the two triangles ABL, FGM, having two angles equal, are equiangular; and consequently their like sides are proportional (th. 84); hence AB : FG :: the diameter AL : the diameter FM.

In like manner, each side BC, CD, &c. has to each side GH, HI, &c. the same ratio of AL to FM; and consequently the sums of them are still in the same ratio, viz. AB + BC + CD, &c. : FG + GH + HI, &c. :: the diam. AL : the diam. FM (th. 72). Q. E. D.

THEOREM XCI.

Similar figures inscribed in circles, are to each other as the squares of the diameters of those circles.

Let ABCDE, FGHIK, be two similar figures, inscribed in the circles whose diameters are AL and FM; then the surface of the polygon ABCDE will be to the surface of the polygon FGHIK, as AL^2 to FM^2.

For, the figures being similar, are to each other as the squares of their like sides, AB^2 to FG^2 (th. 88). But, by the last theorem, the sides AB, FG, are as the diameters AL, FM; and therefore the squares of the sides AB^2 to FG^2, as the squares of the diameters AL^2 to FM^2 (th. 74). Consequently the polygons ABCDE, FGHIK, are also to each other as the squares of the diameters AL^2 to FM^2 (ax. 1). Q. E. D. [See fig. th. xc.]

THEOREM XCII.

The circumferences of all circles are to each other as their diameters.

Let D, d, denote the diameters of two circles, and C, c, their circumferences; then will $D : d :: C : c$, or $D : C :: d : c$.

For (by theor. 90), similar polygons inscribed in circles have their perimeters in the same ratio as the diameters of those circles.

Now, as this property belongs to all polygons, whatever the number of the sides may be; conceive the number of the sides to be indefinitely great, and the length of each infinitely small. till they coincide with the circumference of the circle, and be equal to it, indefinitely near. Then the perimeter of the polygon of an indefinite number of sides, is the same thing as the circumference of the circle. Hence it appears that the circumferences of the circles, being the same as the perimeters of such polygons, are to each other in the same ratio as the diameters of the circles. Q. E. D.

THEOREM XCIII.

The areas or spaces of circles, are to each other as the squares of their diameters, or of their radii.

Let A, a, denote the areas or spaces of two circles, and D, d, their diameters; then $A : a :: D^2 : d^2$.

For (by theorem 91), similar polygons inscribed in circles are to each other as the squares of the diameters of the circles.

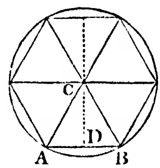
Hence, conceiving the number of the sides of the polygons to be increased more or more, or the length of the sides to become less and less, the polygon approaches nearer and nearer to the circle, till at length, by an infinite approach, they coincide, and become in effect equal; and then it follows, that the spaces of the circles, which are the same as of the polygons, will be to each other as the squares of the diameters of the circles. Q. E. D.

Corol. The spaces of circles are also to each other as the squares of the circumferences; since the circumferences are in the same ratio as the diameters (by theorem 92).

THEOREM XCIV.

The area of any circle, is equal to the rectangle of half its circumference and half its diameter.

Conceive a regular polygon to be inscribed in a circle; and radii drawn to all the angular points, dividing it into as many equal triangles as the polygon has sides, one of which is ABC, of which the altitude is the perpendicular CD from the centre to the base AB.



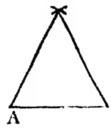
Then the triangle ABC, being equal to a rectangle of half the base and equal altitude (th. 26, cor. 2), is equal to the rectangle of the half base AD and the altitude CD; consequently, the whole polygon, or all the triangles added together which compose it, is equal to the rectangle of the common altitude CD, and the halves of all the sides, or the half perimeter of the polygon.

Now, conceive the number of sides of the polygon to be indefinitely increased; then will its perimeter coincide with the circumference of the circle, and consequently the altitude CD will become equal to the radius, and the whole polygon equal to the circle. Consequently, the space of the circle, or of the polygon in that state, is equal to the rectangle of the radius and half the circumference. Q. E. D.

PROBLEMS.

PROBLEM I.

To make an equilateral triangle on a given line AB.

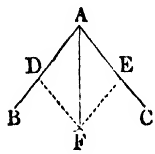


From the centres A and B, with the distance AB, describe arcs, intersecting in C. Draw AC, BC, and ABC will be the equilateral triangle.

For the equal radii AC, BC, are, each of them, equal to AB.

PROBLEM II.

To bisect a given angle BAC.



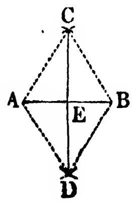
From the centre A, with any radius, describe an arc, cutting off the equal lines AD, AE; and from the two centres D, E, with the same radius, describe arcs intersecting in F; then draw AF, which will bisect the angle A as required.

Join DF, EF. Then the two triangles ADF, AEF, having the two sides AD, DF, equal to the two AE, EF (being equal radii), and the side AF common, they are mutually equilateral; consequently, they are also mutually equiangular (th. 5), and have the angle BAF equal to the angle CAF.

Scholium. In the same manner is an arc of a circle bisected.

PROBLEM III.

To bisect a given line AB.



From the two centres A and B, with any equal radii, describe arcs of circles, intersecting each other in C and D; and draw the line CD, which will bisect the given line AB in the point E.

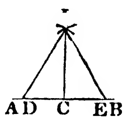
Draw the radii AC, BC, AD, BD. Then, because all these four radii are equal, and the side CD common, the two triangles ACD, BCD, are mutually equilateral; consequently, they are also mutually equiangular (th. 5), and have the angle ACE equal to the angle BCE.

Hence, the two triangles ACE, BCE, having the two sides AC, CE, equal to

the two sides BC, CE, and their contained angles equal, are identical (th. 1), and therefore have the side AE equal to EB.

PROBLEM IV.

At a given point C, in a line AB, to erect a perpendicular.

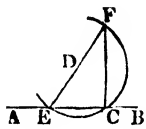


From the given point C, with any radius, cut off any equal parts CD, CE, of the given line; and, from the two centres D and E, with any one radius, describe arcs intersecting in F; then join CF, which will be perpendicular as required.

Draw the two equal radii DF, EF. Then the two triangles CDF, CEF, having the two sides CD, DF, equal to the two CE, EF, and CF common, are mutually equilateral; consequently they are also mutually equiangular (th. 5), and have the two adjacent angles at C equal to each other; therefore, the line CF is perpendicular to AB (def. 11).

OTHERWISE.

When the point C is near the end of the line.

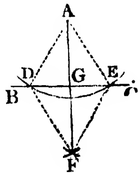


From any point D, assumed above the line, as a centre, through the given point C describe a circle, cutting the given line at E; and through E and the centre D, draw the diameter EDF; then join CF, which will be the perpendicular required.

For the angle at C, being an angle in a semicircle, is a right angle, and therefore the line CF is a perpendicular (by def. 15).

PROBLEM V.

From a given point A, to let fall a perpendicular on a given line BC.



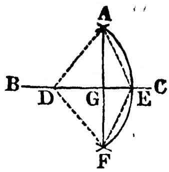
From the given point A as a centre, with any convenient radius, describe an arc, cutting the given line at the two points D and E; and from the two centres D, E, with any radius, describe two arcs, intersecting at F; then draw AGF, which will be perpendicular to BC as required.

Draw the equal radii AD, AE, and DF, EF. Then the two triangles ADF, AEF, having the two sides AD, DF, equal to the two AE, EF, and AF, common, are mutually equilateral; consequently, they are also mutually equiangular (th. 5), and have the angle DAG equal the angle EAG. Hence then, the two triangles ADG, AEG, having the two sides AD, AG, equal to the two AE, AG, and their included angles equal, are therefore equiangular (th. 1), and have the angles at G equal; consequently AG is perpendicular to BC (def. 11).

OTHERWISE.

When the point is nearly opposite the end of the line.

From any point D, in the given line BC, as a centre, describe the arc of a circle through the given point A, cutting BC in E; and from the centre E, with the radius EA, describe another arc, cutting the former in F; then draw AGF, which will be perpendicular to BC as required.

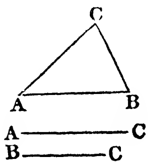


Draw the equal radii DA, DF, and EA, EF. Then the two triangles DAE, DFE, will be mutually equilater, consequently, they are also mutually equiangular (th. 5), and have the angles at D equal. Hence, the two triangles DAG, DFG, having the two sides DA, DG, equal to the two DF, DG, and the included angles at D equal, have also the angles at G equal (th. 1); consequently, those angles at G are right angles, and the line AG is perpendicular to DG.

PROBLEM VI.

To make a triangle with three given lines AB, AC, BC.

With the centre A, and distance AC, describe an arc. With the centre B, and distance BC, describe another arc, cutting the former in C. Draw AC, BC, and ABC will be the triangle required.



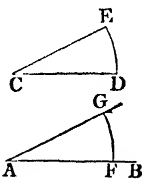
For the radii, or sides of the triangle, AC, BC, are equal to the given lines AC, BC, by construction.

Note. If any two of the lines are not together greater than the third, the construction is impossible.

PROBLEM VII.

At a given point A, *in a line* AB, *to make an angle equal to a given angle* C.

From the centres A and C, with any one radius, describe the arcs DE, FG. Then, with radius DE, and centre F, describe an arc, cutting FG in G. Through G draw the line AG, and it will form the angle required.

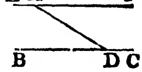


Conceive the equal lines or radii, DE, FG, to be drawn. Then the two triangles CDE, AFG, being mutually equilateral, are mutually equiangular (th. 5), and have the angle at A equal to the angle at C

PROBLEM VIII.

Through a given point A, *to draw a line parallel to a given line* BC.

From the given point A draw a line AD to any point in the given line BC. Then draw the line EAF making the angle at A equal to the angle at D (by prob. 5); so shall EF be parallel to BC as required.

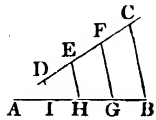


For, the angle D being equal to the alternate angle A, the lines BC, EF, are parallel, by th. 13.

PROBLEM IX.

To divide a line AB *into any proposed number of equal parts.*

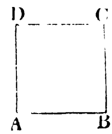
Draw any other line AC, forming any angle with the given line AB; on which set off as many of any equal parts AD, DE, EF, FC, as the line AB is to be divided into. Join BC; parallel to which draw the other lines FG, EH, DI: then these will divide AB in the manner required. — For those parallel lines divide both the sides AB, AC, proportionally, by th. 82.



PROBLEM X.

To make a square on a given line AB.

Raise AD, BC, each perpendicular and equal to AB; and join DC: so shall ABCD be the square sought.

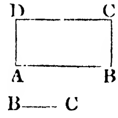


For all the three sides AB, AD, BC, are equal, by the construction, and DC is equal and parallel to AB (by th. 24); so that all the four sides are equal, and the opposite ones are parallel. Again, the angle A or B, of the parallelogram, being a right angle, the angles are all right ones (cor. 1, th. 22). Hence, then, the figure, having all its sides equal, and all its angles right, is a square (def. 34).

PROBLEM XI.

To make a rectangle, or a parallelogram, of a given length and breadth, AB, BC.

Erect AD, BC, perpendicular to AB, and each equal to BC; then join DC, and it is done.



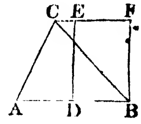
The demonstration is the same as the last problem.

And in the same manner is described any oblique parallelogram, only drawing AD and BC to make the given oblique angle with AB, instead of perpendicular to it.

PROBLEM XII.

To make a rectangle equal to a given triangle ABC.

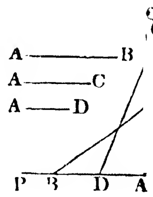
Bisect the base AB in D; then raise DE and BF perpendicular to AB, and meeting CF parallel to AB, at E and F; so shall DF be the rectangle equal to the given triangle ABC (by cor. 2, th. 26).



PROBLEM XIII.

To make a square equal to the sum of two or more given squares.

Let AB and AC be the sides of two given squares. Draw two indefinite lines AP, AQ at right angles to each other; in which place the sides AB, AC, of the given squares; join BC: then a square described on BC will be equal to the sum of the two squares described on AB and AC (th. 34).

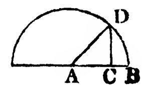


In the same manner, a square may be made equal to the sum of three or more given squares. For, if AB, AC, AD be taken as the sides of the given squares, then, making AE = BC, AD = AD, and drawing DE, it is evident that the square on DE will be equal to the sum of the three squares on AB, AC, AD. And so on for more squares.

PROBLEM XIV.

To make a square equal to the difference of two given squares.

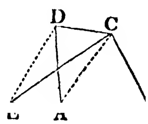
Let AB and AC, taken in the same straight line, be equal to the sides of the two given squares. From the centre A, with the distance AB, describe a circle · and make CD perpendicular to AB, meeting the circumference in D: so shall a square described on CD be equal to $AD^2 - AC^2$, or $AB^2 - AC^2$, as required (cor. th. 34).



PROBLEM XV.

To make a triangle equal to a given quadrilateral ABCD.

Draw the diagonal AC, and parallel to it DE, meeting BA produced at E, and join CE; then will the triangle CEB be equal to the given quadrilateral ABCD.

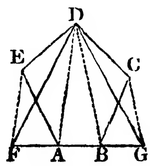


For, the two triangles ACE, ACD, being on the same base AC, and between the same parallels AC, DE, are equal (th. 25); therefore, if ABC be added to each, it will make BCE equal to ABCD (ax. 2).

PROBLEM XVI.

To make a triangle equal to a given pentagon ABCDE.

Draw DA and DB, and also EF, CG, parallel to them, meeting AB produced at F and G; then draw DF and DG; so shall the triangle DFG be equal to the given pentagon ABCDE.

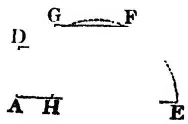


For, the triangle DFA = DEA, and the triangle DGB = DCB (th. 25); therefore, by adding DAB to the equals, the sums are equal (ax. 2), that is, DAB + DAF + DBG = DAB + DAE + DBC, or the triangle DFG = to the pentagon ABCDE.

PROBLEM XVII.

To make a square equal to a given rectangle ABCD

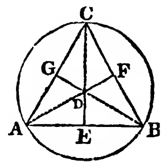
Produce one side AB, till BE be equal to the other side BC. On AE as a diameter describe a circle meeting BC produced at F: then will BF be the side of the square BFGH, equal to the given rectangle BD, as required; as appears by cor. th. 87, and th. 77.



PROBLEM XVIII.

To describe a circle about a given triangle ABC.

Bisect any two sides with two of the perpendiculars DE, DF, DG, and D will be the centre.

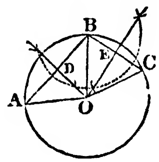


For, Join DA, DB, DC. Then the two right-angled triangles DAE, DBE, have the two sides, DE, EA, equal to the two DE, EB, and the included angles at E equal: these two triangles are therefore identical (th. 1), and have the side DA equal to DB. In like manner it is shown, that DC is also equal to DA or DB. So that all the three, DA, DB, DC, being equal, they are radii of a circle passing through A, B, and C.

Note.—The problem is the same in effect when it is required—

To describe the circumference of a circle through three given points A, B, C.

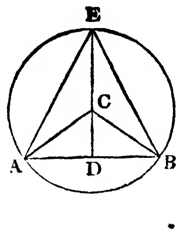
Then, from the middle point B draw chords BA, BC, to the two other points, and bisect these chords perpendicularly by lines meeting in O, which will be the centre. Again, from the centre O, at the distance of any one of the points, as OA, describe a circle, and it will pass through the two other points, B, C, as required. The demonstration is evidently as above.



PROBLEM XIX.

An isosceles triangle ABC *being given, to describe another on the same base* AB, *whose vertical angle shall be only half the vertical angle* C.

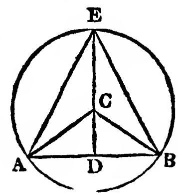
From C as a centre, with the distance CA, describe the circle ABE. Bisect AB in D, join DC, and produce to the circumference E, join EA and EB, and ABE shall be the isosceles triangle required.



For, since in the triangle EDA, EDB, AD is equal to DB, and DE common to both, and the right angle EDA, equal to the right angle EDB, the side EA must be equal to the side EB, the triangle AEB, is therefore isosceles, and the angle ACB at the centre, must be double of the angle AEB at the circumference for they both stand on the same segment AB.

PROBLEM XX.

Given an isosceles triangle AEB, *to erect another on the same base* AB, *which shall have double the vertical angle* E.

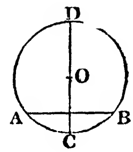


Describe a circle about the triangle AEB, find its centre C, and join CA, CB, and ACB is the triangle required.

The angle C at the centre is double of the angle E at the circumference, and the triangle ACB is isosceles for the sides CA, CB being radii of the same circle are equal.

PROBLEM XXI.

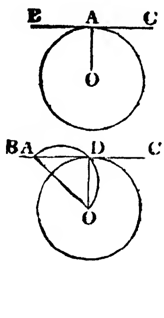
To find the centre of a given circle.



Draw any chord AB; and bisect it perpendicularly with the line CD: this (th. 41, cor.) will be a diameter. Therefore bisect CD in O, which will be the centre, as required.

PROBLEM XXII.

To draw a tangent to a circle, through a given point A.



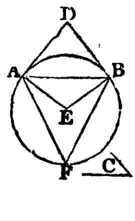
1. When the given point A is in the circumference of the circle: join A and the centre O; perpendicular to which draw BAC, and it will be the tangent, by th. 46.

2. When the given point A is out of the circle: draw AO to the centre O; on which as a diameter describe a semicircle, cutting the given circumference in D; through which draw BADC, which will be the tangent as required.

For, join DO. Then the angle ADO, in a semicircle, is a right angle, and consequently AD is perpendicular to the radius DO, or is a tangent to the circle (th. 46.)

PROBLEM XXIII.

On a given line AB *to describe a segment of a circle, to contain a given angle* C.



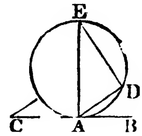
At the ends of the given line make angles DAB, DBA, each equal to the given angle C. Then draw AE, BE, perpendicular to AD, BD; and with the centre E, and radius EA or EB, describe a circle; so shall AFB be the segment required, as any angle F made in it will be equal to the given angle C.

For, the two lines AD, BD, being perpendicular to the radii EA, EB (by construction), are tangents to

the circle (th. 46); and the angle A or B, which is equal to the given angle C by construction, is equal to the angle F in the alternate segment AEB (th. 53).

PROBLEM XXIV.

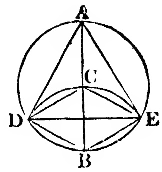
To cut off a segment from a circle, that shall contain a given angle C.



Draw any tangent AB to the given circle; and a chord AD to make the angle DAB equal to the given angle C; then DEA will be the segment required, any angle E made in it being equal to the given angle C.

PROBLEM XXV.

To inscribe an equilateral triangle in a given circle.

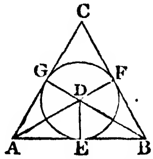


Through the centre C draw any diameter AB. From the point B as a centre, with the radius BC of the given circle, describe an arc DCE. Join AD, AE, DE, and ADE is the equilateral sought.

Join DB, DC, EB, EC. Then DCB is an equilateral triangle, having each side equal to the radius of the given circle. In like manner, BCE is an equilateral triangle. But the angle ADE is equal to the angle ABE or CBE, standing on the same arc AE; also the angle AED is equal to the angle CBD, on the same arc AD; hence the triangle DAE has two of its angles, ADE, AED, equal to the angles of an equilateral triangle, and therefore the third angle at A is also equal to the same; so that the triangle is equiangular, and therefore equilateral.

PROBLEM XXVI.

To inscribe a circle in a given triangle ABC.



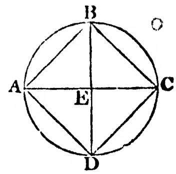
Bisect any two angles A and B, with the two lines AD, BD. From the intersection D, which will be the centre of the circle, draw the perpendiculars DE, DF, DG, and they will be the radii of the circle required.

For, since the angle DAE is equal to the angle DAG, and the angles at E, G, right angles (by construction), the two triangles, ADE, ADG, are equiangular; and, having also the side AD common, they are identical, and have the sides DE, DG, equal (th. 2). In like manner it is shown, that DF is equal to DE or DG.

Therefore, if with the centre D, and distance DE, a circle be described, it will pass through all the three points, E, F, G, in which points also it will touch the three sides of the triangle (th. 46), because the radii DE, DF DG, are perpendicular to them.

PROBLEM XXVII.

To inscribe a square in a given circle.



Draw two diameters AC, BD, crossing at right angles in the centre E. Then join the four extremities A, B, C, D, with right lines, and these will form the inscribed square ABCD.

For the four right-angled triangles AEB, BEC, CED, DEA, are identical, because they have the sides EA, EB, EC, ED, all equal, being radii of the circle, and the four included angles at E all equal, being right angles, by the construction. Therefore, all their third sides AB, BC, CD, DA, are equal to one another, and the figure ABCD is equilateral. Also, all its four angles, A, B, C, D, are right ones, being angles in a semicircle. Consequently the figure is a square.

PROBLEM XXVIII.

To describe a square about a given circle.

Draw two diameters AC, BD, crossing at right angles, in the centre E. Then through their four extremities draw FG, IH, parallel to AC, and FI, GH, parallel to BD, and they will form the square FGHI.

For, the opposite sides of parallelograms being equal, FG and IH are each equal to the diameter AC, FI and GH each equal to the diameter BD; so that the figure is equilateral. Again, because the opposite angles of parallelograms are equal, all the four angles F, G, H, I, are right angles, being equal to the opposite angles at E. So that the figure FGHI, having its sides equal, and its angles right ones, is a square, and its sides touch the circle at the four points A B, C, D, being perpendicular to the radii drawn to those points.

PROBLEM XXIX.

To inscribe a circle in a given square.

Bisect the two sides FI, FG, in the points A and B (last fig.) Then, through these two points draw AC parallel to FG or IH, and BD parallel to FI or GH. Then the point of intersection E will be the centre, and the four lines EA, EB, EC, ED, radii of the inscribed circle.

For, because the four parallelograms EF, EG, EH, EI, have their opposite sides and angles equal, therefore all the four lines EA, EB, EC, ED, are equal, being each equal to half a side of the square. So that a circle described from the centre E, with the distance EA, will pass through all the points A, B, C, D, and will be inscribed in the square, or will touch its four sides in those points, because the angles there are right ones.

PROBLEM XXX.

To describe a circle about a given square.

(See fig. Prob. xxvii.)

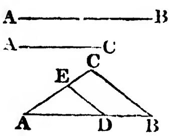
Draw the diagonals AC, BD, and their intersection E will be the centre.

As the diagonals of a square bisect each other (th. 40), then will EA, EB, EC, ED, be all equal, and consequently these are radii of a circle passing through the four points A, B, C, D.

PROBLEM XXXI.

To find a third-proportional to two given lines, AB, AC.

Place the two given lines AB, AC (or two lines equal to them), to form any angle at A; and in AB set off AD = AC. Join BC, and draw DE parallel to it; so will AE, on the line AC, be the third proportional sought.

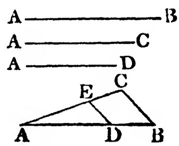


For, since DE is parallel to BC, the two lines AB, AC, are cut proportionally by DE (th. 82): hence, AB : AC :: AD (= AC) : AE, and AE is, therefore, the third proportional required.

PROBLEM XXXII.

To find a fourth proportional to three given lines, AB, AC, AD.

Place two of the given lines AB, AC, or their equals, to make any angle at A; and on AB set off, or place, the other line AD, or its equal. Join BC, and parallel to it draw DE: so shall AE be the fourth proportional as required.

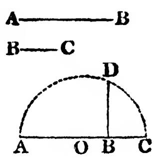


For, because of the parallels BC, DE, the two sides AB, AC, are cut proportionally (th. 82): so that AB : AC :: AD : AE.

PROBLEM XXXIII.

To find a mean proportional between two lines AB, BC.

Place AB, BC, joined in one straight line AC; on which, as a diameter, describe the semicircle ADC; to meet which erect the perpendicular BD; and it will be the mean proportional sought, between AB and BC (by cor. th. 87).

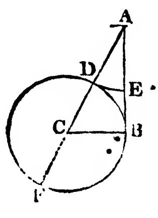


PROBLEM XXXIV.

To divide a given line in extreme and mean ratio.

Let AB be the given line to be divided in extreme and mean ratio, that is, so that the whole line may be to the greater part, as the greater is to the less part.

Draw BC perpendicular to AB, and equal to half AB. Join AC; and with centre C and distance CB, describe the circle BD; then with centre A and distance AD, describe the arc DE; so shall AB be divided in E in extreme and mean ratio, or so that AB : AE :: AE : EB.

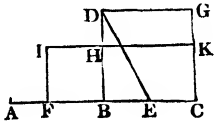


Produce AC to the circumference at F. Then, ADF

being a secant, and AB a tangent, because B is a right angle: therefore the rectangle AF·AD is equal to AB^2 (cor. 1, th. 61); consequently the means and extremes of these are proportional (th. 77), viz. AB : AF or AD + DF :: AD : AB. But AE is equal to AD by construction, and AB = 2 BC = DF; therefore, AB : AE + AB :: AE : AB or AE + EB; and by divition, AB : AE :: AE : EB.

PROBLEM XXXV.

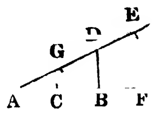
To cut a given line AB *in a point* F, *so that the square of the one part* BF *may be equal to the rectangle of the whole line* AB *and the other part* AF.



Produce AB till BC be equal to it, erect the perpendicular BD equal to AB or BC, bisect BC in E, join ED and make EF equal to it; the square of the segment BF is equivalent to the rectangle contained by the whole BA and its remaining segment AF. The line AB is then said to be divided by medial section at the point F.

For on BC construct the square BG, make BH equal to BF, and draw IHK and FI parallel to AC and BD. Since AB is equal to BD, and BF to BH; the remainder AF is equal to HD: and it is further evident, that FH is a square, and IC and DK are rectangles. But BC being bisected in E and produced to F, the rectangle under CF, FB, or the rectangle IC, together with the square of BE, is equivalent to the square of EF or DE. But the square of DE is equivalent to the squares of DB and BE; whence the rectangle IC, with the square of BE, is equivalent to the squares of DB and BE; or, omitting the common square of BE, the rectangle IC is = to the square of DB. Take away from both the rectangle BK, and there remains the square BI, or the square of BF, = to the rectangle HG, or the rectangle contained by BA and AF.

Cor. Hence also the construction of another problem of the same nature; in which it is required to produce a straight line AB, such that the rectangle contained by the whole line thus produced and the part produced, shall be equivalent to the square of the line AB itself.



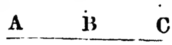
Bisect AB in C, draw the perpendicular BD = BC, join AD and continue it until DE = DB or BC, and on AB produced take AF = AE: the line AF is the required extension of AB. For make DG = DB or BC; and because the rectangle EA, AG together with the square of DG or DB, is equivalent to the square of DA or to the squares of AB and DB; the rectangle EA, AG, or FA, AC, is equivalent to the square of AB.

PROBLEM XXXVI.

Given either one of the sides AB, *or the base* a b, *to construct an isosceles triangle, so that each of the angles at the base may be double of its vertical angle.*

First, let one of the sides AB be given. By the last problem divide it into two parts, AC, CB, such that $CB^2 = AB \times AC$. Construct the triangle, having the base = CB, and each of the two sides = AB.

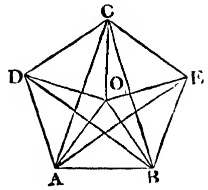
Next, if the base AB be given, by the second case of the foregoing proposition, produce AB to C, so that $AC \times CB = AB^2$ then will AB be the base, and AC the length of each of the two sides.



PROBLEM XXXVII

To describe a regular pentagon on a given line AB.

On AB erect the isosceles triangle ACB having each of the angles at the base double of its vertical angle, on AB again construct another isosceles triangle whose vertical angle AOB is double of ACB, and about the vertex O place the isosceles triangles AOD, DOC, COE, and EOB; these triangles, with AOB, will compose a regular pentagon.



For the angle AOB, being the double of ACB, which is the fifth part of two right angles, must be equal to the fifth part of four right angles; and consequently five angles, each of them equal to AOB, will adapt themselves about the point O. But the bases of those central triangles, and which form the sides of the pentagon, are all equal; and the angles at their bases being likewise equal, they are equal in the collective pairs which constitute the internal angles of the figure. It is therefore a regular pentagon.

PROBLEM XXXVIII.

To describe a hexagon upon a given line AB.

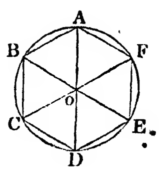
From A and B as centres, with AB as radius, describe arcs intersecting in O (fig. to the next problem). From O as a centre, with the same radius, describe a circle ABCDEF. Within this circle set off from B, the chords BC, CD, DE, EF, FA, in succession, each equal to AB: they will, together with AB, form the hexagon required.

The demonstration is analogous to that of the following problem.

PROBLEM XXXIX.

To inscribe a regular hexagon in a circle.

Apply the radius AO of the given circle as a chord, AB, BC, CD, &c. quite round the circumference, and it will complete the regular hexagon ABCDEF.



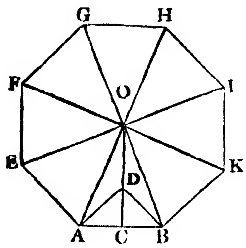
For, draw the radii AO, BO, CO, DO, EO, FO, completing six equal triangles; of which any one, as ABO, being equilateral (by constr.), its three angles are all equal (cor. 2, th. 3), and any one of them, as AOB, is one-third of the whole, or of two right angles (th. 17), or one-sixth of four right angles. But the whole circumference is the measure of four right angles (cor. 4, th. 6). Therefore the arc AB is one-sixth of the

circumference of the circle, and consequently its chord AB one side of an equilateral hexagon incribed in the circle. And the same of the other chords.

Cor. The side of a regular hexagon is equal to the radius of the circumscribing circle, or to the chord of one-sixth part of the circumference.

PROBLEM XL.

On a given line AB *to construct a regular octagon.*



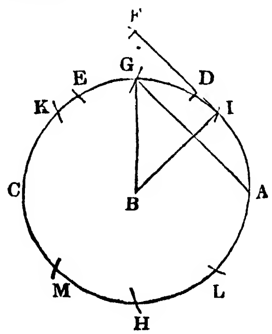
Bisect AB by the perpendicular CD, which make = CA or CB, join DA and DB, produce CD making DO = DA or DB, draw AO and BO, thus forming an angle equal to the half of ADB, and about the vertex O repeat the equal triangles AOB, AOE, EOF, FOG, GOH, HOI, IOK, and KOB to compose the octagon.

For the distances AD, BD are evidently equal; and because CA, CD, and CB are all equal, the angle ADB is contained in a semicircle, and is, therefore, a right angle. Consequently AOB is equal to the half of a right angle, and eight such angles will adapt themselves about the point O. Whence the figure BAEFGHIK, having eight equal sides and equal angles, is a regular octagon.

PROBLEM XLI.

To divide the circumference of a given circle successively into 4, 8, 12, *and* 24 *equal parts.*

1. Insert the radius AB three times from A to D, E, and C; from the extremities of the diameter AC, and with a distance equal to the double chord AE, describe arcs intersecting in the point F; and from A, with the distance BF, cut the circumference on opposite sides at G and H: AG, GC, CH, and HA are quadrants.



2. From the point F with the radius AB, cut the circle in I and K, and from A and C inflect the chord AI from L and M; the circumference is divided into eight equal portions by the points A, I, G, K, C, M, H, and L.

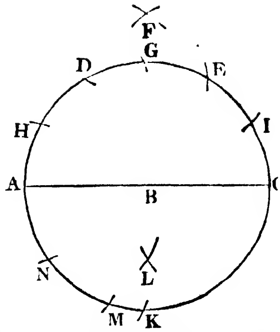
3. The arc DG, on being repeated, will form twelve equal sections of the circumference,

4. The arc ID is the twenty-fourth part of the circumference.

PROBLEM XLII.

To divide the circumference of a given circle successively into 5, 10, *and* 20 *equal parts.*

Mark out the semicircumference ADEC by the triple insertion of the radius, from A and C with the double chord AE describe arcs intersecting in F, from

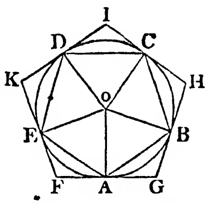


A with the distance BF cut the circle in G and H, inflect the chords GH and GI equal to the radius AB, and from the points H and I, with distance BF or AG, describe arcs intersecting in L.

For BL is the greater segment of the radius BH divided by a medial section; wherefore AL is equal to the side of the inscribed pentagon, and BL, to that of the decagon inscribed in the given circle. Hence AL may be inflected five times in the circumference, and BL ten times; and consequently the arc MK, or the excess of the fourth above the fifth, is equal to the twentieth part of the whole circumference.

PROBLEM XLIII.

To describe a regular pentagon, hexagon, or octagon, about a circle.



In the given circle inscribe a regular polygon of the same name or number of sides, as ABCDE, by one of the foregoing problems. Then to all its angular points draw tangents (by prob. 22), and these will form the circumscribing polygon required.

For all the chords, or sides of the inscribed figure, AB, BC, &c., being equal; and all the radii OA, OB, &c., being equal; all the vertical angles about the point O are equal. But the angles OEF, OAF, OAG, OBG, made by the tangents and radii, are right angles; therefore OEF + OAF = two right angles, and OAG + OBG = two right angles; consequently, also, AOE + AFE = two right angles, and AOB + AGB = two right angles (cor. 2, th. 18). Hence, then, the angles AOE + AFE being = AOB + AGB, of which AOB is = AOE; consequently, the remaining angles F and G are also equal. In the same manner it is shown, that all the angles F, G, H, I, K, are equal.

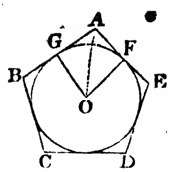
Again, the tangents from the same point FE, FA, are equal, as also the tangents AG, GB (cor. 2, th. 61); and the angles F and G of the isosceles triangles AFE, AGB, are equal, as well as their opposite sides AE, AB; consequently, those two triangles are identical (th. 1), and have their other sides EF, FA, AG, GB, all equal, and FG equal to the double of any one of them. In like manner it is shown, that all the other sides GH, HI, IK, KF, are equal to FG, or double of the tangents GB, BH, &c.

Hence, then, the circumscribed figure is both equilateral and equiangular; which was to be shown.

Cor.—The inscribed circle touches the middle of the sides of the polygon.

PROBLEM XLIV.

To inscribe a circle in a regular polygon.

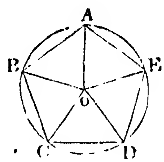


Bisect any two sides of the polygon by the perpendiculars GO, FO, and their intersection O will be the centre of the inscribed circle and OG or OF will be the radius.

For the perpendiculars to the tangents AF, AG, pass through the centre (cor., th. 47); and the inscribed circle touches the middle point F, G, by the last corollary. Also, the two sides AG, AO, of the right-angled triangle AOG, being equal to the two sides AF, AO, of the right-angled triangle AOF, the third sides OF, OG, will also be equal (cor., th. 45). Therefore, the circle described with the centre O and radius OG will pass through F, and will touch the sides in the points G and F. And the same for all the other sides of the figure.

PROBLEM XLV.

To describe a circle about a regular polygon.

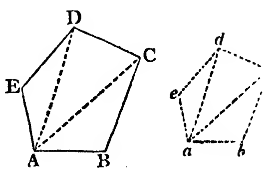


Bisect any two of the angles C and D with the lines CO, DO; then their intersection O will be the centre of the circumscribing circle; and OC, or OD, will be the radius.

For, draw OB, OA, OE, &c., to the angular points of the given polygon. Then the triangle OCD is isosceles, having the angles at C and D equal, being the halves of the equal angles of the polygon BCD, CDE; therefore, their opposite sides CO, DO, are equal (th. 4). But the two triangles OCD, OCB, having the two sides OC, CD, equal to the two OC, CB, and the included angles OCD, OCB, also equal, will be identical (th. 1), and have their third sides BO, OD, equal. In like manner it is shown, that all the lines OA, OB, OC, OD, OE, are equal. Consequently, a circle described with the centre O and radius OA, will pass through all the other angular points, B, C, D, &c., and will circumscribe the polygon.

PROBLEM XLVI.

On a given line to construct a rectilinear figure similar to a given rectilinear figure.



Let *abcde* be the given rectilinear figure, and AB the side of the proposed similar figure that is similarly posited with *ab*.

Place AB in the prolongation of *ab*, or parallel to it. Draw AC, AD, AE, &c., parallel to *ac*, *ad*, *ae*, respectively. Draw BC parallel to *bc*, meeting AC in C; CD parallel to *cd*, and meeting AD in D; DE parallel to *de*, and meeting AE in E; and so on, till the figure is completed. Then ABCDE will be similar to *abcde*, from the nature of parallel lines and similar figures (th. 89).

Miscellaneous Exercises in Plane Geometry.

(1.) *From two given points, to draw two equal straight lines, which shall meet in the same point of a line given in position.*

(2.) From two given points, on the same side, or opposite sides of a line given in position, to draw two lines, which shall meet in that line, and make equal angles with it.

(3.) To trisect a given finite straight line.

(4.) If from the extremities of the diameter of a semicircle, perpendiculars be let fall on any line cutting the semicircle, the parts intercepted between those perpendiculars and the circumference are equal.

(5.) If on each side of any point in a circle any number of equal arcs be taken, and the extremities of each pair joined, the sum of the chords so drawn will be equal to the last chord produced to meet a line drawn from the given point through the extremity of the first arc.

(6.) If one circle touch another externally or internally, any straight line drawn through the point of contact will cut off similar segments.

(7.) If two circles touch each other, and also touch a straight line, the part of the line between the points of contact is a mean proportional between the diameters of the circles.

(8.) From two given points in the circumference of a given circle, to draw two lines to a point in the circumference, which shall cut a line given in position, so that the part of it intercepted by them may be equal to a given line.

(9.) If from any point within an equilateral triangle perpendiculars be drawn to the sides, they are, together, equal to a perpendicular drawn from any of the angles to the opposite side.

(10.) If the three sides of a triangle be bisected, the perpendiculars drawn to the sides, at the three points of bisection, will meet in the same point.

(11.) If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.

(12.) The three straight lines which bisect the three angles of a triangle, meet in the same point.

(13.) If from the angles of a triangle perpendiculars be drawn to the opposite sides, they will intersect in the same point.

(14.) If any two chords be drawn in a circle, to intersect at right angles, the sum of the squares of the four segments is equal to the square of the diameter of the circle.

(15.) In a given triangle to inscribe the *greatest* square.

(16.) In a given triangle to inscribe a rectangle, whose sides shall have a given ratio.

(17.) The two sides of a triangle are, together, greater than the double of the straight line which joins the vertex and the bisection of the base.

(18.) If in the sides of a square, at equal distances from the four angles, four other points be taken, one in each side, the figure contained by the straight lines which join them shall also be a square.

(19.) If the sides of an equilateral and equiangular pentagon be produced to meet, the angles formed by these lines are, together, equal to two right angles.

(20.) If the sides of an equilateral and equiangular hexagon be produced to meet, the angles formed by these lines are, together, equal to four right angles.

(21.) If squares be described on the three sides of a right-angled triangle, and the extremities of the adjacent sides be joined, the triangles so formed are equal to the given triangle, and to each other.

(22.) If squares be described on the hypothenuse and sides of a right-angled triangle, and the extremities of the sides of the former, and the adjacent sides of the others, be joined, the sum of the squares of the lines joining them will be equal to five times the square of the hypothenuse.

(23.) To bisect a triangle by a line drawn parallel to one of its sides.

(24.) To divide a circle into any number of concentric equal annuli.

(25.) To inscribe a square in a given semicircle.

(26.) If in a right-angled triangle a perpendicular be drawn from the right angle to the hypothenuse, and circles inscribed in the triangles on each side of it, their diameters will be to each other as the subtending sides of the right-angled triangle.

(27.) If on one side of an equilateral triangle, as a diameter, a semicircle be described, and from the opposite angle two straight lines be drawn to trisect that side, these lines produced will trisect the semi-circumference.

(28.) Draw straight lines across the angles of a given square, so as to form an equilateral and equiangular octagon.

(29.) The square of the side of an equilateral triangle, inscribed in a circle, is equal to three times the square of the radius.

(30.) To draw straight lines from the extremities of a chord to a point in the circumference of the circle, so that their sum shall be equal to a given line.

(31.) In a given triangle to inscribe a rectangle of a given magnitude.

(32.) Given the perimeter of a right-angled triangle, and the perpendicular from the right angle upon the hypothenuse, to construct the triangle.

(33.) Describe a circle touching a given straight line, and also passing through two given points.

(34.) In an isosceles triangle to inscribe three circles, touching each other, and each touching two of the three sides of the triangle.

GEOMETRY OF PLANES.

DEFINITIONS.

1. A PLANE is a surface in which, if any two points be taken, the straight line which joins these points will be wholly in that surface.

2. A straight line is said to be perpendicular to a plane, when it is perpendicular to all the straight lines in the plane which pass through the point in which it meets the plane.

This point is called the foot of the perpendicular.

3. The inclination of a straight line to a plane, is the acute angle contained by the straight line, and another straight line drawn from the point in which the first meets the plane, to the point in which a perpendicular to the plane, drawn from any point in the first line, meets the plane.

4. A straight line is said to be parallel to a plane when it cannot meet the plane, to whatever distance both be produced.

5. It will be proved in Prop. 2, that the common intersection of two planes is a straight line; this being premised,

The angle contained by two planes, which cut one another, is measured by the angle contained by two straight lines drawn from any point in the common intersection of the planes perpendicular to it, one in each of the planes.

This angle may be acute, right, or obtuse.

If it be a right angle, the planes are said to be perpendicular to each other.

6. Two planes are parallel to each other, when they cannot meet, to whatever distance both be produced.

PROP. I.

A straight line cannot be partly in a plane, and partly out of it.

For, by def. (1), when a straight line has two points common to a plane, it lies wholly in that plane.

PROP. II.

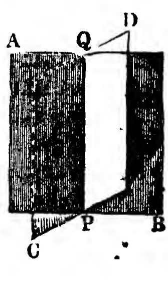
If two planes cut each other, their common intersection is a straight line.

Let the two planes, AB, CD, cut one another, and let P, Q, be two points in their common section.

Join P, Q;

Then, since the points P, Q, are in the same plane AB, the straight line PQ which joins them must lie wholly in that plane.

For a similar reason, PQ must lie wholly in the plane CD.



∴ The straight line PQ is common to the two planes, and is ∴ their common intersection.

PROP. III.

Any number of planes may be drawn through the same straight line.

For let a plane, drawn through a straight line, be conceived to revolve round the straight line as an axis. Then the different positions assumed by the revolving plane will be those of different planes drawn through the straight line.

PROP. IV.

One plane, and one plane only, can be drawn,

1°. *Through a straight line, and a point not situated in the given line.*

2°. *Through three points which are not in the same straight line.*

3°. *Through two straight lines which intersect each other.*

4° *Through two parallel straight lines.*

1. For if a plane be drawn through the given line, and be conceived to revolve round it as an axis, it must in the course of a complete revolution pass through the given point, and so assume the position enounced in 1°.

Also, one plane only can answer these conditions, for if we suppose a second plane passing through the same straight line and point, it must have at least two intersections with the first, which is impossible.

2. Join two of the points, this case is then reduced to the last.

3. Take a point in each of the lines which is not the point of intersection, join these two points; the case is now the same as the two former.

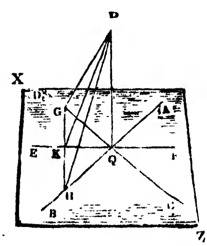
4. Parallel straight lines are, by their definition, in the same plane, and, by the first case, one plane only can be drawn through either of them, and a point assumed in the other.

Cor. Hence, the position of a plane is determined by,

1. *A straight line, and a point not in the given straight line.*
2. *A triangle, or three points not in the same straight line.*
3. *Two straight lines which intersect each other.*
4. *Two parallel straight lines.*

PROP. V.

If a straight line be perpendicular to two other straight lines which intersect at its foot in a plane, it will be perpendicular to every other straight line drawn through its foot in the same plane, and will therefore be perpendicular to the plane.



Let XZ be a plane, and let the straight line PQ be perpendicular to the two straight lines AB, CD which intersect in Q in the plane XZ.

Draw any straight line EF through Q;

Then PQ will be perpendicular to EF.

Draw through any point K in QF a straight line GH, such, that $GK = KH$.

Join P, G; P, K; P, H;

Then, since GH, the base of the $\triangle$ GQH, is bisected in K;

$$\therefore GQ^2 + HQ^2 = 2GK^2 + 2QK^2 \ldots\ldots\ldots(1)$$

Similarly, since GH, the base of $\triangle$ GPH, is bisected in K;

$$\therefore\ GP^2 + HP^2 = 2GK^2 + 2PK^2.$$

But the angles PQG, PQH, are right angles, $\therefore$ the above becomes,

$$PQ^2 + GQ^2 + PQ^2 + HQ^2 = 2GK^2 + 2PK^2 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Taking (1) from (2), there remains,

$$2PQ^2 = 2PK^2 - 2QK^2$$

$$\therefore\ PQ^2 + QK^2 = PK^2$$

Hence, the angle PQK is a right angle.

In like manner, it may be proved that PQ is at right angles to every other straight line passing through Q in the plane XZ

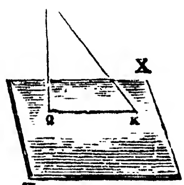
PROP. VI.

A perpendicular is the shortest line which can be drawn from any point to a plane.

Let PQ be perpendicular to the plane XZ;

From P draw any other straight line PK to the plane XZ;

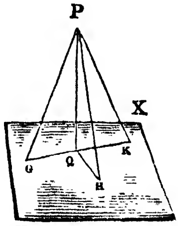
Then PQ $\angle$ PK.



In the plane XZ draw the straight line QK, joining the points Q, K.

Then, since the line PQ is perpendicular to the plane XZ, the angle PQK is a right angle; and $\therefore$ PQ is less than any other line PK. (Geom. Theor. xxi.)

Cor. 1. Hence, oblique lines equally distant from the perpendicular are equal, and, if two oblique lines be unequally distant from the perpendicular, the more distant is the larger.

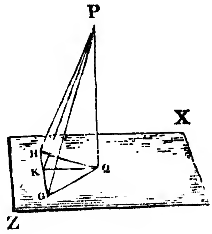


That is, if QG, QH, QK, ... are all equal, then PG, PH, PK, ... are all equal; and if QH be greater than QG, then PH is greater than PG.

Cor. 2. A perpendicular measures the distance of any point from a plane. The distance of one point from another is measured by the straight line joining them, because this is the shortest line which can be drawn from one point to another. So also, the distance from a point to a line, is measured by a perpendicular, because this line is the shortest that can be drawn from the point to the line. In like manner, the distance from a point to a plane, must be measured by a perpendicular drawn from that point to the plane, because this is the shortest line that can be drawn from the point to the plane.

PROP. VII.

Let PQ *be a perpendicular on the plane* XZ, *and* GH *a straight line in that plane; if from* Q, *the foot of the perpendicular,* QK *be drawn perpendicular to* GH, *and* P, K, *be joined; then* PK *will be perpendicular to* GH.



Take KG = KH, join P, G; P, H; Q, G; Q, H;

∵ KG = KH, and KQ common to the triangles GQK, HQK, and angle GKQ = angle HKQ, each being a right angle.

∴ QG = QH

∴ PG = PH Cor. to last Prop.

Hence, the two triangles GKP, HKP, have the two sides GK, KP, equal to the two sides HK, KP, and the remaining side GP, equal to the remaining side HP.

∴ Angle GKP = angle HKP, and ∴ each of them is a right angle.

Cor. GH is perpendicular to the plane PQK, for GH is perpendicular to each of the two straight lines KP, KQ.

Remark.—The two straight lines PQ, GH, present an example of two straight lines which do not meet, because they are not situated in the same plane.

The shortest distance between these two lines is the straight line QK, which is perpendicular to each of them.

For, join any two other points, as P, G;

$$\left.\begin{array}{ll}\text{Then,} & PG > PK \\ \text{And,} & KP > KQ\end{array}\right\} \text{last Prop.}$$

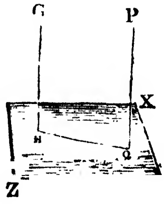
$$\therefore \quad PG > KQ$$

The two lines PQ, GH, although not situated in the same plane, are considered to form a right angle with each other. For PQ, and a straight line drawn through any point in PQ parallel to GH, would form a right angle.

In like manner, PG, and QK, which represent any two straight lines not situated in the same plane, are considered to form with each other the same angle which PG would make with any parallel to QK, drawn through a point in PG.

PROP. VIII.

If two straight lines be perpendicular to the same plane, they will be parallel to each other.



Let each of the straight lines PQ, GH, be perpendicular to the plane XZ.

Then, PQ will be parallel to GH.

In the plane XZ draw the straight line QH, joining the points Q, H.

Then, since PQ, GH, are perpendicular to the plane XZ; they are perpendicular to the straight line QH in that plane; and, since PQ, GH, are both perpendicular to the same line QH, they are parallel to each other. (Geom. theor. 13, cor.)

Cor. 1. Conversely, if two straight lines be parallel, and if one of them be perpendicular to any plane, the other will also be perpendicular to the same plane.

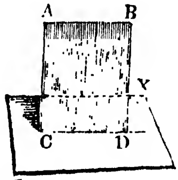
Cor. 2. Two straight lines parallel to a third, are parallel to each other.

For, conceive a plane perpendicular to any one of them, then the other two being parallel to the first, will be perpendicular to the same plane; hence, b. the Prop. they will be parallel to each other.

The three straight lines are not supposed to be in the same plane, in this case the Proposition has been already demonstrated.

PROP. IX.

If a straight line, without a given plane, be parallel to a straight line in th plane, it will be parallel to the plane.



Let AB, lying without the plane XZ, be parallel to CD, lying in the plane,

Then AB is parallel to the plane XZ.

Through the parallels AB, CD, draw the plane ABCD.

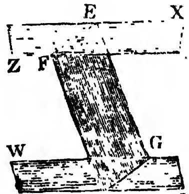
If the line AB can meet the plane XZ, it must meet it in some point of the line CD, which is the common intersection of the two planes.

But AB cannot meet CD, because AD is parallel to CD.

Hence, AB cannot meet the plane XZ, *i. e.* AB is parallel to the plane XZ.

PROP. X.

The sections made by a plane cutting two parallel planes, are parallel.



Let FE, GH, be the sections made by the plane GF which cuts the parallel planes XZ, WY;

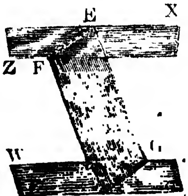
Then, FE will be parallel to GH.

For if the lines FE, GH, which are situated in the same plane, be not parallel, they will meet if produced. Therefore, the planes XZ, WY, in which these lines lie, will meet if produced, and ∴ cannot be parallel, which is contrary to the hypothesis.

∴ FE is parallel to GH.

PROP. XI.

Parallel straight lines included between two parallel planes are equal.



Let the parallels EG, FH, be cut by the parallel planes XZ, WY, in the points G, H, E, F,

Then, EG = FH,

Through the parallels EG, FH, draw the plane EGHF, intersecting the parallel planes in GH, FE.

Then, GH is parallel to FE, by last Prop.

And, GE is parallel to HF;

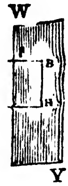
$\therefore$ GHFE is a parallelogram; and therefore,

$$EG = FH$$

Cor. Two parallel planes are every where equidistant.

PROP. XII.

If two planes be parallel to each other, a straight line which is perpendicular to one of the planes, will be perpendicular to the other also.



Let the two planes XZ, WY, be parallel, and let the straight line AB, be perpendicular to the plane XZ;

Then, AB will be perpendicular to WY.

For, from any point H in the plane WY, draw HG perpendicular to the plane XZ, and draw AG, BH.

Then, since BA, HG, are both perpendicular to XZ; $\therefore$ the angles A, G, are right angles.

And, since the planes XZ, WY, are parallel, $\therefore$ the perpendiculars BA, HG, are equal.

Hence AG is parallel to BH, and AB being perpendicular to AG, is perpendicular to BH also.

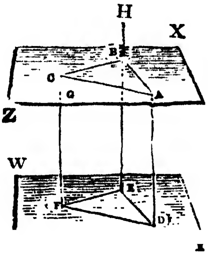
In like manner, it may be proved, that AB is perpendicular to all other lines which can be drawn from B in the plane WY.

$\therefore$ AB is perpendicular to the plane WY.

Cor. Conversely, if two planes be perpendicular to the same straight line, they will be parallel to each other.

PROP. XIII.

If two straight lines which form an angle, be parallel to two other straight lines which form an angle in the same direction, although not in the same plane with the former, the two angles will be equal, and their planes will be parallel.



Let the two straight lines AB, BC, in the plane XZ, be parallel to the two DE, EF, in the plane WY;

Then, angle ABC = angle DEF.

For, make AB = DE, BC = EF; join A, C; D, F; A, D; B, E; C, F;

Then, the straight lines AD, BE, which join the equal and parallel straight lines AB, DE, are themselves equal and parallel.

For the same reason, CF, BE, are equal and parallel.

$\therefore$ AD, CF, are equal and parallel, and $\therefore$ AC, DF, are, also, equal and parallel.

Hence, the two triangles ABC, DEF, having all their sides equal, each to each, have their angles also equal.

$\therefore$ angle ABC = angle DEF.

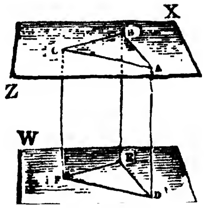
Again, the plane XZ is parallel to the plane WY.

For, if not, let a plane drawn through A parallel to DEF, meet the straight lines FC, EB, in G and H.

Then, DA = EH = FG Prop.
But, DA = EB = FC
∴ EH = EB, FG = FC

which is absurd; hence,

Cor. 1. If two parallel planes XZ, WY, are met by two other planes ADEB, CFEB, the angles ABC, DEF, formed by the intersection of the parallel planes, will be equal.



For the section AB is parallel to the section DE, Prop.

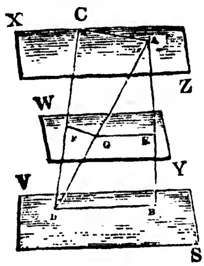
So also, the section BC is parallel to the section EF.

∴ angle ABC = angle DEF.

Cor. 2. If three straight lines AD, BE, CF, not situated in the same plane, be equal and parallel, the triangles ABC, DEF, formed by joining the extremities of these straight lines, will be equal, and their planes will be parallel.

PROP. XIV.

If two straight lines be cut by parallel planes, they will be cut in the same ratio.



Let the straight lines AB, CD, be cut by the parallel planes XZ, WY, VS, in the points A, E, B; C, F, D;

Then, AE : EB : : CF : FD.

Join A, C; B, D; A, D; and let AD meet the plane WY in G; join E, G; G, F;

Then, the intersections EG, BD, of the parallel planes WY, VS, with the plane ED, are parallel. (Prop. x.)

∴ AE : EB : : AG : GD

Again, the intersections AC, GF, of the parallel planes XZ, YW, with the plane CG, are parallel,

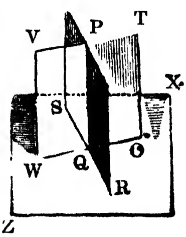
∴ AG : GD : : CF : FD

∴ comparing this with the first proportion,

AE : EB : : CF : FD

PROP. XV.

If a straight line be at right angles to a plane, every plane which passes through it will be at right angles to that plane.



Let the straight line PQ be at right angles to the plane XZ.

Through PQ draw any plane PO, intersecting XZ in the line OQW.

Then, the plane PO is perpendicular to the plane XZ.

Draw RS, in the plane XZ, perpendicular to WQO.

Then, since the straight line PQ is perpendicular to the plane XZ, it is *perpendicular to the two straight lines* RS, OW, *which pass through its foot in that plane.*

But the angle PQR, contained between PQ, QR which are perpendiculars to OW, the common intersection of the planes XZ, PO, measures the angle of the two planes (Def. 5); hence, since this angle is a right angle, the two planes are perpendicular to each other.

Cor. If three straight lines, such as PQ, RS, OW, be perpendicular to each other, each will be perpendicular to the plane of the two others, and the three planes will be perpendicular to each other.

PROP. XVI.

If two planes be perpendicular to each other, a straight line drawn in one of the planes perpendicular to their common section, will be perpendicular to the other plane.

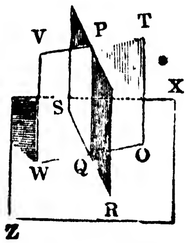
Let the plane VO be perpendicular to the plane XZ and let OW be their common section.

In the plane VO draw PQ perpendicular to OW;

Then PQ is perpendicular to the plane XZ.

From the point Q, draw QR in the plane XZ, perpendicular to OW

Then, since the two planes are perpendicular, the angle PQR is a right angle.



∴ The straight line PQ, is perpendicular to the straight lines QR, QO, which intersect at its foot in the plane XZ.

∴ PQ is perpendicular to the plane XZ.

Cor. If the plane VO be perpendicular to the plane XZ, and if from any point in OW, their common intersection, we erect a perpendicular to the plane XZ, that straight line will lie in the plane VO.

For if not, then we may draw from the same point a straight line in the plane VO, perpendicular to OW, and this line, by the Prop. will be perpendicular to the plane XZ.

Thus we should have two straight lines drawn from the same point in the plane XZ, each of them perpendicular to the given plane, which is absurd.

PROP. XVII.

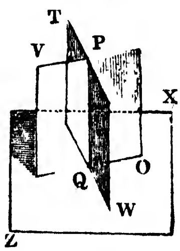
If two planes which cut each other, be each of them perpendicular to a third plane, their common section will be perpendicular to the same plane.

Let the two planes VO, TW, whose common section is PQ, be both perpendicular to the plane XZ.

Then, PQ is perpendicular to the plane XZ.

For, from the point Q, erect a perpendicular to the plane XZ.

Then, by Cor. to last Prop., this straight line must be situated at once in the planes VO and TW, and is ∴ their common section.



SOLID ANGLES.

DEFINITION.

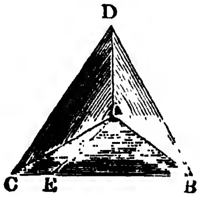
A *solid angle* is the angular space contained between several planes which meet in the same point.

Three planes, at least, are required to form a solid angle.

A solid angle is called a *trihedral, tetrahedral,* &c. *angle*, according as it is formed by *three, four,* . . . . plane angles.

If a solid angle be contained by three plane angles, the sum of any two of these angles will be greater than the third.

It is unnecessary to demonstrate this proposition except in the case where the plane angle, which is compared with the two others, is greater than either of them.



Let A be a solid angle, contained by the three plane angles BAC, CAD, DAB, and let BAC be the greatest of these angles;

Then, $\quad CAD + DAB > BAC.$

In the plane BAC draw the straight line AE, making the angle BAE = angle BAD.

Make AE = AD, and through E draw any straight line BEC, cutting AB, AC, in the points B, C; join D, B; D, C;

Then, ∵ AD = AE, and AB is common to the two triangles DAB, BAE, and the angle DAB = angle BAE.

$$\therefore \quad BD = BE$$

$$\text{But,} \quad BD + DC > BE + EC,$$

$$\therefore \quad DC > EC$$

Again, ∵ AD = AE, and AC is common to the two triangles DAC, EAC, but the base DC > base EC.

$$\therefore \quad \text{angle } DAC > \text{angle } EAC$$

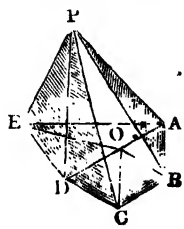
$$\text{But, angle } DAB = \text{angle } BAE$$

$$\therefore \quad \text{angle } CAD + \text{angle } DAB > \text{angle } BAE + \text{angle } EAC$$

$$> \text{angle } BAC.$$

PROP. II.

The sum of the plane angles which form a solid angle, is always less than four right angles.



Let P be a solid angle contained by any number of plane angles APB, BPC, CPD, DPE, EPA.

Let the solid angle P be cut by any plane ABCDE.

Take any point O in this plane; join A, O; B, O; C, O; D, O; E, O;

Then, since the sum of all the angles of every triangle is always equal to two right angles, the

sum of all the angles of the triangles ABP, BPC, ... about the point P, will be equal to the sum of all the angles of the equal number of triangles AOB, BOC, ... about the point O.

Again, by the last Prop., angle ABC $\angle$ angle ABP + angle CBP; in like manner, angle BCD $\angle$ angle BCP + angle DCP, and so for all the angles of the polygon ABCDE.

Hence, the sum of the angles at the bases of the triangles whose vertex is O, is less than the sum of the angles at the bases of the triangles whose vertex is P.

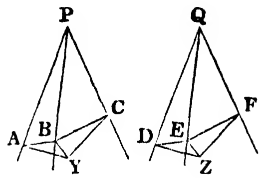
∴ The sum of the angles about the point O, must be greater than the sum of the angles about the point P.

But, the sum of the angles about the point O, is four right angles.

∴ The sum of the angles about the point P, is less than four right angles.

PROP. III.

If two solid angles be formed by three plane angles which are equal, each to each the planes in which these angles lie will be equally inclined to each other.



Let P, Q, be two solid angles, each contained by three plane angles;

Let angle APC = angle DQF, angle APB = angle DQE, and angle BPC = angle EQF.

Then, the inclination of the planes APC, APB, will be equal to the inclination of the planes DQF, DQE.

Take any point B in the intersection of the planes APB, CPB.

From B draw BY perpendicular to the plane APC, meeting the plane in Y.

From Y draw YA, YC, perpendiculars on PA, PC; join A, B; B, C;

Again, take QE = PB, from E draw EZ perpendicular to the plane DQF, meeting the plane in Z, from Z draw ZD, ZF, perpendiculars on QD, QF; join D, E; E, F.

The triangle PAB is right angled at A, and the triangle QDE is right angled at D. (Geom. of Planes, Prop. VII.)

Also, the angle APB = angle DQE, by construction.

∴ angle PBA = angle QED

But, the side PB = side QE, ∴ the two triangles APB, DQF, are equal and similar.

∴ PA = QD, and, AB = DE

In like manner, we can prove that,

PC = QF, and, BC = EF

We can now prove that the quadrilateral PAYC, is equal to the quadrilateral QDZF.

For, let the angle APC be placed upon the equal angle DQF, then the point A will fall upon the point D, and the point C on the point F, because PA = QD, and PC = QF.

At the same time, AY, which is perpendicular to PA, will fall upon DZ, which is perpendicular to QD; and in like manner, CY will fall upon FZ.

Hence, the point Y will fall on the point Z, and we shall have,

AY = DZ, and, CY = FZ

But, the triangles AYB, DZE, are right angled in Y and Z, the hypothenuse AB = hypothenuse DE, and the side AY = side DZ; hence, these two triangles are equal.

$\therefore$ angle YAB = angle ZDE

The angle YAB is the inclination of the planes APC, APB; and,
The angle ZDE is the inclination of the planes DQF, DQE.
$\therefore$ These planes are equally inclined to each other.

In the same manner, we prove that angle YCB = angle ZFE, and consequently, the inclination of the planes APC, BPC, is equal to the inclination of the planes DQF, EQF.

We must, however, observe, that the angle A of the right angled triangle YAB, is not, properly speaking, the inclination of the two planes APC, APB, except when the perpendicular BY falls upon the same side of PA as PC does; if it fall upon the other side, then the angle between the two planes will be obtuse, and, added to the angle A of the triangle YAB, will make up two right angles. But, in this case, the angle between the two planes DQF, DQE, will also be obtuse, and, added to the angle D of the triangle ZDE, will make up two right angles.

Since, then, the angle A will always be equal to the angle D, we infer that the inclination of the two planes APC, APB, will always be equal to the inclination of the two planes DQF, DQE. In the first case, the inclination of the plane is the angle A or D; in the second case, it is the supplement of those angles.

Scholium.—If two solid trihedral angles have the three plane angles of the one equal to the three plane angles of the other, each to each, and at the same time the corresponding angles *arranged in the same manner* in the two solid angles, then these two solid angles will be equal; and if placed one upon the other, they will coincide. In fact, we have already seen, that the quadrilateral PAYC will coincide with the quadrilateral QDZF. Thus, the point Y falls upon the point Z, and, in consequence of the equality of the triangles AYB, DZE, the straight line YB, perpendicular to the plane APC, is equal to the straight line, ZE perpendicular to the plane DQE; moreover, these perpendiculars lie in the same direction; hence, the point B will fall upon the point E, the straight line PB on the straight line QE, and the two solid angles will entirely coincide with each other.

This coincidence, however, cannot take place, except we suppose the equal plane angles to be *arranged in the same manner* in the two solid angles; for if the equal plane angles be *arranged in an inverse order*, or, which comes to the same thing, if the perpendiculars YB, ZE, instead of being situated both on the same side of the planes APC, DQF, were situated on opposite sides of these planes, then it would be impossible to make the two solid angles coincide with each other. It would not, however, be less true, according to the above theorem, that the planes, in which the equal angles lie, would be equally inclined to each other; so that the two solid angles would be equal in all their constituent parts, without admitting of superposition. This species of equality, which is not absolute, or *equality of coincidence*, has received from Legendre a particular description. He terms it *equality of symmetry*.

Thus, the two solid trihedral angles in question, which have the three plane angles of the one, equal to the three plane angles of the other, each to each, but

arranged in an inverse order, are termed *angles equal by symmetry*, or simply, *symmetrical angles.*

The same observation applies to solid angles formed by more than three plane angles. Thus, a solid angle formed by the plane angles A, B, C, D, E, and another solid angle formed by the same angles in an inverse order, A, E, D, C, B, may be such that the planes in which the equal angles are situated are equally inclined to each other. These two solid angles, which would in this case be equal, although not admitting of superposition, would be termed *solid angles equal by symmetry*, or *symmetrical solid angles.*

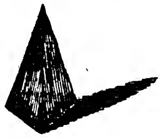
In plane figures, there is no species of equality to which this designation can belong, for all those cases to which the term might seem to apply, are cases of *absolute equality*, or *equality of coincidence.* The reason of this is, that the position of a plane figure may be altered at pleasure, and one may take the upper part for the under, and vice versa. This, however, does not hold in solids. in which the third dimension may be taken in two different directions.

SOLID GEOMETRY.

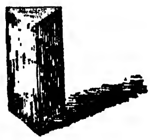
DEFINITIONS.

1. SIMILAR solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

2. A pyramid is a solid figure contained by planes that are constituted betwixt one plane and one point above it in which they meet.

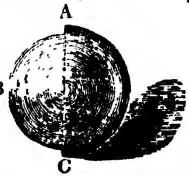


3. A prism is a solid figure contained by plane figures, of which, two that are opposite are equal, similar, and parallel to each other; and the others are parallelograms.



4. A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

Thus, the inner side of the semicircle ABC revolving round the diameter AC, which remains fixed, generates a sphere.



5. The axis of a sphere is the fixed right line about which the semicircle revolves.

Thus AC, in the figure above, is the axis of the sphere.

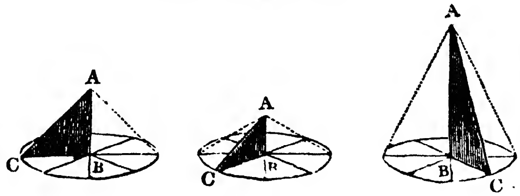
6. The centre of a sphere is the same with that of the semicircle.

7. The diameter of a sphere is any right line which passes through the centre, and is terminated both ways by the superficies of the sphere.

8. A right cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled; and if greater, an acute-angled cone.

Thus, the side AC, revolving round AB, one of the sides which contains the right angle and remains fixed, generates a cone.

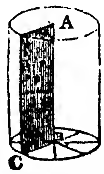


9. The axis of a cone is the fixed right line about which the triangle revolves. In figure above, AB is the axis.

10. The base of a cone is the circle described by that side containing the right angle which revolves.

11. A cylinder is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed.

Thus, the revolution of the parallelogram AC about its side AB, which remains fixed, generates a cylinder.



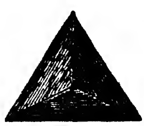
12. The axis of a cylinder is the fixed right line about which the parallelogram revolves.

13. The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

14. Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

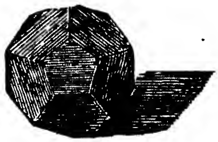
15. A cube is a solid figure contained by six equal squares.

16. A tetrahedron is a solid figure contained by four equal and equilateral triangles.

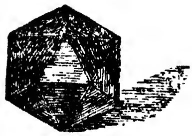


17. An octahedron is a solid figure contained by eight equal and equilateral triangles.

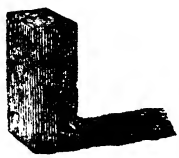
18. A dodecahedron is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.



19. An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

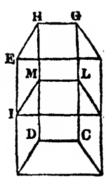


20. A parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.



PROPOSITIONS.

If a prism be cut by a plane parallel to its base, the section will be equal and like to the base.

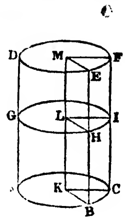


Let AG be any prism, and IL a plane parallel to the base AC; then will the plane IL be equal and like to the base AC, or the two planes will have all their sides and all their angles equal.

For, the two planes, AC, IL, being parallel, by hypothesis; and two parallel planes, cut by a third plane, having parallel sections; therefore, IK is parallel to AB, KL to BC, LM to CD, and IM to AD. But AI and BK are parallels, by Def. 3; consequently, AK is a parallelogram; and the opposite sides, AB, IK, are equal. In like manner, it is shown that KL is = BC and LM = CD, and IM = AD, or the two planes, AC, IL, are mutually equilateral. But these two planes, having their corresponding sides parallel, have the angles contained by them also equal; namely, the angle A = the angle I, the angle B = the angle K, the angle C = the angle L, and the angle D = the angle M. So that the two planes, AC, IL, have all their corresponding sides and angles equal, or are equal and like. Q. E. D.

PROP. II.

If a cylinder be cut by a plane parallel to its base, the section will be a circle, equal to the base.



Let AF be a cylinder, and GHI any section parallel to the base ABC; then will GHI be a circle, equal to ABC.

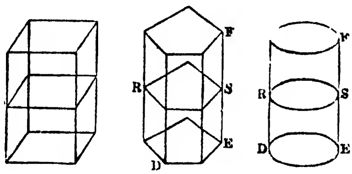
For, let the planes KE, KF, pass through the axis of the cylinder MK, and meet the section GHI in the three points H, I, L; and join the points as in the figure.

Then, since KL, CI, are parallel; and the plane KI, meeting the two parallel planes ABC, GHI, makes the two sections KC, LI, parallel; the figure KLIC is therefore a parallelogram, and consequently has the opposite sides LI, KC equal, where KC is a radius of the circular base.

In like manner, it is shown that LH is equal to the radius KB; and that any other lines, drawn from the point L to the circumference of the section GHI, are all equal to radii of the base; consequently, GHI is a circle, and equal to ABC. Q. E. D.

PROP. III.

All prisms, and a cylinder, of equal bases and altitudes, are equal to each other.



Let AC, DF, be two prisms, and a cylinder, upon equal bases AB, DE, and having equal altitudes; then will the solids AC, DF, be equal.

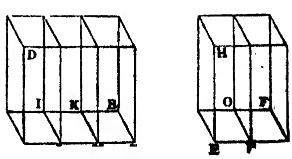
For, let PQ, RS, be any two sections parallel to the bases, and equidistant from them. Then, by the last two propositions, the section PQ is equal to the base AB, and the section RS equal the base DE. But the bases AB, DE, are equal by the hypothesis; therefore the sections PQ, RS, are also equal. And in like manner, it may be shown, that any other corresponding sections are equal to one another.

Since, then, every section in the prism AC, is equal to its corresponding section in the prism, or cylinder RS, the prisms and cylinder themselves, which are composed of those sections, must also be equal. Q. E. D.

Corol. Every prism, or cylinder, is equal to a rectangular parallelopipedon, of an equal base and altitude.

PROP. IV.

Rectangular parallelopipedons, of equal altitudes, have to each other the same proportion as their bases.



Let AC, EG, be two rectangular parallelopipedons, having the equal altitudes AD, EH; then will AC be to EG as the base AB is to the base EF

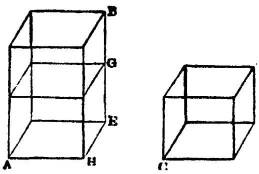
For, let the proportion of the base AB to the base EF, be that of any one number m (3) to any other number n (2).

And conceive AB to be divided into m equal parts, or rectangles, AI, LK, MB (by dividing AN into that number of equal parts, and drawing IL, KM, parallel to BN). And let EF be divided, in like manner, into *n* equal parts, or rectangles, EO, PF. all of these parts of both bases being mutually equal among themselves. And through the lines of division let the plane sections LR, MS, PV, pass parallel to AQ, ET.

Then the parallelopipedons AR, LS, MC, EV, PG, are all equal, having equal bases and heights. Therefore, the solid AC is to the solid EG, as the number of parts in AC to the number of equal parts in EG, or as the number of parts in AB to the number of equal parts in EF; that is, as the base AB to the base EF. Q. E. D.

Corol. From this proposition, and the corollary to the last, it appears, that all prisms and cylinders of equal altitudes, are to each other as their bases; every prism and cylinder being equal to a rectangular parallelopipedon of an equal base and height.

Rectangular parallelopipedons, of equal bases, are in proportion to each other as their altitudes.



Let AB, CD, be two rectangular parallelopipedons standing on the equal bases AE, CF; then will AB be to CD as the altitude EB is to the altitude DF.

For, let AG be a rectangular parallelopipedon on the base AE, and its altitude EG equal to the altitude FD of the solid CD.

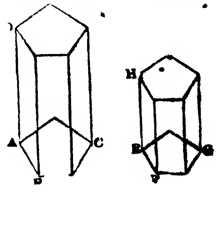
Then, AG and CD are equal, being prisms of equal bases and altitudes. But if HB, HG, be considered as bases, the solids AB, AG, of equal altitude AH, will be to each other as those bases HB, HG. But these bases HB, HG, being parallelograms of equal altitude HE, are to each other as their bases EB, EG; and therefore the two prisms AB, AG, are to each other as the lines EB, EG. But AG is equal CD, and EG equal FD; consequently, the prisms AB, CD, are to each other as their altitudes EB, FD; that is, AB : CD :: EB : FD. Q. E. D.

Corol. 1. From this proposition, and the corollary to Prop. III., it appears, that all prisms and cylinders, of equal bases, are to one another as their altitudes.

Corol. 2. Because, by corol. 1, prisms and cylinders are as their altitudes, when their bases are equal. And, by the corollary to the last theorem, they are as their bases, when their altitudes are equal. Therefore, universally, when neither are equal, they are to one another as the product of their bases and altitudes. And hence, also, these products are the proper numeral measures of their quantities or magnitudes.

PROP. VI.

Similar prisms and cylinders are to each other as the cubes of their altitudes, or of any other like linear dimensions.



Let ABCD, EFGH, be two similar prisms; then will the prism CD be to the prism GH, as AB^3 to EF^3, or as AD^3 to EH^3.

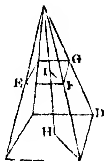
For, the solids are to each other as the product of their bases and altitudes (Prop. v., cor. 2), that is, as AC . AD to EG . EH. But the bases, being

similar planes, are to each other as the squares of their like sides, that is, AC to EG as AB^2 to EF^2; therefore, the solid CD is to the solid GH as $AB^2 . AD$ to $EF^2 . EH$. But BD and FH, being similar planes, have their like sides proportional, that is, $AB : EF :: AD : EH$, or $AB^2 : EF^2 :: AD^2 : EH^2$; therefore, $AB^2 . AD : EF^2 . EH :: AB^3 : EF^3$, or $:: AD^3 : EH^3$; and consequently, the solid CD : solid GH $:: AB^3 : EF^3 :: AD^3 : EH^3$. Q. E. D.

PROP. VII

In a pyramid, a section parallel to the base is similar to the base, and these two planes will be to each other as the squares of their distances from the vertex.

Let ABCD be a pyramid, and EFG a section parallel to the base BCD, also AIH a line perpendicular to the two planes at H and I; then will BD, EG, be two similar planes, and the plane BD will be to the plane EG as AH^2 to AI^2.



For, join CH, FI. Then, because a plane cutting two parallel planes, makes parallel sections, therefore the plane ABC, meeting the two parallel planes BD, EG, makes the sections BC, EF, parallel;—in like manner, the plane ACD makes the sections CD, FG, parallel. Again, because two pair of parallel lines make equal angles, the two EF, FG, which are parallel to BC, CD, make the angle EFG equal the angle BCD. And, in like manner, it is shown, that each angle in the plane EG is equal to each angle in the plane BD, and consequently those two planes are equiangular.

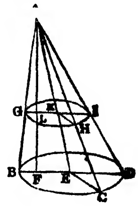
Again, the three lines AB, AC, AD, making with the parallels BC, EF, and CD, FG, equal angles; and the angles at A being common, the two triangles ABC, AEF, are equiangular, as also the two triangles ACD, AFG and have therefore their like sides proportional, namely, $AC : AF :: BC : EF . : CD : FG$. And, in like manner, it may be shown, that all the lines in the plane EG are proportional to all the corresponding ones in the base BD. Hence, these two planes, having their angles equal and their sides proportional, are similar.

But, similar planes being to each other as the squares of their like sides, the plane $BD : EG :. BC^2 : EF^2 :$ or $:: AC^2 : AF^2$, by what is shown above. But the two triangles AHC, AIF, having the angles H and I right ones, and the angle A common, are equiangular, and have therefore their like sides proportional, namely, $AC : AF :: AH : AI$, or $AC^2 : AF^2 :: AH^2 : AI^2$. Consequently the two planes BD, EG, which are as the former squares AC^2, AF^2, will be also as the latter squares AH^2, AI^2, that is, $BD : EG :: AH^2 : AI^2$.

PROP. VIII.

In a right cone a section parallel to the base is a circle; and this section is to the base as the squares of their distances from the vertex.

Let ABCD be a right cone, and GHI a section parallel to the base BCD; then will GHI be a circle, and BCD, GHI, will be to each other as the squares of their distances from the vertex.



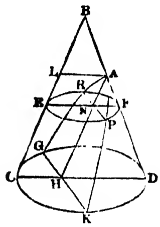
For, draw ALF perpendicular to the two parallel planes; and let the planes ACE, ADE, pass through the axis of the cone AKE, meeting the section in the three points H, I, K.

Then, since the section GHI is parallel to the base BCD, and the planes CK, DK, meet them, HK is parallel to CE, and IK to DE. And because the triangles formed by these lines are equianglar, KH : EC :: AK : AE :: KI : ED. But EC is equal to ED, being radii of the same circle; therefore, KI is also equal to KH. And the same may be shown of any other lines drawn from the point K to the circumference of the section GHI, which is therefore a circle.

Again, by similar triangles AL : AF :: AK : AE or :: KI : ED, hence $AL^2 : AF^2 :: KI^2 : ED^2$; but, $KI^2 : ED^2$ :: circle GHI : circle BCD; therefore, $AL^2 : AF^2$ :: circle GHI : circle BCD. Q. E. D.

PROP. IX.

If a right cone BCD *be cut by a plane* AGK *which is parallel to a plane touching the cone along the slant side* BC, *the section* AGK *is a parabola.*



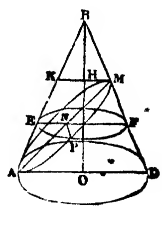
Let BCD be that position of the generating triangle, which is perpendicular to the cutting plane AGK; AH their common section, which is parallel to BC. Draw AL parallel to CD. Then, since the plane BCD passes through the axis, it is perpendicular to the base CKD and to every circular section EPF parallel to the base; it is also perpendicular to AGK. Hence, the common section PR of the planes AGK, EPF, is perpendicular to BCD and therefore to AH and EF.

But, AN : NF :: BC : CD, which is a constant ratio, therefore $AN \propto NF \propto EN \times NF$ (for EN is equal and parallel to AL, and constant) $\propto NP^2$ by the property of the circle. Hence, the curve is a parabola, whose axis is AH. (See Conic Sections, *infra*, Parabola, Prop. VII.)

Cor. If L be the latus rectum of the parabola GAK, $L \times AN = NP^2 = EN \times NF$.

$$L = EN \times \frac{NF}{AF} = AL \times \frac{AL}{BL} = \frac{AL^2}{BL}.$$

If a right cone BAD *be cut by a plane* AMP *through both slant sides, the section is an ellipse.*



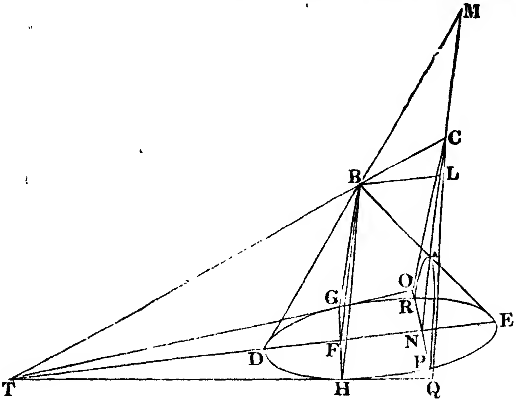
Let BAD be that position of the generating triangle which is perpendicular to the cutting plane; EPF any circular section. Draw MHK parallel to AD and therefore bisected by the axis BO.

$$\begin{aligned} \text{Then,}\quad AN : EN &:: AM : MK \\ NM : NF &:: AM : AD; \\ \therefore\ AN \times NM : EN \times NF\ (NP^2) &:: AM^2 : AD \times MK \end{aligned}$$

which is the property of an ellipse, one of whose axes is AM and the other a mean proportional between AD and MK. (Conic Sections, *infra*, Ellipse, Prop. XII.)

PROP. XI.

If a right cone BED *be cut through one side* BE *by a plane* RAP *which being produced backwards cuts the other side* DB *produced, the section is an hyperbola.*



Let DGEH be any circular section, BGH a triangular section through the vertex B of the cone parallel to the plane RAP.

$$\text{Then, } AN : EN :: BF : EF$$
$$NM : ND :: BF : FD$$
$$\therefore \quad AN \times NM : EN \times ND\,(NP^2) :: BF^2 : EF \times FD\,(FH^2)$$

which is the property of an hyperbola, whose axis major is AM and whose conjugate axis is to AM as FH to BF.

Cor. If GT, HT, be tangents to the circle at G, H; and planes passing through GT, HT, respectively, touch the cone along the lines BG, BH; also if TB, the common section of the planes, meet AM in C: then the common sections CO, CQ, of the plane RAP extended to meet the tangent planes are the asymptotes of the hyperbola.

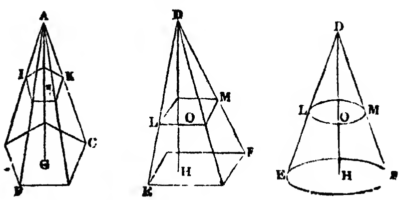
Draw BL parallel to DE, meeting AM in L. Then the axes of the hyperbola being in the proportion of BF to FH, the angle GBH or the equal angle OCQ is the angle between the asymptotes.

Now, by similar triangles ALB, BFE, and CLB, BFT; AL : CL :: TF : FE, and therefore AC : CL :: TE : FE. In like manner, by similar triangles MLB, BFD, and CLB, BFT; ML : CL :: TF : DF, and therefore CM : CL :: TD : DF. But by the property of the circle, TE : FE :: TD : DF. Therefore, CA = CM. Hence C is the centre of the hyperbola, and CO, CQ, are the asymptotes. (Conic Sections, *infra*, Hyperbola, Prop. XII.)

PROP. XII.

All pyramids and right cones of equal bases and altitudes are equal to one another

Let ABC, DEF, be any pyramids and cone, of equal bases BC, EF, and equal altitudes AG, DH; then will the pyramids and cone ABC and DEF, be equal.



For, parallel to the bases, and at equal distances AN, DO, from the vertices, suppose the planes IK. LM, to be drawn.

Then, by Props. VII. and VIII.,

$$DO^2 : DH^2 :: LM : EF,$$
$$\text{and } AN^2 : AG^2 :: IK : BC.$$

But, since AN^2, AG^2, are equal to DO^2, DH^2; therefore, IK : BC :: LM : EF

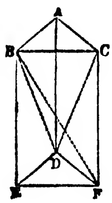
But BC is equal to EF, by hypothesis; therefore, IK is also equal to LM.

In the same manner, it is shown that any other sections, at equal distance from the vertex, are equal to each other.

Since, then, every section in the cone, is equal to the corresponding section in the pyramids, and the heights are equal, the solids ABC, DEF, composed of those sections, must be equal also. Q. E. D.

PROP. XIII.

Every pyramid of a triangular base, is the third part of a prism of the same base and altitude.



Let ABCDEF be a prism, and BDEF a pyramid, upon the same triangular base DEF: then will the pyramid BDEF be a third part of the prism ABCDEF.

For, in the planes of the three sides of the prism, draw the diagonals BF, BD, CD. Then the two planes BDF, BCD, divide the whole prism into the three pyramids BDEF, DABC, DBCF; which are proved to be all equal to one another as follows:

Since the opposite ends of the prism are equal to each other, the pyramid whose base is ABC and vertex D, is equal to the pyramid whose base is DEF and vertex B (Prop. XII.), being pyramids of equal base and altitude.

But the latter pyramid, whose base is DEF and vertex B, is the same solid as the pyramid whose base is BEF and vertex D, and this is equal to the third pyramid, whose base is BCF and vertex D, being pyramids of the same altitude and equal bases BEF, BCF.

Consequently, all the three pyramids which compose the prism, are equal to each other, and each pyramid is the third part of the prism, or the prism is triple of the pyramid. Q. E. D.

Corol. 1. Every pyramid, whatever its figure may be, is the third part of a prism of the same base and altitude; since the base of the prism, whatever be its figure, may be divided into triangles, and the whole solid into triangular prisms and pyramids.

Cor. 2. Any right cone is the third part of a cylinder, or of a prism, of equal base and altitude; since it has been proved that a cylinder is equal to a prism, and a cone equal to a pyramid, of equal base and altitude.

SCHOLIUM.—Whatever has been demonstrated of the proportionality of prisms, or cylinders, holds equally true of pyramids or cones,—the former being always triple the latter; viz. that similar pyramids or cones, are as the cubes of their like linear sides, or diameters, or altitudes, &c.

PROP. XIV.

If a sphere be cut by a plane, the section will be a circle.

Because the radii of the sphere are all equal, each of them being equal to the radius of the describing semicircle, it is evident that if the section pass through the centre of the sphere, then the distance from the centre to every point in the periphery of that section will be equal to the radius of the sphere, and the section will therefore be a circle of the same radius as the sphere. But if the plane do not pass through the centre, draw a perpendicular to it from the centre, and draw any number of radii of the sphere to the intersection of its surface with the plane; then these radii are evidently the hypothenuses of a corresponding number of right-angled triangles, which have the perpendicular from the centre on the plane of the section, as a common side; consequently their other sides are all equal, and therefore the section of the sphere by the plane is a circle, whose centre is the point in which the perpendicular cuts the plane.

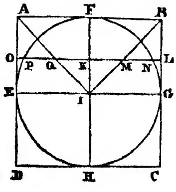
Cor. If two spheres intersect one another, the common section is a circle.

SCHOLIUM.

All the sections through the centre are equal to one another, and are greater than any other section which does not pass through the centre. Sections through the centre are called *great circles*, and the other sections *small* or *less circles.* Also, a straight line drawn through the centre of a circle of the sphere perpendicular to the plane of the circle is a diameter of the sphere, and the extremities of this diameter are called the *poles* of the circle. Hence it is evident that the arcs of great circles between the pole and circumference are equal, for the chords drawn in the sphere from either pole of a circle to the circumference are all equal.

PROP. XV.

Every sphere is two-thirds of its circumscribing cylinder.



Let ABCD be a cylinder circumscribing the sphere EFGH; then will the sphere EFGH be two-thirds of the cylinder ABCD. For let the plane AC be a section of the sphere and cylinder through the centre I, and join AI, BI. Let FIH be parallel to AD or BC, and EIG and KL parallel to AB or DC, the base of the cylinder; the latter line KL meeting BI in M, and the circular section of the sphere in N.

Then, if the whole plane HFBC be conceived to revolve about the line HF as an axis, the square FG will describe a cylinder AG, and the quadrant IFG will describe a hemisphere EFG, and the triangle IFB will describe a cone IAB. Also, in the rotation, the three lines, or parts, KL, KN, KM, as radii, will describe corresponding circular sections of these solids, viz. KL a section of the cylinder, KN a section of the sphere, and KM a section of the cone.

Now, FB being equal to FI, or IG, and KL parallel to FB, then by similar triangles IK=KM (Geom. Theor. 82), and IKN is a right-angled triangle; hence IN^2 is equal to IK^2+KN^2 (Theor. 34). But KL is equal to the radius IG or IN, and KM=IK; therefore KL^2 is equal to KM^2+KN^2, or the square of the longest radius of the said circular sections, is equal to the sum of the squares of the two others. Now circles are to each other as the squares of their diameters, or of their radii, therefore the circle described by KL is equal to both the circles described by KM and KN; or the section of the cylinder is equal to both the corresponding sections of the sphere and cone. And as this is always the case in every parallel position of KL, it follows that the cylinder EB, which is composed of all the former sections, is equal to the hemisphere EFG and cone IAB, which are composed of all the latter sections.

But the cone IAB is a third part of the cylinder EB (Prop. XIII. Cor. 2) consequently the hemisphere EFG is equal to the remaining two-thirds, or the whole sphere EFGH is equal to two-thirds of the whole cylinder ABCD.

Corol. 1. A cone, hemisphere, and cylinder of the same base and altitude are to each other as the numbers 1, 2, 3.

Corol. 2. All spheres are to each other as the cubes of their diameters; all these being like parts of their circumscribing cylinders.

Corol. 3. From the foregoing demonstration it appears that the spherica zone or frustum EGNP is equal to the difference between the cylinder EGLO and the cone IMQ, all of the same common height IK. And that the spherical segment PFN is equal to the difference between the cylinder ABLO and the conic frustum AQMB, all of the same common altitude FK.

Scholium.

By the scholium to Prop. XIII. we have

$$\text{cone AIB} : \text{cone QIM} :: IF^3 : IK^3 :: FH^3 : (FH-2FK)^3$$

$$\therefore \text{cone AIB} : \text{frustum ABMQ} :: FH^3 : FH^3-(FH-2FK)^3$$

$$:: FH^3 : 6FH^2FK-12FH.FK^2+8FK^3;$$

but cone AIB = one-third of the cylinder ABGE; hence

$$\text{cylinder AG} : \text{frustum ABMQ} :: 3FH^3 : 6FH^2.FK-12FH.FK^2+8FK^3$$

$$\text{Now cylinder AL} : \text{cylinder AG} :: FK : FI$$

$$\therefore \text{cylinder AL} : \text{frustum ABMQ} :: 6FH^2 : 6FH^2-12FH.FK+8FK^2$$

$$\therefore \text{cylinder AL} : \text{segment PFN} :: 6FH^2 : 12FH.FK-8FK^2, \text{ dividendo}$$

$$:: \tfrac{3}{2}FH^2 : FK(3FH-2FK).$$

But cylinder AL = circular base whose diameter is AB or FH multiplied by the height FK; hence cylinder AL = circle EFGH × FK.

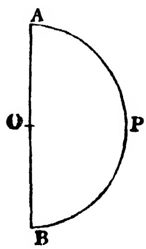
$$\therefore \text{segment PFN} = \frac{2}{3} \cdot \frac{\text{circle EFGH}}{FH^2}(3FH-2FK)FK^2.$$

SPHERICAL GEOMETRY.

DEFINITIONS.

1. A SPHERE is a solid terminated by a curve surface, and is such that all the points of the surface are equally distant from an interior point, which is called the *centre* of the sphere.

We may conceive a sphere to be generated by the revolution of a semicircle APB about its diameter AB; for the surface described by the motion of the curve ABP will have all its points equally distant from the centre O



2. The *radius* of a sphere is a straight line drawn from the centre to any point on the surface.

The *diameter* or *axis* of a sphere is a straight line drawn through the centre, and terminated both ways by the surface.

It appears from Def. 1, that all the radii of the same sphere are equal, and that all the diameters are equal, and each double of the radius.

3. It will be demonstrated, (Prop. I.), that every section of a sphere, made by a plane, is a circle; this being assumed,

A *great circle* of a sphere is the section made by a plane passing through the centre of the sphere.

A *small circle* of a sphere is the section made by a plane which does not pass through the centre of the sphere.

4. The *pole* of a circle of a sphere is a point on the surface of the sphere equally distant from all the points in the circumference of that circle.

It will be seen, (Prop. II.), that all circles, whether great or small, have two poles.

5. A *spherical triangle* is the portion of the surface of a sphere included by the arcs of three great circles.

6. These arcs are called the *sides* of the triangle, and each is supposed to be less than half of the circumference.

7. The *angles* of a spherical triangle are the angles contained between the planes in which the sides lie.

8. A plane is said to be a tangent to a sphere, when it contains only one point in common with the surface of the sphere.

PROP. I.

Every section of a sphere made by a plane is a circle.

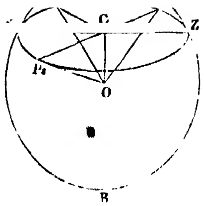
Let AZBX be a sphere whose centre is O.

Let XPZ be a section made by the plane XZ.

From O draw OC perpendicular to the plane XZ.

In XPZ take any points P_1, P_2, P_3, ...

Join CP_1; CP_2; CP_3; ...... also, OP_1; OP_2; OP_3; ......



Then, since OC is perpendicular to the plane XZ, it will be perpendicular to all straight lines passing through its foot in that plane. (Geometry of Planes.)

Hence, the angles OCP_1, OCP_2, OCP_3, ... are right angles,

$$\begin{aligned} \therefore\ OP_1^2 &= CP_1^2 + OC^2 \\ OP_2^2 &= CP_2^2 + OC^2 \\ OP_3^2 &= CP_3^2 + OC^2 \end{aligned}$$

But, since P_1, P_2, P_3, ... are all points upon the surface of the sphere, . by Def. 1, $OP_1 = OP_2 = OP_3 = \ldots\ldots$

$$CP_1 = CP_2 = CP_3$$

Hence, XPZ is a circle whose centre is C, and every other section of a sphere made by a plane may, in like manner, be proved to be a circle.

Cor. 1. If the plane pass through the centre of the sphere, then $OC = o$, and the radius of the circle will be equal to the radius of the sphere.

Cor. 2. Hence, all great circles are equal to one another, since the radius of each is equal to the radius of the sphere.

Cor. 3. Hence, also, two great circles always bisect each other, for their common intersection passing through the centre is a diameter.

Cor. 4. The centre of a small circle and that of the sphere, are in a straight line, which is perpendicular to the plane of the small circle.

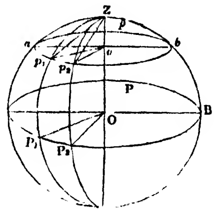
Cor. 5. We can always draw one, and only one, great circle through any two points on the surface of a sphere, for the two given points and the centre of the sphere give three points, which determine the position of a plane.

If, however, the two given points are the extremities of a diameter, then these two points and the centre of the sphere are in the same straight line, and an infinite number of great circles may be drawn through the two points.

Distances on the surface of a sphere are measured by the arcs of great circles. The reason for this is, that the shortest line which can be drawn upon the surface of a sphere, between any two points, is the arc of a great circle joining them.

PROP. II.

If a diameter be drawn perpendicular to the plane of a great circle, the extremities of the diameter will be the poles of that circle, and of all the small circles whose planes are parallel to it.



Let APB be a great circle of the sphere whose centre is O.

Draw ZN a diameter perpendicular to the plane of circle APB.

Then, Z and N, the extremities of this diameter, are the poles of the great circle APB, and all the small circles, such as *apb*, whose planes are parallel to that of APB.

Take any points P_1, P_2, ... in the circumference of APB.

Through each of these points respectively, and the points Z and N, describe great circles, ZP_1N, ZP_2N.

Join OP_1, OP_2, ...

Then, since ZO is perpendicular to the plane of APB, it is perpendicular to all the straight lines OP_1, OP_2, ... drawn through its foot in that plane.

Hence, all the angles ZOP_1, ZOP_2, ... are right angles, and ∴ the arcs ZP_1, ZP_2, ... are quadrants.

Thus, it appears that the points Z and N are equally distant from all the points in the circumference of APB, and are ∴ the poles of that great circle.

Again, since ZO is perpendicular to the plane APB, it is also perpendicular to the plane *apb*, which is parallel to the former.

Hence, the oblique lines Zp_1, Zp_2, ... drawn to p_1, p_2, in the circumference of *apb*, will be equal to each other. (Geometry of Planes.)

∴ The chords Zp_1, Zp_2, ... being equal, the arcs Zp_1, Zp_2, ... which they subtend, will also be equal.

∴ The point Z is the pole of the circle *apb*; and, for the same reason, the point N is also a pole.

Cor. 1. Every arc P_1Z drawn from a point in the circumference of a great circle to its pole, is a quadrant, and this arc P_1Z makes a right angle with the arc AP_1B. For, the straight line ZO being perpendicular to the plane APB, every plane which passes through this straight line will be perpendicular to the plane APB (Geometry of Planes); hence, the angle between these planes is a right angle, or, by (Def. 7), the angle of the arcs AP_1 and ZP_1 is a right angle.

Cor. 2. In order to find the pole of a given arc AP_1 of a great circle, take P_1Z equal to a quadrant, and perpendicular to AP_1, the point Z will be a pole of the arc AP_1; or, from the points A and P_1 draw two arcs AZ and P_1Z perpendicular to AP_1, the point Z in which they meet is a pole of AP_1.

Cor. 3. Reciprocally, if the distance of the point Z from each of the points A and P_1 is equal to a quadrant, then the point Z is the pole of AP_1, and each of the angles ZAP_1, ZP_1A, is a right angle.

For, let O be the centre of the sphere, draw the radii OA, OP_1, OZ;

Then, since the angles AOZ, P_1OZ, are right angles, the straight line OZ perpendicular to the straight lines OA, OP_1, and is ∴ perpendicular to the

plane; hence, by Prop., the point Z is the pole of AP_1, and ∴ the angle ZAP_1, ZP_1A, are right angles.

Cor. 4. Great circles, such as ZA, ZP_1, whose planes are at right angles to the plane of another great circle, as APB, are called its *secondaries;* and it appears from the foregoing corollaries, that,

1. The planes of all secondaries pass through the axis, and their circumferences through the poles of their primary; and that the poles of any great circle may always be determined by the intersection of any two of its secondaries.

2. The arcs of all secondaries intercepted between the primary and its poles are = 90°.

3. A secondary bisects all circles parallel to its primary.

Cor. 5. Let the radius of the sphere = R, radius of small circle parallel to it = r. Distance of two circles, or $Oo = \delta$.

Join Op_1, arc $P_1p_1 = \varphi$.

$$R^2 = r^2 + \delta^2$$
$$r = R \cos. \varphi$$
$$\delta = R \sin. \varphi$$

Cor. 6. Two secondaries intercept similar arcs of circles parallel to their primary, and these arcs are to each other as the cosines of the arcs of the secondaries between the parallels and the primary.

For the arcs of the parallels subtend at their respective centres, angles equal to the inclinations of the planes of the secondaries, and these arcs will therefore be similar.

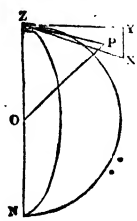
Also, if r_1, r_2, be the radii of two small parallels, the rest of notation as before,

$$\frac{\text{circumference } p_1 p_2}{\text{circumference } q_1 q_2} = \frac{\text{whole circumference of 1st}}{\text{whole circumference of 2d}}$$
$$= \frac{r_1}{r_2}$$
$$= \frac{R \cos. \varphi}{R \cos. \varphi'}$$
$$= \frac{\cos. \varphi}{\cos. \varphi'}$$

PROP. III.

Every plane perpendicular to a radius at its extremity, is a tangent to the sphere in that point.

Let ZXY be a plane perpendicular to the radius OZ.



Then, ZXY touches the sphere in Z.

Take any point P in the plane, join ZP; OP,

Then, since OZP is a right angled triangle,

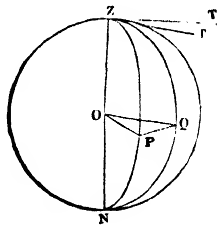
∴ The side OP is > side OZ.

Hence, the point P is without the sphere; and, in like manner, it may be shown, that every point in XYZ, except Z, is without the sphere.

Therefore, the plane XYZ is a tangent to the sphere.

PROP. IV.

The angle formed by two arcs of great circles, is equal to the angle contained by the tangents drawn to these arcs at their point of intersection, and is measured by the arc described from their point of intersection or pole, intercepted by the arcs containing the angle.



Let ZPN, ZQN, arcs of great circles, intersect in Z.

Draw ZT, ZT', tangents to the arcs at the point Z.

With Z as pole, describe the arc PQ.

Take O the centre of the sphere, and join OP, OQ.

Then, the spherical angle PZQ is equal to the angle TZT', and is measured by the arc PQ.

For the tangent ZT drawn in the plane ZPN, is perpendicular to radius OZ.

And the tangent ZT' drawn in the plane ZQN, is perpendicular to radius OZ.

Hence, the angle TZT' is equal to the angle contained by these two planes, that is, to the spherical angle PZQ. (Geom. of Planes).

Again, since the arcs ZP, ZQ, are each of them equal to a quadrant;

∴ Each of the angles ZOP, ZOQ, is a right angle,

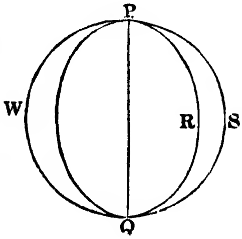
∴ The angle QOP is the angle contained by the planes ZPN, ZQN, and is = TZT'.

∴ The arc PQ, which measures the angle POQ, measures the angle between the planes, that is, the spherical angle PZQ.

Cor. 1. The angle under two great circles is measured by the distance between their poles. For the axis of the great circles drawn through their poles being perpendicular to the planes of the circles, the angles under these axes will be equal to the angle between the circles; but the angle under the axes is obviously measured by the arc which joins their extremities, that is, by the distance between their poles.

Cor. 2. The angle under two great circles is measured by the arc of a common secondary intercepted between them.

For, since the secondary passes through the poles of both, taking away from the equal quadrants of the secondary between each circle and its pole, the common arc intercepted between one circle and the pole of the other, the remainders are the intercept of the common secondary between the two circles, and the distance between their poles, and these are therefore equal. But the latter is, by the last Cor., the measure of the angle.

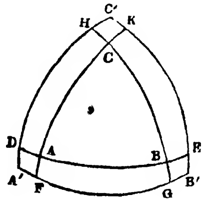


Cor. 3. Vertical spherical angles, such as QPW, QPS, are equal, for each of them is the angle formed by the planes QPS, WPR.

Also, when two arcs cut each other, the two adjacent angles QPW, QPR, when taken together, are always equal to two right angles.

PROP. V

If from the angular points of a spherical triangle considered as poles, three arcs be described forming another triangle, then, reciprocally, the angular points of this last triangle will be the poles of the sides opposite to them in the first.



Let ABC be a spherical triangle.

From the points A, B, C, considered as poles, describe the arcs B′C′, A′C′, A′B′, forming the spherical triangle A′B′C′.

Then, A′ will be the pole of BC, B′ of AC, and C′ of AB.

For, since B is the pole of A′C′, the distance from B to A′ is a quadrant.

And, since C is the pole of A′B′, the distance from C to A′ is a quadrant.

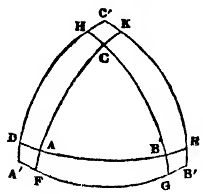
Thus, it appears that the point A′ is distant by a quadrant from the points B and C.

$\therefore$ A′ is the pole of the arc BC.

Similarly, it may be shown that B′ is the pole of AC, and C′ the pole of AB.

PROP. VI.

The same things being given as in the last proposition, each angle in either of the triangles will be measured by the supplement of the side opposite to it in the other triangle.



Produce the sides of the first triangle to D, E, F, G, H, K.

Then, since A is the pole of B′C′, the angle A is measured by the arc EK.

For the same reason, the angles B and C are measured by the arcs DH and FG respectively.

Because B′ is the pole of FK, the arc B′K is a quadrant.

Because C′ is the pole of DE, the arc C′E is a quadrant.

$$\therefore \quad B'K + C'E = 180^\circ$$
$$\text{or,} \quad B'C' + EK = 180^\circ$$
$$\therefore \quad EK = 180^\circ - B'C'$$
$$\text{Similarly,} \quad DH = 180^\circ - A'C'$$
$$FG = 180^\circ - A'B'$$

But the arcs EK, DH, FG, are the measures of the angles A, B, C, respectively, $\therefore$ $180^\circ - B'C'$, $180^\circ - A'C'$, $180^\circ - A'B'$, or the supplements of B′C′, A′C′, and A′B′, are the measures of these angles

Again, since A′ is the pole of HG, the angle A′ is measured by GH.

For the same reason, the angles B′, C′, are measured by the arcs FK and DE respectively.

Because B is the pole of A′C′, the arc BH is a quadrant.

Because C is the pole of A B′, the arc CG is a quadrant.

$$\therefore \quad BH + CG = 180^\circ$$
$$\text{or,} \quad GH + BC = 180^\circ$$
$$\therefore \quad GH = 180^\circ - BC$$
$$\text{Similarly,} \quad FK = 180^\circ - AC$$
$$DE = 180^\circ - AB$$

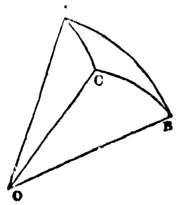
And GH, FK, DE, are the measures of the angles A′, B′, C′, respectively.

These triangles ABC, A′B′C′, are, from their properties, usually called *Polar triangles*, or *Supplemental triangles.*

PROP. VII.

In any spherical triangle any one side is less than the sum of the two others.

Let ABC be a spherical triangle, O the centre of the sphere. Draw the radii OA, OB, OC.



Then the three plane angles AOB, AOC, BOC, form a solid angle at the point O, and these three angles are measured by the arcs AB, AC, BC.

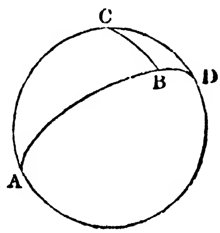
But each of the plane angles which form the solid angle, is less than the sum of the two others.

Hence each of the arcs AB, AC, BC, which measures these angles, is less than the sum of the two others.

PROP. VIII.

The sum of the three sides of a spherical triangle is less than the circumference of a great circle.

Let ABC be any spherical triangle.



Produce the sides AB, AC, to meet in D.

Then, since two great circles always bisect each other (Prop. 1, cor.) the arcs ABD, ACD, are semicircles.

Now, in the triangle BCD,

$$BC \angle BD + DC, \text{ by Prop. VII.;}$$
$$\therefore AB + AC + BC \angle AB + BD + AC + DC$$
$$\angle ABD + ACD$$
$$\angle \text{ circumference of great circle.}$$

CONIC SECTIONS.

There are three curves, whose properties are extensively applied in Mathematical investigations, which, being the sections of a cone made by a plane in different positions, are called the *Conic Sections* (see page 437). These are,

1. The Parabola.
2. The Ellipse.
3. The Hyperbola.

Before entering upon the discussion of their properties, it may be useful to enumerate the more useful theorems of proportion which have been proved in the treatises on Algebra and Geometry, or which are immediately deducible from those already established. For convenience in reference, they may be arranged in the following

TABLE.

If	A	:	B	::	C	:	D
Then	A	:	C	::	B	:	D
Or	B	:	A	::	D	:	C
...	$A+B$	:	B	::	$C+D$	:	D
...	$A-B$	:	B	::	$C-D$	:	D
...	A	:	$A+B$	::	C	:	$C+D$
...	A	:	$A-B$	::	C	:	$C-D$
...	$A+B$	:	$A-B$	::	$C+D$	:	$C-D$
...	mA	:	mB	::	nC	:	nD
...	mA	:	nB	::	mC	:	nD
...	$\frac{A}{m}$	:	$\frac{B}{m}$	::	$\frac{C}{n}$	:	$\frac{D}{n}$
...	$\frac{A}{m}$	:	$\frac{B}{n}$	::	$\frac{C}{m}$	:	$\frac{D}{n}$
...	A^n	:	B^n	::	C^n	:	D^n
Also if	A	:	B	::	D	:	E
And	B	:	C	::	E	:	F
Then	A	:	C	::	D	:	F
And if	A	:	B	::	E	:	F
And	B	:	C	::	D	:	E
Then	A	:	C	::	D	:	F
If	A	:	B	::	C	:	D
And	E	:	F	::	G	:	H
And	K	:	L	::	M	:	N
Then	$A.E.K$	:	$B.F.L$	::	$C.G.M$	:	$D.H.N$

PARABOLA.

DEFINITIONS.

1. A PARABOLA is a plane curve, such, that if from any point in the curve two straight lines be drawn; one to a given fixed point, the other perpendicular to a straight line given in position: these two straight lines will always be equal to one another.

2. The given fixed point is called the *focus* of the parabola.

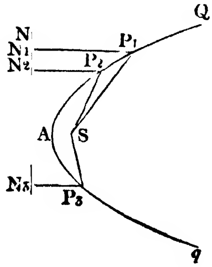
3. The straight line given in position, is called the *directrix* of the parabola.

Thus, let QAq be a parabola, S the focus, Nn the directrix;

Take any number of points, P_1, P_2, P_3, ... in the curve;

Join S, P_1; S, P_2; S, P_3; .... and draw P_1N_1, P_2N_2, P_3N_3, ... perpendicular to the directrix; then

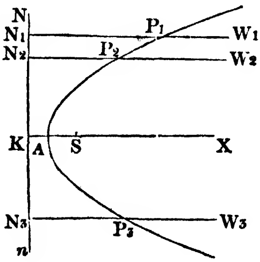
$SP_1 = P_1N_1$, $SP_2 = P_2N_2$, $SP_3 = P_3N_3$, ...



4. A straight line drawn perpendicular to the directrix, and cutting the curve, is called a *diameter;* and the point in which it cuts the curve is called the *vertex of the diameter.*

5. The diameter which passes through the focus is called the *axis*, and the point in which it cuts the curve is called the *principal vertex.*

Thus: draw $N_1P_1W_1$, $N_2P_2W_2$, $N_3P_3W_3$, KASX, through the points P_1, P_2, P_3, S, perpendicular to the directrix; each of these lines is a diameter; P_1, P_2, P_3, A, are the vertices of these diameters; ASX is the axis of the parabola, A the principal vertex.



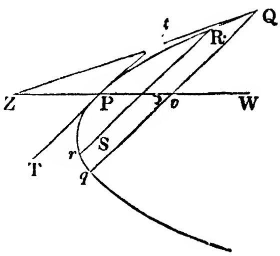
6. A straight line which meets the curve in any point, but which, when produced both ways, does not cut it, is called a *tangent* to the curve at that point.

7. A straight line drawn from any point in the curve, parallel to the tangent at the vertex of any diameter, and terminated both ways by the curve, is called an *ordinate* to that diameter.

8. The ordinate which passes through the focus, is called the *parameter* of that diameter.

9. The part of a diameter intercepted between its vertex and the point in which it is intersected by one of its own ordinates, is called the *abscissa* of the diameter.

10. The part of a diameter intercepted between one of its own ordinates and its intersection with a tangent, at the extremity of the ordinate, is called the *sub-tangent* of the diameter.



Thus: let TPt be a tangent at P, the vertex of the diameter PW.

From any point Q in the curve draw Qq parallel to Tt and cutting PW in v. Through S draw RSr parallel to Tt.

Let QZ, a tangent at Q, cut WP, produced in Z.

Then Qq is an ordinate to the diameter PW; Rr is the parameter of PW.

Pv is the abscissa of PW, corresponding to the point Q.

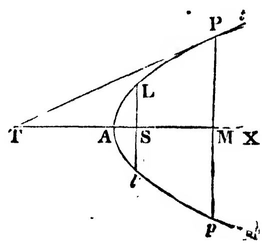
vZ is the sub-tangent of PW, corresponding to the point Q.

11. A straight line drawn from any point in the curve, perpendicular to the axis, and terminated both ways by the curve, is called an *ordinate to the axis.*

12. The ordinate to the axis which passes through the focus is called the *principal parameter,* or *latus rectum* of the parabola.

13. The part of the axis intercepted between its vertex and the point in which it is intersected by one of its own ordinates, is called the *abscissa of the axis.*

14. The part of the axis intercepted between one of its own ordinates, and its intersection with a tangent at the extremity of the ordinate, is called the *sub-tangent of the axis.*



Thus: from any point P in the curve draw Pp perpendicular to AX and cutting AX in M. Through S draw LSl perpendicular to AX.

Let PT, a tangent at P, cut XA produced in T.

Then, Pp is an ordinate to the axis; Ll is the latus rectum of the curve.

AM is the abscissa of the axis corresponding to the point P.

MT is the sub-tangent of the axis corresponding to the point P.

It will be proved in Prop. 3, that the tangent at the principal vertex is perpendicular to the axis; hence, the four last definitions are in reality included in the four which immediately precede them.

Cor. It is manifest from def. 1, that the parts of the curve on each side of the axis are similar and equal, and that every ordinate Pp is bisected by the axis.

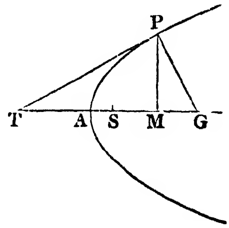
15. If a tangent be drawn at any point, and a straight line be drawn from the point of contact perpendicular to it, and terminated by the curve, that straight line is called a *normal*.

16. The part of the axis intercepted between the intersections of the normal and the ordinate, is called the *sub-normal*.

Thus: let TP be a tangent at any point P.

From P draw PG perpendicular to the tangent, and PM perpendicular to the axis.

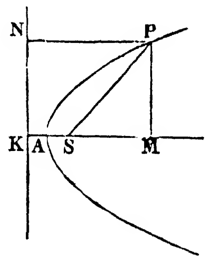
Then PG is the normal corresponding to the point P; MG is the sub-normal corresponding to the point P.



PROP. I.

The distance of the focus from any point in the curve, is equal to the sum of the abscissa of the axis corresponding to that point, and the distance from the focus to the vertex. That is,

$$SP = AM + AS.$$



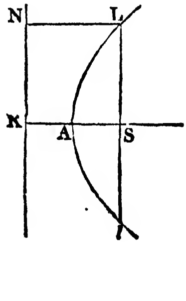
For,

SP $=$ PN by Def. (1)
$=$ KM $\because$ NM is a parallelogram.
$=$ AM $+$ AK
$=$ AM $+$ AS $\because$ AK $=$ AS, by Def. (1).

PROP. II.

The latus rectum is equal to four times the distance from the focus to the vertex. That is,

$$Ll = 4\,AS.$$



For,

Ll $=$ 2 LS, Def. (14) cor.
$=$ 2 LN
$=$ 2 SK
$=$ 4 AS $\because$ AS $=$ AK.

PROP. III.

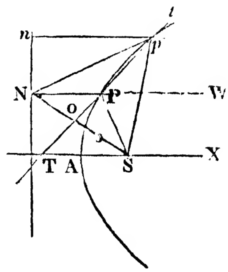
To draw a tangent to the parabola at any point.

Let P be the given point.

Join S, P; draw PN perpendicular to the directrix.

Bisect the angle SPN by the straight line T*t*.

T*t* is a tangent at the point P.



For if T*t* be not a tangent, let T*t* cut the curve in some other point *p*.

Join S, *p*; draw *pn* perpendicular to the directrix; join S, N.

Since SP = PN, PO common to the triangles SPO, NPO, and angle SPO = angle NPO by construction,

$$\therefore \text{SO} = \text{NO}, \quad \text{and angle SOP} = \text{angle NOP}.$$

Again, since SO = NO, O*p* common to the triangles SO*p*, NO*p*, and angle SO*p* = angle NO*p*,

$$\therefore Sp = Np.$$

But since *p* is a point in curve, and *pn* is drawn perpendicular to the directrix,

$$Sp = pn$$
$$\therefore pN = pn.$$

That is, the hypothenuse of a right-angled triangle equal to one of the sides, which is impossible, ∴ *p* is not a point in the curve; and in the same manner it may be proved that no point in the straight line T*t* can be in the curve, except P.

∴ T*t* is a tangent to the curve at P.

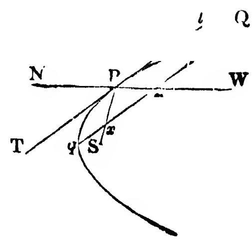
Cor. 1. A tangent at the vertex A, is perpendicular to the axis.

Cor. 2. SP = ST

For, since NW is parallel to TX

$$\therefore \angle \text{STP} = \angle \text{NPT}$$
$$= \angle \text{SPT by construction,}$$
$$\therefore \text{SP} = \text{ST}$$

Cor. 3. Let Q *q* be an ordinate to the diameter PW, cutting SP in *x*.

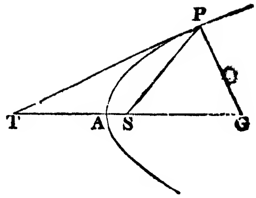


Then, P*x* = P*v*

For, since Q*q* is parallel to T*t*

$$\therefore \angle Pxv = \angle xPT$$
$$= \angle \text{NPT by construction,}$$
$$= \angle Pvx \text{ interior opposite angle,}$$
$$\therefore Px = Pv$$

Cor. 4. Draw the normal PG.



Then, $SP = SG$

For since $\angle$ GPT is a right angle,

$$\angle GPT = \angle PGT + \angle PTG$$
$$= \angle PGT + \angle SPT$$

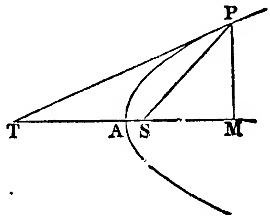
Take away the common $\angle$ SPT and there remains

$$\angle SPG = \angle SGP$$
$$\therefore SP = SG$$

PROP. IV.

The subtangent to the axis is equal to twice the abscissa. That is,

$$MT = 2\,AM$$



For, $MT = MS + ST$

$= MS + SP$. Prop. 3. cor. 2.

$= MS + SA + AM$. Prop. 1.

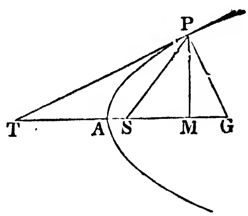
$= 2\,AM$.

Cor. MT is bisected in A.

PROP. V.

The subnormal is equal to one half of the latus rectum. That is,

$MG = \frac{L}{2}$ if we denote the latus rectum by L.



For, $MG = SG - SM$

$= SP - SM$. Prop. 3. cor. 4.

$= AS + AM - SM$. Prop. 1.

$= AS + AS + SM - SM$

$= 2\,AS$

$= \frac{L}{2}$ Prop. 2.

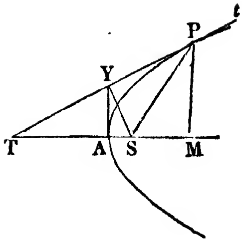
PROP. VI.

If a straight line be drawn from the focus perpendicular to the tangent at any point, it will be a mean proportional between the distance from the focus to that point, and the distance from the focus to the vertex.

That is, if SY be a perpendicular let fall from S upon Tt the tangent at any point P

$$SP : SY :: SY : SA.$$

Join A, Y.



Since SP = ST, and SY is drawn perpendicular to the line PT,

$\therefore$ TY = YP.

Also by Prop. 4.,

TA = AM

$\therefore$ Since AY cuts the sides of $\triangle$ TPM proportionally, AY is parallel to MP,

$\therefore$ AY is perpendicular to AM.

Hence the $\triangle$s SYA, SYT, are similar,

$\therefore$ ST : SY :: SY : SA

or, SP : SY :: SY : SA $\because$ SP = ST by Prop. 3. cor. 2.

Cor. 1. Multiplying extremes and means,

$$SY^2 = SP \,.\, SA.$$

Cor. 2. SP : SA :: SP² : SY²

Cor. 3. By Cor. 1,

$$SP = \frac{SY^2}{SA}.$$

And since SA is constant for the same parabola,

$$SP \propto SY^2.$$

Cor. 4. By Cor. 1.,

$$\begin{aligned} SY^2 &= AS \,.\, SP \\ \therefore 4 \cdot SY^2 &= 4\, AS \,.\, SP \\ &= L \,.\, SP. \quad \text{Prop. 2} \end{aligned}$$

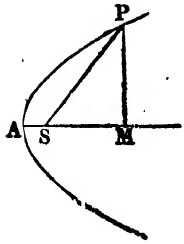
PROP. VII.

The square of any semi-ordinate to the axis is equal to the rectangle under the latus rectum and the abscissa.

That is, if P be any point in the curve

$$PM^2 = L \,.\, AM.$$

For,



$$\begin{aligned} PM^2 &= SP^2 - SM^2 \qquad \text{Geom. Theor. 34.} \\ &= (AM+AS)^2 - (AM-AS)^2 \end{aligned}$$

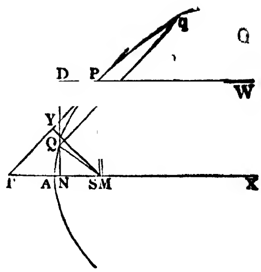
$\because$ SP = AM + AS (Prop. 1), & SM = AM — AS

$$\begin{aligned} &= 4\, AS \,.\, AM. \quad \text{Geom. Theor. 31 \& 32.} \\ &= L \,.\, AM. \qquad \text{Prop. 1.} \end{aligned}$$

Cor. 1. Since L is constant for the same parabola $PM^2 \propto AM$,

That is, *The abscissæ are proportional to the squares of the ordinates.*

PROP. VIII.



If Qq *be an ordinate to the diameter* PW *and* Pv, *the corresponding abscissa, then,*

$$Qv^2 = 4SP \times Pv.$$

Draw PM an ordinate to the axis.

Join S, Q; and through Q draw DQN perpendicular to the axis.

From S let fall SY perpendicular on the tangent at P.

The triangles SPY, QDv, are similar.

$$Qv^2 : QD^2 :: SP^2 : SY^2$$
$$:: SP : SA, \quad \text{Prop. VI. Cor. 2.}$$

The triangles PTM, QDv, are also similar;

$$\therefore \quad QD : Dv :: PM : MT$$
$$:: PM^2 : PM.MT$$
$$:: 4AS.AM : 2PM.AM$$
$$:: 4AS : 2PM$$
$$\therefore \quad 2PM.QD = 4AS.Dv$$

But,

$$PM^2 - QN^2 = 4AS.AM - 4AS.AN = 4AS(AM - AN) = 4AS.MN$$

And, $PM^2 - QN^2 = (PM + QN)(PM - QN)$
$= (PM + QN).QD$

$\therefore (PM + QN).QD = 4AS.MN = 4AS.DP$

But, $2PM.QD = 4AS.Dv$

$\therefore (PM - QN).QD = 4AS.Pv$

Or, $QD^2 = 4AS.Pv$

$\therefore \quad Qv^2 : 4AS.Pv :: SP : SA.$

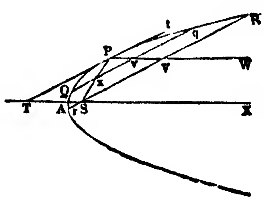
$\therefore \quad Qv^2 = 4SP.Pv.$

Cor. 1. In like manner it may be proved, that

$$qv^2 = 4SP \times Pv.$$

Hence, $Qv = qv$; and since the same may be proved for any ordinate, it follows, that

A diameter bisects all its own ordinates.



Cor. 2. Let Rr be the parameter to the diameter PW.

Then, by Prop. III. Cor. 3.

$$Px = Pv$$
$$\therefore PS = PV$$

Now, by the Proposition,

$$\begin{aligned} RV^2 &= 4SP \cdot PV \\ &= 4SP^2 \\ \therefore\ 4RV^2 \text{ or } Rr^2 &= 16SP^2 \\ Rr &= 4SP. \end{aligned}$$

Hence the Proposition may be thus enunciated:

The square of the semi-ordinate to any diameter is equal to the rectangle under the parameter and abscissa.

It will be seen, that Prop. VII. is a particular case of the present proposition.

ELLIPSE.

DEFINITIONS.

1. AN ELLIPSE is a plane curve, such that, if from any point in the curve two straight lines be drawn to two given fixed points, the sum of these straight lines will always be the same.

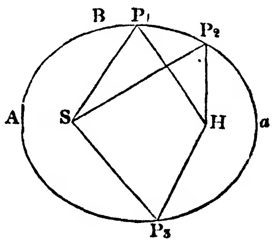
2. The two given fixed points are called the *foci.*

Thus, let ABa be an ellipse, S and H the foci.

Take any number of points in the curve $P_1, P_2, P_3, \text{-----}$

Join S,P_1, H,P_1; S,P_2, H,P_2; S,P_3, H,P_3; ----- then,

$SP_1 + HP_1 = SP_2 + HP_2 = SP_3 + HP_3 = \text{-----}$



3. If a straight line be drawn joining the foci and bisected, the point of bisection is called the *centre.*

4. The distance from the centre to either focus is called the *eccentricity.*

5. Any straight line drawn through the centre, and terminated both ways by the curve, is called a *diameter.*

6. The points in which any diameter meets the curve are called the *vertices* of that diameter.

7. The diameter which passes through the foci is called the *axis major*, and the points in which it meets the curve are called the *principal vertices.*

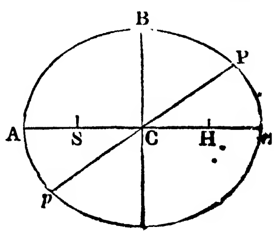
8. The diameter at right angles to the axis major is called the *axis minor.*

Thus, let ABa be an ellipse, S and H the foci.

Join S,H; bisect the straight line SH in C, and produce it to meet at the curve in A and *a.*

Through C draw any straight line P*p*, terminated by the curve in the points P, *p.*

Through C draw B*b* at right angles to A*a.*



Then, C is the centre, CS or CH the eccentricity, Pp is a diameter, P and p its vertices, Aa is the major axis, Bb is the minor axis.

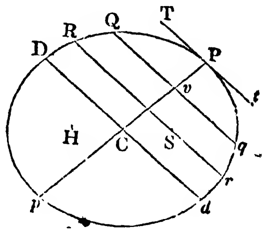
9. A straight line which meets the curve in any point, but which, being produced both ways, does not cut it, is called a *tangent* to the curve at that point.

10. A diameter drawn parallel to the tangent at the vertex of any other diameter, is called the *conjugate diameter* to the latter, and the two diameters are called a *pair of conjugate diameters.*

11. Any straight line drawn parallel to the tangent at the vertex of any diameter and terminated both ways by the curve, is called an *ordinate* to that diameter.

12. The segments into which any diameter is divided by one of its own ordinates are called the *abscissæ* of the diameter.

13. The ordinate to any diameter, which passes through the focus, is called the *parameter* of that diameter.



Thus, let Pp be any diameter, and Tt a tangent at P.

Draw the diameter Dd parallel to Tt.

Take any point Q in the curve, draw Qq parallel to Tt, cutting Pp in v.

Through S draw Rr parallel to Tt.

Then, Dd is the conjugate diameter to Pp.

Qq is the ordinate to the diameter Pp, corresponding to the point Q.

Pv, vp are the abscissæ of the diameter Pp, corresponding to the point Q.

Rr is the parameter of the diameter Pp.

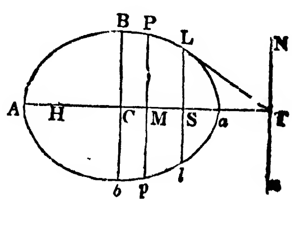
14. Any straight line drawn at right angles to the major axis, and terminated both ways by the curve, is called an *ordinate to the axis.*

15. The segments into which the major axis is divided by one of its own ordinates are called the *abscissæ of the axis.*

16. The ordinate to the axis which passes through either focus is called the *latus rectum.*

(It will be proved in Prop. IV., that the tangents at the principal vertices are perpendicular to the major axis; hence, definitions 14, 15, 16, are in reality included in the three which immediately precede them.)

17. If a tangent be drawn at the extremity of the latus rectum and produced to meet the major axis, and if a straight line be drawn through the point of intersection at right angles to the major axis, the tangent is called the *focal tangent*, and the straight line the *directrix.*



Thus, from P any point in the curve, draw PMp perpendicular to Aa, cutting Aa in M.

Through S draw Ll perpendicular to Aa.

Let LT, a tangent at L, cut Aa produced in T.

Through T draw Nn perpendicular to Aa.

Then, Pp is the ordinate to the axis, corresponding to the point P

AM, Ma are the abscissæ of the axis, corresponding to the point P.

Ll is the latus rectum.

LT is the focal tangent.

Nn is the directrix.

18. A straight line drawn at right angles to a tangent from the point of contact, and terminated by the major axis, is called a *normal.*

The part of the major axis intercepted between the intersections of the normal and the ordinate, is called the *subnormal.*

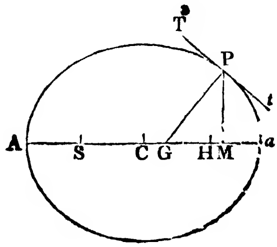
Let Tt be a tangent at any point P.

From P draw PG perpendicular to Tt meeting Aa in G.

From P draw PM perpendicular to Aa.

Then, PG is the normal corresponding to the point P.

MG is the subnormal corresponding to the point P.



PROP. I.

The sum of two straight lines drawn from the foci to any point in the curve is equal to the major axis. That is, if P be any point in the curve,

$$SP + HP = Aa.$$

For,

$SP + HP = AS + AH$

$= 2\ AS + SH,$ Def. 1.

And,

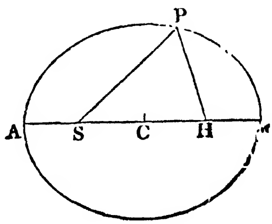
$SP + HP = aS + aH$

$= 2\ aH + SH,$

$\therefore 2\,(SP + HP) = 2\,(AS + SH + Ha)$

Or,

$SP + HP = Aa.$



Cor. 1. The centre bisects the axis major, for

$$2\ AS + SH = 2\ aH + SH$$
$$\therefore AS = aH$$
$$\text{And, } SC = CH \text{ by definition 3.}$$
$$\therefore AC = aC.$$

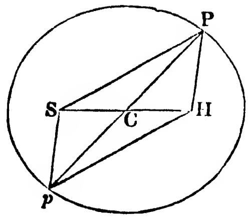
$$SP + HP = 2\ AC$$
$$\therefore SP = 2\ AC - HP$$
$$HP = 2\ AC - SP$$
$$SP - HP = 2\ AC - 2\ HP$$

PROP. II.

The centre bisects all diameters.

Take any point P in the curve.
Join S,P; H,P; S,H;
Complete the parallelogram SPHp,
Join C,p; C,P;
Then, since the opposite sides of a parallelogram are equal,

$$SP = Hp \;,\; HP = Sp$$
$$\therefore \; SP + PH = Sp + pH$$
$\therefore$ p is a point in the curve.



Again, since the diagonals of a parallelogram bisect each other, and since SH is bisected in C,

$\therefore$ Pp is a straight line, and a diameter, and is bisected in C.

And in like manner, it may be proved that every other diameter is bisected in C.

PROP. III.

The distance of either focus from the extremity of the axis minor is equal to the semi-axis major.

That is,

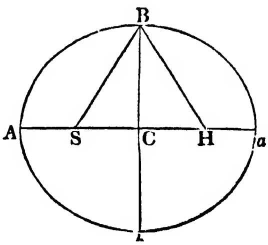
$$SB \text{ or } HB = AC.$$

Since SC = HC, and CB is common to the two right-angled triangles SCB, HCB,

$$\therefore \; SB = HB.$$

But,

$$SB + HB = 2\,AC. \quad \text{Prop. 1.}$$
$$\therefore \; SB = HB = AC.$$



Cor. 1. $BC^2 = AS \, . \, Sa.$

For,

$$\begin{aligned} BC^2 &= SB^2 - SC^2 \\ &= AC^2 - SC^2 \\ &= (AC + SC) \, . \, (AC - SC) \\ &= AS \, . \, Sa. \end{aligned}$$

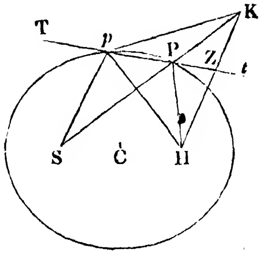
Cor. 2. The square of the eccentricity is equal to the difference of the squares of the semi-axes;

For,

$$\begin{aligned} SC^2 &= SB^2 - BC^2 \\ &= AC^2 - BC^2. \end{aligned}$$

PROP. IV.

To draw a tangent to the ellipse at any point.



Let P be the given point.

Join S,P; H,P; produce SP.

Bisect the exterior angle HPK by the straight line T*t*.

T*t* is a tangent to the curve at P.

For, if T*t* be not a tangent, let T*t* cut the curve in some other point *p*.

Join S, *p*; H, *p*; make PK = PH; join *p*,K; H,K cutting T*t* in Z.

Since HP = PK, PZ common to the triangles HPZ, KPZ, and the angle HPZ = angle KPZ by construction,

$$\therefore \text{HZ} = \text{KZ}, \text{ and the angle HZP} = \text{angle KZP}.$$

Again, since HZ = KZ, Z*p* common to the triangles HZ*p*, KZ*p*, and angle HZ*p* = angle KZ*p*,

$$\therefore p\text{K} = p\text{H}.$$

But, since any two sides of a triangle are greater than the third side,

$$\begin{aligned} Sp + p\text{K} &> \text{SK} \\ &> \text{SP} + \text{PK} \\ &> \text{SP} + \text{PH} \quad \because \text{PK} = \text{PH by construction,} \\ &> Sp + p\text{H}, \text{ by definition 1,} \\ \therefore p\text{K} &> p\text{H}. \end{aligned}$$

But we have just proved that $p\text{K} = p\text{H}$, which is absurd, $\therefore$ *p* is not a point in the curve, and in the same manner it may be proved that no point in the straight line T*t* can be in the curve except P.

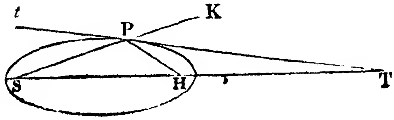
$$\therefore \text{T}t \text{ is a tangent to the curve at P.}$$

Cor. 1. Hence, tangents at A and *a*, are perpendicular to the major axis, and tangents at B and *b* are perpendicular to the minor axis.

Cor. 2. SP and HP make equal angles with every tangent.

Cor. 3. Since HPK, the exterior angle of the triangle SPH, is bisected by the straight line T*t*, cutting the base SH produced in T

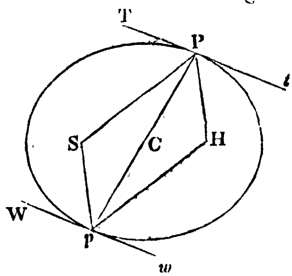
$$\therefore \text{ST} : \text{HT} :: \text{SP} : \text{HP}.$$



PROP. V.

Tangents drawn at the vertices of any diameter are parallel.

Let Tt, Ww, be tangents at P, p, the vertices of the diameter PCp.



Join S, P; P, H; S, p; p, H;

Then, by Prop. 2, SH is a parallelogram, and since the opposite angles of parallelograms are equal,

$\therefore \angle SPH =$ angle SpH

supplement of $\angle SPH =$ supplement of $\angle SpH$

or,

$\angle SPT + \angle HPt = \angle SpW + \angle Hpw$

$$\left.\begin{array}{l}\text{But } \angle SPT = \angle HPt \\ \text{And } \angle SpW = \angle Hpw\end{array}\right\} \text{by Prop. 4. Cor. 2.}$$

Hence, these four angles are all equal,

$\therefore \angle SPT = \angle Hpw.$

And since SP is parallel to Hp,

$\angle SPp = \angle PpH,$

$\therefore$ whole $\angle TPp =$ whole $\angle wpP$, and they are alternate angles,

$\therefore Tt$ is parallel to Ww.

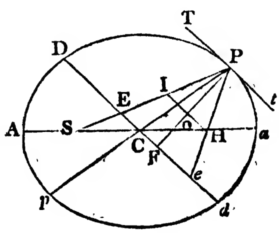
Cor. Hence, if tangents be drawn at the vertices of any two diameters, they will form a parallelogram circumscribing the ellipse.

PROP. VI.

If straight lines be drawn from the foci to a vertex of any diameter, the distance from the vertex to the intersection of the conjugate diameter, with either focal distance, is equal to the semi-axis, major.

That is, if Dd be a diameter conjugate to Pp, cutting SP in E, and HP in e,

$$PE \text{ or } Pe = AC.$$



Draw PF perpendicular to Dd, and HI parallel to Dd or Tt, cutting PF in O,

Then, since the angles at O are right angles, the $\angle IPO = \angle HPO$, and PO common to the two triangles HPO, IPO,

$\therefore IP = HP.$

Also, since $SC = HC$, and CE is parallel HI, the base of $\triangle SHI$,

$\therefore SE = EI.$

Hence,

$$\begin{aligned} 2\,PE &= 2\,EI + 2\,IP \\ &= SE + EI + IP + HP \\ &= SP + HP \\ &= 2\,AC \end{aligned}$$

$\therefore PE = AC.$

Also, $\angle PEe = \angle PeE. \quad \therefore PE = Pe$, and $Pe = AC.$

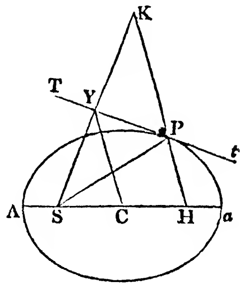
PROP. VII.

Perpendiculars, from the foci upon the tangent at any point, intersect the tangent in the circumference of a circle, whose diameter is the major axis.

From S let fall SY perpendicular on Tt a tangent at P.

Join S,P; H,P; produce HP to meet SY produced in K.

Join CY;



Then, since angle SPY = angle KPY (Prop. IV.), and the angles at Y are right angles, and PY common to the two triangles, SPY, KPY,

$\therefore$ SP = PK

And SY = YK.

Again, since SY = YK, and HC = CS, CY cuts the sides of the triangle HSK proportionally,

$\therefore$ CY is parallel to HK.

Al, since CY is parallel to HK, SY = YK, HC = CS,

$$\begin{aligned} \therefore \text{CY} &= \tfrac{1}{2}\,\text{HK} \\ &= \tfrac{1}{2}\,(\text{HP} + \text{PK}) \\ &= \tfrac{1}{2}\,(\text{HP} + \text{SP}) \\ &= \tfrac{1}{2}\,\text{A}a \\ &= \text{AC}. \end{aligned}$$

Hence, a circle described with centre C and radius CA will pass through Y.

And in like manner, if HZ be drawn perpendicular to Tt, it may be proved that the same circle will pass through Z also

PROP. VIII.

The rectangle, contained by the perpendiculars, from the foci upon the tangent at any point, is equal to the square of the semi-axis, minor.

That is,

$$\text{SY} \cdot \text{HZ} = \text{BC}^2.$$

Let Tt be a tangent at any point P.

On Aa describe a circle cutting Tt in Y and Z.

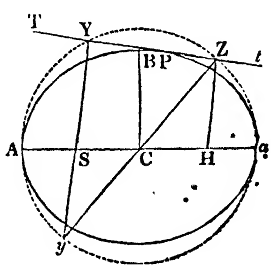
Join S,Y; H,Z;

Then, by the last Prop. SY, HZ are perpendicular to Tt.

Produce YS to meet the circumference in y.

Join C,y; C, Z;

Since yYZ is a right angle, the segment in which it lies is a semicircle; and Z,y, are the extremities of a diameter.



$\therefore yCZ$ is a straight line and a diameter.

Hence the triangles SCy, HCZ, are in every respect equal.

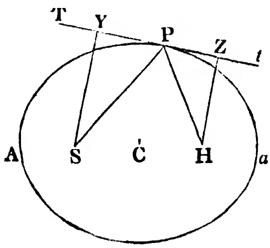
$$\therefore Sy = HZ$$
$$\therefore SY \cdot HZ = YS \cdot Sy$$
$$= AS \cdot Sa$$
$$= BC^2 \text{ (Prop. 3. Cor. 1.)}$$

PROP. IX.

Perpendiculars let fall from the foci upon the tangent at any point are to each other as the focal distance of the point of contact.

That is,

$$SY : HZ :: SP : HP.$$



For the triangles SPY, HPZ, are manifestly similar,

$$\therefore SY : HZ :: SP : HP.$$

Cor. Hence,

$$SY = HZ \cdot \frac{SP}{HP}$$
$$SY^2 = SY \cdot HZ \cdot \frac{SP}{HP}$$
$$= BC^2 \cdot \frac{SP}{HP} \text{ last Prop.}$$
$$= BC^2 \cdot \frac{SP}{2\,AC - SP}.$$

So also,

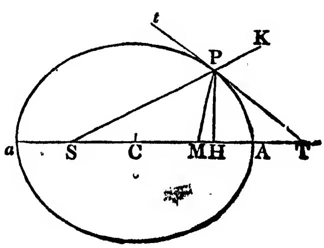
$$HZ^2 = BC^2 \cdot \frac{HP}{SP}$$
$$= BC^2 \cdot \frac{HP}{2\,AC - HP}.$$

PROP. X.

If a tangent be applied at any point, and from the same point an ordinate to the axis be drawn, the semi-axis major is a mean proportional between the distance from the centre to the intersection of the ordinate with the axis, and the distance from the centre to the intersection of the tangent with the axis.

That is,

$$CT : CA :: CA : CM.$$



Since the exterior angle HPK is bisected by T*t*, Proposition, 4

$\therefore$ ST : HT :: SP : HP. (Geom. Theor. 83, pt. 2.)
$\therefore$ ST + HT : ST — HT :: SP + HP . SP — HP
or, 2 CT : SH :: 2 AC : SP — HP
$\therefore$ 2 CT : 2 AC :: SH : SP — HP..............................(1)

But since PM is drawn from the vertex of $\triangle$ SPH perpendicular on base SH,

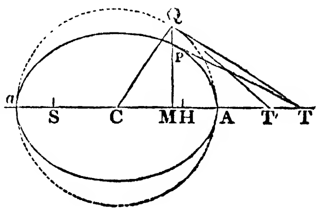
$\therefore$ SM + HM : SP — HP :: SP + HP : SM — HM
or, SH : SP — HP :: 2 AC : 2 CM..............................(2)

Comparing this with the proportion marked (1), we have

2 CT : 2 AC :: 2 AC : 2 CM
or, CT : CA :: CA : CM.

PROP. XI.

If a circle be described on the major axis of an ellipse, and if any ordinate to the axis be produced to meet the circle, tangents drawn to the ellipse and circle, at the points in which they are intersected by the ordinate, will cut the major axis in the same point.



Let AQ*a*, be a circle described on A*a*,

Take any point P in the ellipse, draw PM perpendicular to A*a*, and produce MP to meet the circle in Q, join C, Q,

Draw PT a tangent to the ellipse at P cutting CA produced in T.

Join TQ.

Then QT is a tangent to the circle at Q.

For if TQ be not a tangent, draw QT′ a tangent at Q cutting CA in T′.

Then CQT′ is a right angle.

$\therefore$ Since QM is drawn from the right angle CQT′ perpendicular on the hypothenuse,

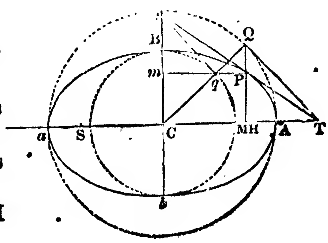
$\therefore$ CT′ : CQ :: CQ : CM. (Geom. Theor. 87.)
or, CT′ : CA :: CA : CM, $\because$ CQ = CA.

But, by the last proposition,

CT : CA :: CA : CM,
$\therefore$ CT = CT′,

which is absurd, therefore QT′ is not a tangent at Q; in the same manner it may be proved that no line but QT can be a tangent at Q,

$\therefore$ &c.



Cor. 1. Describe a circle on the minor axis.

Draw P*m* an ordinate to the minor axis cutting the circle in *q*.

Let a tangent at P cut the minor axis produced in *t*.

Then, since P*m* is parallel to AC, and PM to BC,

$$\begin{aligned} Ct : Cm &:: CT : MT \\ &:: CT^2 : QT^2 \\ &:: Cq^2 : Cm^2 \quad \because \text{the } \Delta^s\ CQT,\ Cmq \text{ are similar} \\ &:: BC^2 : Cm^2 \\ \therefore Ct : CB &:: CB : Cm. \end{aligned}$$

Which is analogous to the property proved in the last Prop. for the major axis.

Cor. 2. Join tq.

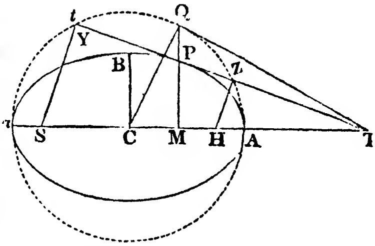
We can prove, as above, that tq is a tangent to the circle Bqb.

PROP. XII.

The square of any semi-ordinate to the axis, is to the rectangle under the abscissæ, as the square of the semi-axis minor is to the square of the semi-axis major.

That is, if P be any point in the curve,

$$PM^2 : AM \,.\, Ma :: BC^2 : AC^2.$$



Describe a circle on Aa, and produce MP to meet it in Q.

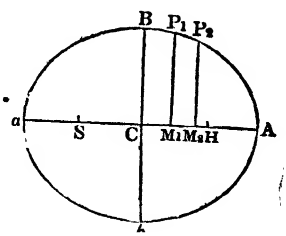
At the points P and Q draw the tangents PT, QT, which will intersect the axis in the same point T. (Prop. 11.)

Let the tangent to the ellipse intersect the circle in Y, Z.

Join S, Y ; H, Z ; SY and HZ are perpendicular to Tt. (Prop. 7.)

Hence the triangles PMT, SYT, HZT, are similar to each other.

$$\begin{aligned} \therefore PM : SY &:: MT : TY \\ \text{and } PM : HZ &:: MT : TZ \\ \therefore PM^2 : SY \,.\, HZ &:: MT^2 : TY \,.\, TZ \\ \text{or, } PM^2 : BC^2 &:: MT^2 : TQ^2 \quad \text{(Prop. 11, and Geom. Theor. 61.)} \\ &:: QM^2 : CQ^2 \quad \because MQT,\ MQC \text{ are similar } \Delta^s. \\ &:: AM \,.\, Ma : AC^2 \\ \therefore PM^2 : AM \,.\, Ma &:: BC^2 : AC^2. \end{aligned}$$



Cor. 1.

Let $P_1 M_1$, $P_2 M_2$, -- be ordinates to the axis from any points P_1, P_2 ---

Then by Prop.

$$\begin{aligned} P_1 M_1^2 : AM_1 \,.\, M_1 a &:: BC^2 : AC^2 \\ P_2 M_2^2 : AM_2 \,.\, M_2 a &:: BC^2 : AC^2 \\ P_1 M_1^2 : P_2 M_2^2 &:: AM_1 \,.\, M_1 a : AM_2 \,.\, M_2 a. \end{aligned}$$

That is, the squares of the ordinates to the axis are to each other as the rectangles of their abscissæ.

Cor. 2. By the fifth proportion in Prop.

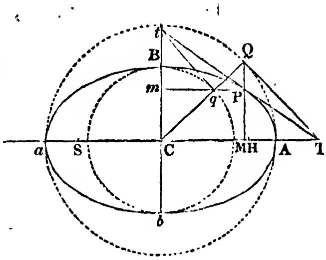
$$PM : QM :: BC : AC.$$

Cor. 3. By Prop.

$$PM^2 . AM . Ma :: BC^2 : AC^2$$

But $AM = AC + CM$, $Ma = AC - CM$,

$$\therefore PM^2 : (AC + CM)(AC - CM) :: BC^2 : AC^2.$$
$$PM^2 : AC^2 - CM^2 :: BC^2 : AC^2.$$



Cor. 4. Describe a circle on Bb, draw Pm, an ordinate to the minor axis cutting the circle in q.

Then, $Pm = CM$, $PM = Cm$.
Then by Cor. 3,

$$\begin{array}{lllll} & AC^2 - Pm^2 : & AC^2 & :: \quad Cm^2 & : BC^2 \\ & Pm^2 \quad : & AC^2 & :: \quad BC^2 - Cm^2 & : BC^2 \\ & & & :: (BC + Cm)(BC - Cm) & : BC^2 \\ & & & :: \quad Bm . mb & : BC^2 \\ \text{or,} & Pm^2 & : Bm . mb & :: \quad AC^2 & : BC^2. \end{array}$$

Which is analogous to the property proved in the Prop. for the major axis.

Cor. 5.

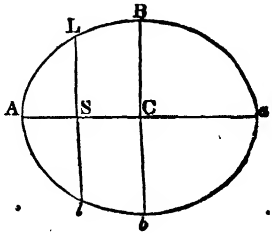
$$Pm : qm :: AC : BC.$$

PROP. XIII.

The latus rectum is a third proportional to the axis major and minor.

That is,

$$Aa : Bb :: Bb : Ll.$$



Since LS is a semiordinate to the axis,

$AC^2 : BC^2 :: AS . Sa : LS^2$, Prop. 12.
$:: BC^2 : LS^2$, Prop. 3. Cor. 1.

$\therefore AC : BC :: BC : LS$

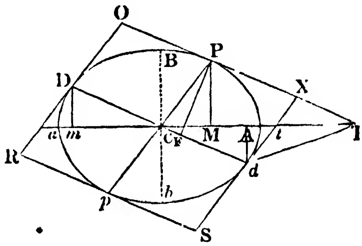
And,

$Aa : Bb :: Bb : Ll.$

PROP. XIV.

The area of all the parallelograms, circumscribing an ellipse, formed by drawing tangents at the extremities of two conjugate diameters, is constant, each being equal to the rectangle under the axes.

Let P*p*, D*d*, be any two conjugate diameters, SROX a parallelogram circumscribing the ellipse formed by drawing tangents at P, D, *p*, *d*; then P*p*, D*d*, divide the parallelogram SROX into four equal parallelograms.



Draw PM, *dm*, ordinates to the axis; PF perpendicular to D*d*.

Produce CA to meet PX in T and S*d* in *t*.

$$\begin{aligned}
\text{Then, } & CT : CA :: CA : CM \\
\text{And, } & Ct : CA :: CA : Cm \\
\therefore\ & CT : Ct :: Cm : CM \\
\text{But, } & CT : Ct :: TM : Cm, \text{ by similar triangles.} \\
\therefore\ & MT : Cm :: Cm : CM,
\end{aligned}$$

$$\therefore CM \,.\, MT = Cm^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\begin{aligned}
\text{Again, } & CM : CA :: CA : CT \\
\therefore\ & CM : CA :: MA : AT, \text{ dividendo.} \\
\text{Or, } & CM : Ma :: MA : MT, \text{ componendo.}
\end{aligned}$$

$$\therefore AM \,.\, Ma = CM \,.\, MT = Cm^2 \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\begin{aligned}
\text{But, } & AC^2 : BC^2 :: AM \,.\, Ma\,(Cm^2) : PM^2, \text{ Prop. 12.} \\
\therefore\ & AC : BC :: Cm : PM \\
\text{Similarly, } & AC : BC :: CM : dm \\
\text{Or, } & BC : dm :: CA : CM \\
\text{But, } & CT : CA :: CA : CM \\
\therefore\ & CT : CA :: BC : dm \\
\text{But, } & PF : CT :: dm : Cd, \text{ for } \triangle\, CdT = \tfrac{1}{2} \square\, CPXd \\
\therefore\ & PF : CA :: BC : Cd.
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ rectangle } PF \,.\, Cd &= \text{rectangle } AC \,.\, BC \\
\text{or, parallelogram } CX &= \text{rectangle } AC \,.\, BC \\
\therefore \text{ parallelogram } SROX &= 4\, AC : BC \\
&= Aa \,.\, Bb.
\end{aligned}$$

Cor. By (2),

$$\begin{aligned}
Cm^2 &= AM \,.\, Ma \\
&= (CA + CM) \,.\, (CA - CM) \\
&= CA^2 - CM^2 \\
\therefore CA^2 &= CM^2 + Cm^2 \\
\text{And similarly, } CB^2 &= PM^2 + dm^2.
\end{aligned}$$

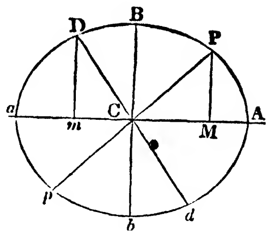
PROP. XV.

The sum of the squares of any two conjugate diameters, is equal to the same constant quantity, namely, the sum of the squares of the two axes.

That is,

If Pp, Dd, be any two conjugate diameters,

$$Pp^2 + Dd^2 = Aa^2 + Bb^2.$$



Draw PM, Dm, ordinates to the axis.

Then, by Cor. to Prop. 14,

$$\begin{aligned} AC^2 + BC^2 &= CM^2 + Cm^2 + PM^2 + dm^2 \\ &= CP^2 + CD^2 \end{aligned}$$

$$\therefore 4AC^2 + 4BC^2 = 4CP^2 + 4CD^2$$

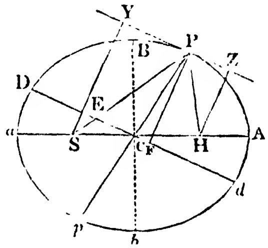
Or, $Aa^2 + Bb^2 = Pp^2 + Dd^2$.

PROP. XVI.

The rectangle under the focal distances of any point is equal to the square of the semi-conjugate.

That is, if CD be conjugate to CP,

$$SP \cdot HP = CD^2.$$



Draw SY, HZ, perpendiculars to the tangent at P, and PF perpendicular on CD.

Then by similar triangles SPY, PEF,

$$\begin{aligned} &SP : SY :: PE : PF \\ \text{Or, } &SP : SY :: AC : PF \quad \because PE = AC, \text{ by Prop. 6} \\ \text{Similarly, } &HP : HZ :: AC : PF \\ \therefore\ &SP \cdot HP : SY \cdot HZ :: AC^2 : PF^2, \\ &:: CD^2 : BC^2, \text{ Prop. 14.} \end{aligned}$$

But $SY \cdot HZ = BC^2$, by Prop. 8.

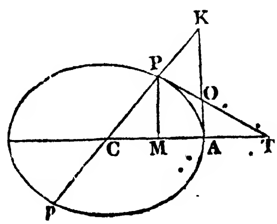
$\therefore SP \cdot HP = CD^2$.

PROP. XVII.

If two tangents be drawn, one at the principal vertex, the other at the vertex of any other diameter, each meeting the other diameter produced, the two tangential triangles thus formed, will be equal.

That is,

$$\triangle CPT = \triangle CAK.$$



Draw the ordinate PM, then,

CM : CA :: CP : CK, by similar $\triangle$s

But, CM : CA :: CA : CT, Prop. x.

$\therefore$ CA : CT :: CP : CK.

The two triangles CPT, CAK, have thus the angle C common, and the sides about that angle reciprocally proportional; these triangles are therefore equal.

Cor. 1. From each of the equal triangles CPT, CAK, take the common space CAOP; there remains,

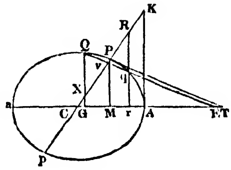
triangle OAT = triangle OKP.

Cor. 2. Also from the equal triangles CPT, CAK, take the common triangle CPM; there remains,

triangle MPT = trapez. AKPM.

PROP. XVIII.

The same being supposed, as in last proposition, then any straight lines, QG, QE, *drawn parallel to the two tangents shall cut off equal spaces.*



That is,

triangle GQE = trapez. AKXG
triangle rqE = trapez. AKRr

Draw the ordinate PM.

The three similar triangles CAK, CMP, CGX, are to each other as CA^2, CM^2, CG^2,

$\therefore$ trap. AKPM : trap. AKXG :: $CA^2 - CM^2$: $CA^2 - CG^2$, dividendo.
But, PM^2 : QG^2 :: $CA^2 - CM^2$: $CA^2 - CG^2$,
$\therefore$ trap. AKPM : trap. AKXG :: PM^2 : QG^2
But, trian. MPT : trian. GQE :: PM^2 : QG^2, $\because$ the triangles are similar.
$\therefore$ trap. AKPM : trian. MPT :: trap. AKXG : trian. GQE.

But by Prop. 17., Cor. 2,

trap. AKPM = triangle MPT
$\therefore$ trap. AKXG = triangle GQE
And similarly, trap. AKRr = triangle rqE

Cor. 1. The three spaces AKXG, TPXG, GQE, are all equal.

Cor. 2. From the equals AKXG, EQG, take the equals AKRr, Eqr; there remains,

RrXG = rqQG.

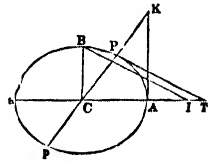
Cor. 3. From the equals RrXG, rqQG, take the common space rqvXG there remains,

triangle vQX = triangle vqR.

Cor. 4. From the equals EQG, TPXG, take the common space EvXG; there remains,

TPvE = triangle vQX.

Cor. 5. If we take the particular case in which QG coincides with the minor axis,



The triangle EQG becomes the triangle IBC,
The figure AKXG becomes the triangle AKC,
$\therefore$ triangle IBC = triangle AKC
= triangle CPT.

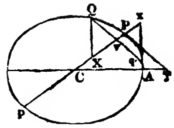
PROP. XIX.

Any diameter bisects all its own ordinates.

That is,

If Qq be any ordinate to a diameter CP,

$$Qv = vq.$$



Draw QX, qx, at right angles to the major axis;

Then triangle vQX = triangle vqx; Prop. 18, Cor. 3.

But these triangles are also equiangular;

$$\therefore \; Qv = vq.$$

Cor. Hence, any diameter divides the ellipse into two equal parts.

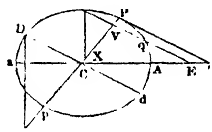
PROP. XX.

The square of the semiordinate to any diameter, is to the rectangle under the abscissæ, as the square of the semi-conjugate to the square of the semi-diameter.

That is,

If Qq be an ordinate to any diameter CP,

$$Qv^2 : Pv \cdot vp :: CD^2 : CP^2$$



Produce Qq to meet the major axis in E;

Draw QX, DW, perpendicular to the major axis, and meeting PC in X and W.

Then, since the triangles CPT, CvE, are similar,

trian. CPT : trian. CvE :: $CP^2 : Cv^2$

or, trian. CPT : trap. TPvE :: $CP^2 : CP^2 - Cv^2$

Again, since the triangles CDW, vQX, are similar,

triangle CDW : triangle vQX :: $CD^2 : vQ^2$

But triangle CDW = triangle CPT; Prop. 18., Cor. 5.

And triangle vQX = trapez. TPvE; Prop. 18., Cor. 3.

$$\therefore \quad CP^2 : CD^2 :: CP^2 - Cv^2 : vQ^2$$

$$\text{Or,} \quad Qv^2 : Pv \cdot vp :: CD^2 : CP^2$$

Cor. 1. The squares of the ordinates to any diameter, are to each other as the rectangles under their respective abscissæ.

Cor. 2. The above proposition is merely an extension of the property already proved in Prop. 12, with regard to the relation between ordinates to the axis and their abscissæ.

HYPERBOLA.

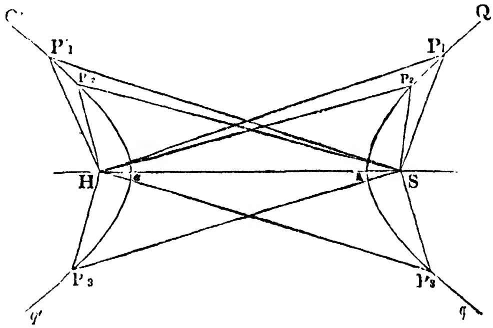
DEFINITIONS.

1. AN HYPERBOLA is a plane curve, such that, if from any point in the curve two straight lines be drawn to two given fixed points, the excess of the straight line drawn to one of the points above the other will always be the same.

2. The two given fixed points are called the *foci.*

Thus, let QAq be an hyperbola, S and H the foci.
Take any number of points in the curve, P_1, P_2, P_3, ...
Join S, P_1, H, P_1; S, P_2, H, P_2; S, P_3, H, P_3;......then,

$$HP_1 - SP_1 = HP_2 - SP_2 = HP_3 - SP_3 = \ldots\ldots$$



If $HP_1 - SP_1$ and $SP'_1 - HP'_1$ ...... be always equal to the same constant quantity, the points $P_1\,P_2\,P_3$ ..... and P'_1, P'_2, P'_3, will lie in two opposite and similar hyperbolas QAq, $Q'aq'$, which in this case are called *opposite hyperbolas.*

3. If a straight line be drawn joining the foci, and bisected, the point of bisection is called the *centre.*

4. The distance from the centre to either focus is called the *eccentricity.*

5. Any straight line drawn through the centre, and terminated by two opposite hyperbolas, is called a *diameter.*

6. The points in which any diameter meets the hyperbolas are called the *vertices* of that diameter.

7. The diameter which passes through the foci is called the *axis major*, and the points in which it meets the curves the *principal vertices.*

8. If a straight line be drawn through the centre at right angles to the major axis, and with a principal vertex as centre, and radius equal to the eccentricity, a circle be described, cutting the straight line in two points, the distance between these points is called the *axis minor.*

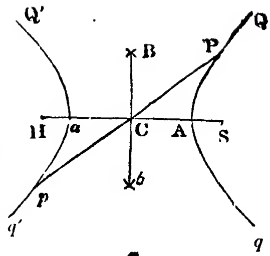
Thus, let Qq, $Q'q'$ be two opposite hyperbolas, S and H the foci, join S, H;

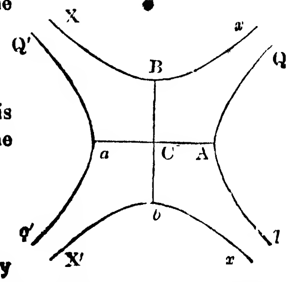
Bisect SH in C, and let SH cut the curves in A, a.

Through C draw any straight line Pp, terminated by the curves in the points P, p.

Through C draw any straight line at right angles to Aa, and with centre A and radius $=$ CS describe a circle cutting the straight line in the points B, b.

Then C is the centre, CS or CH the eccentricity, Pp is a diameter, P and p its *vertices*, Aa is the major axis, Bb is the minor axis.





The hyperbolas Xx, $X'x'$, whose major axis is Bb, and whose minor axis is Aa, are called the *conjugate hyperbolas* to Qq, $Q'q'$.

9. A straight line, which meets the curve in any point, but which, being produced both ways, does not cut it, is called a *tangent* to the curve at that point.

10. A straight line, drawn through the centre, parallel to the tangent, at the vertex of any diameter, is called the *conjugate diameter* to the latter, and the two diameters are called a *pair of conjugate diameters.*

The vertices of the conjugate diameter are its intersections with the conjugate hyperbolas.

11. Any straight line drawn parallel to the tangent at the vertex of any diameter, and terminated both ways by the curve, is called an *ordinate* to that diameter.

12. The segments into which any diameter produced is divided by one of its own ordinates and its vertices, are called the *abscissæ* of the diameter.

13. The ordinate to any diameter, which passes through the focus, is called the *parameter* of that diameter.

Thus, let Pp be any diameter, and Tt a tangent at P;

Draw the diameter Dd parallel to Tt;

Take any point Q in the curve, draw Qq parallel to Tt and cutting Pp produced in v;

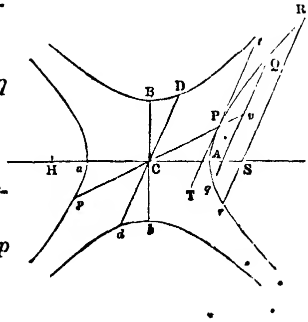
Through S draw Rr parallel to Tt;

Then Dd is the conjugate diameter to Pp,

Qq is the ordinate to the diameter Pp corresponding to the point Q,

Pv, vp, are the abscissæ of the diameter Pp corresponding to the point Q,

Rr is the parameter of the diameter Pp.



14. Any straight line drawn from any point in the curve at right angles to the major axis produced, and terminated both ways by the curve, is called an *ordinate to the axis.*

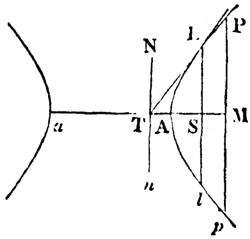
15. The segments into which the major axis produced is divided by one of its own ordinates and its vertices, are called the *abscissæ of the axis.*

16. The ordinate to the axis which passes through the focus, is called the *principal parameter*, or *latus rectum.*

(It will be proved in Prop. 4, that the tangents at the principal vertices are perpendicular to the major axis; hence definitions 14, 15, 16, are in reality included in the three which immediately precede them.)

17. If a tangent be drawn at the extremity of the latus rectum, and produced to meet the major axis; and if a straight line be drawn through the point of intersection, at right angles to the major axis; the tangent is called the *focal tangent*, and the straight line the *directrix.*

Thus, from P, any point in the curve, draw PMp perpendicular to Aa, cutting Aa in M;
Through S draw Ll perpendicular to Aa;
Let LT, a tangent at L, cut Aa in T;
Through T draw Nn perpendicular to Aa:
Then, Pp is the ordinate to the axis corresponding to the point P,
AM, Ma, are the abscissæ of the axis corresponding to the point P,
Ll is the latus rectum,
LT is the focal tangent,
Nn is the directrix.



PROP. I.

The difference of two straight lines drawn from the foci to any point in the curve, is equal to the major axis.

That is, if P be any point in the curve,

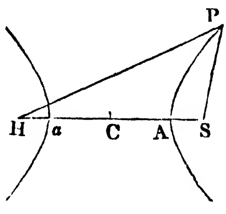
$$HP - SP = Aa;$$

For,

$$\left.\begin{array}{l} HP - SP = AH - AS = Aa + aH - AS \\ \text{And,} \\ HP - SP = aS - aH = Aa - aH + AS \end{array}\right\} \text{Def. 1.}$$

Or,

$$2(HP - SP) = 2Aa$$
$$HP - SP = Aa$$



Cor. I The centre bisects the major axis; for, since

$$\begin{array}{lrcl} & AH - AS & = & aS - aH \\ \text{Or,} & SH - 2AS & = & SH - 2aH \\ \therefore & AS & = & aH \\ \text{And} & CS & = & CH, \text{ by def. 3.} \\ \therefore & AC & = & aC. \end{array}$$

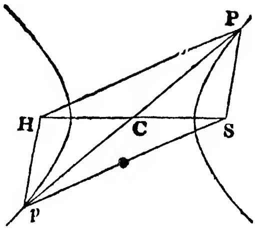
Cor. II.

$$\begin{array}{rcl} HP - SP & = & 2AC \\ \therefore\ HP & = & 2AC + SP \\ SP & = & HP - 2AC \\ HP + SP & = & 2AC + 2SP. \end{array}$$

PROP. II.

The centre bisects all diameters.

Take any point P in the curve;
Join S, P; H, P; S, H;
Complete the parallelogram SPHp;
Join C, p; C, P;



Then, since the opposite sides of parallelograms are equal,

$$HP = Sp, \qquad SP = Hp;$$
$$\therefore \quad HP - SP = Sp - Hp;$$

$\therefore$ p is a point in the opposite hyperbola by definition 2.

Again, since the diagonals of a parallelogram bisect each other and since SH is bisected in C, (def. 3,)

$\therefore$ Pp is a straight line and a diameter, and is bisected in C.

In like manner, it may be proved that any other diameter is bisected in C.

PROP. III.

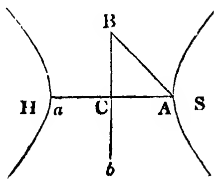
The rectangle under the segments of the major axis produced, made by the focus and its vertices, is equal to the square of the semi-axis, minor.

That is,

$$AS \cdot Sa = BC^2$$

For,

$$\begin{aligned} BC^2 &= AB^2 - AC^2 \\ &= SC^2 - AC^2, \text{ by def. 8,} \\ &= (SC - AC)(SC + AC) \\ &= AS \cdot Sa \end{aligned}$$



Cor. The square of the eccentricity is equal to the sum of the squares of the semi-axes.

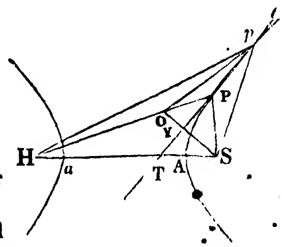
$$\text{For,} \quad \begin{aligned} SC^2 &= AB^2, \text{ def. 8,} \\ &= AC^2 + BC^2. \end{aligned}$$

PROP. IV.

To draw a tangent to the hyperbola at any point.

Let P be the given point;
Join S, P; H, P;
Bisect the angle SPH by the straight line Tt.

Tt is a tangent to the curve at P.



For if Tt be not a tangent, let Tt cut the curve in some other point p.

Join S, p; H, p; draw SYO perpendicular to Tt, meeting HP in O; join p, O.

Since the angles at Y are right angles, and angle SPY = angle OPY by construction, and side YP common to the two triangles SYP, OYP;

$$\therefore \quad SY = OY$$
$$\text{And} \quad SP = OP.$$

Again, since $SY = OY$, and Yp common to the two triangles SYp, OYp, and the angles at Y equal;

$$\begin{aligned} \therefore \quad Sp &= Op \\ \therefore \quad Hp - Op &= Hp - Sp \\ &= HP - SP \\ &= HP - OP \\ &= HO \\ \therefore \quad Hp &= HO + Op \end{aligned}$$

that is, one side of the triangle HOp is equal to the other two, which is absurd; $\therefore$ p is not a point in the curve: and in the same manner, it may be proved that no point in the straight line Tt can be in the curve, except P;

$\therefore$ Tt is a tangent to the curve at P.

Cor. 1. Hence tangents at A and a, are perpendicular to the major axis.

Cor. 2. SP and HP make equal angles with every tangent.

Cor. 3. Since SPH, the verticle angle of $\triangle$ SPH, is bisected by the straight line PT, which cuts the base in T,

$$\therefore HT : TS :: HP : SP.$$

PROP. V.

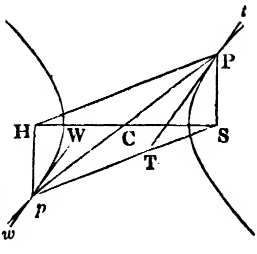
Tangents drawn at the vertices of a diameter are parallel.

Let Tt, Ww, be tangents at P, p, the vertices of the diameter PCB.

Join S, P; H, P; S, p; H, p:

Then, by Prop. 2, SH is a parallelogram, and since the opposite angles of parallelograms are equal,

$$\therefore \text{angle } SPH = \text{angle } SpH.$$



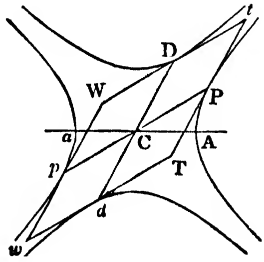
But the tangents Tt, Ww, bisect the angles SPH, SpH, respectively.

$$\begin{aligned} \therefore \text{angle } WpS &= \text{angle HPT} \\ &= \text{angle PTS, which is the exterior opposite angle to } WpS. \end{aligned}$$

$\therefore$ Ww is parallel to Tt.

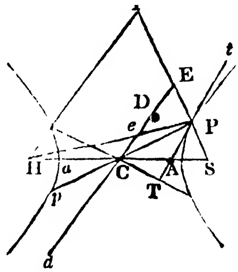
Cor. If Dd be a diameter conjugate to Pp, and terminated by the conjugate hyperbolas, tangents drawn at D and d will be parallel.

Hence tangents drawn at the extremities of conjugate diameters form a parallelogram.



PROP. VI.

If straight lines be drawn from the foci to a vertex of any diameter, the distance from the vertex to the intersection of the conjugate diameter with either focal distance, is equal to the semi-axis major.



That is, if Dd be a diameter conjugate to Pp, cutting SP produced in E, and HP in e,

$$PE \text{ or } Pe = AC.$$

Draw HI parallel to Dd, meeting SP produced in I.

The angle PHI = alternate angle HPT
= angle TPS
= angle HIP
∵ HI is parallel to Dd or Tt
∴ IP = HP.

Also, since SC = HC, and CE is parallel to HI, the base of the △ SHI,

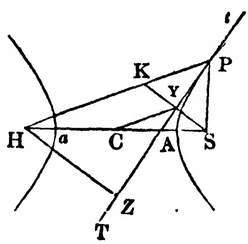
$$\begin{aligned}
\therefore SE &= EI \\
\text{Hence, } \because PE &= PI - EI \\
&= HP - SE \\
&= HP - SP - PE \\
\therefore 2PE &= HP - SP \\
&= 2AC \\
PE &= AC.
\end{aligned}$$

Also angle PEe = angle PeE, ∴ Pe = PE and

$$Pe = AC.$$

PROP. VII.

Perpendiculars from the foci upon the tangent at any point, intersect the tangent in the circumference of a circle whose diameter is the major axis.



From S let fall SY perpendicular on Tt a tangent at P.

Join S, P; H, P; let HP meet SY in K; join C, Y;

Then, since angle SPY = angle KPY, and the angles at Y are right angles, and the side PY common to the two triangles SPY, KPY,

∴ SY = KY
and KP = SP.

Again, since SY = YK, and SC = CH, CY cuts the sides of △ HSK proportionally,

∴ CY is parallel to HP.

Also, since CY is parallel to HP, SY = KY, and SC = CH,

$$\begin{aligned}
\therefore CY &= \tfrac{1}{2} HK \\
&= \tfrac{1}{2}(HP - KP) \\
&= \tfrac{1}{2}(HP - SP) \\
&= \tfrac{1}{2} Aa \\
&= AC.
\end{aligned}$$

Hence, a circle described with centre C and radius = CA, will pass through

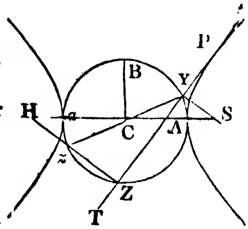
Y; and in like manner, if HZ be drawn perpendicular to Tt, it may be proved that the same circle will pass through Z also.

PROP. VIII.

The rectangle contained by perpendiculars from the foci upon the tangent at any point, is equal to the square of the semi-axis, minor.

That is,

$$SY \cdot HZ = BC^2.$$



Let Tt be a tangent at any point P;

On Aa describe a circle cutting Tt in Y and Z; join S, Y; H, Z.

Then, by last Prop., SY, HZ are perpendicular to Tt.

Let HZ meet the circumference in z;

Join C, z; C, Y;

Since zZY is a right angle, the segment in which it lies is a semicircle, and z, Y, are the extremities of a diameter;

$$\therefore\ zCY \text{ is a straight line and a diameter.}$$

Hence the triangles CYS, CzH, are in every respect equal;

$$\therefore\ SY = Hz$$

$$\therefore\ SY \cdot HZ = Hz \cdot HZ$$

$$= HA \cdot Ha. \quad \text{Geom. Theor. 61.}$$

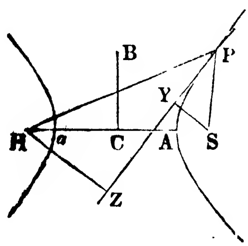
$$= BC^2. \quad \text{Prop. 3.}$$

PROP. IX.

Perpendiculars let fall from the foci upon the tangent at any point, are to each other as the focal distance of the point of contact.

That is,

$$SY : HZ :: SP : HP.$$



For the triangles SPY, HPZ, are manifestly similar;

$$\therefore\ SY : HZ :: SP : HP.$$

Cor. Hence,

$$SY = HZ \cdot \frac{SP}{HP}$$

$$\therefore\ SY^2 = SY \cdot HZ \cdot \frac{SP}{HP}$$

$$= BC^2 \cdot \frac{SP}{HP}, \text{ last Prop.}$$

$$= BC^2 \cdot \frac{SP}{2\,AC + SP}$$

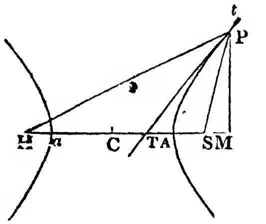
So also,

$$HZ^2 = BC^2 \cdot \frac{HP}{SP}$$

$$= BC^2 \cdot \frac{HP}{HP - 2\,AC}.$$

PROP. X.

If a tangent be applied at any point, and from the same point an ordinate to the axis be drawn, the semi-axis major is a mean proportional between the distance from the centre, to the intersection of the ordinate with the axis, and the distance from the centre to the intersection of the tangent with the axis.



That is,

$$CT : CA :: CA : CM.$$

Since the angle SPH is bisected by PT, which cuts HS, the base of the triangle HPS, in T, $\therefore$

$$\begin{array}{llllllll} & HT & : & ST & :: & HP & : & SP \\ \therefore & HT - ST & : & HT + ST & :: & HP - SP & : & HP + SP \\ \text{or,} & 2\,CT & : & SH & :: & 2\,AC & : & HP + SP \\ \therefore & 2\,CT & : & 2\,AC & :: & SH & : & HP + SP \ldots\ldots\ldots\ldots (1) \end{array}$$

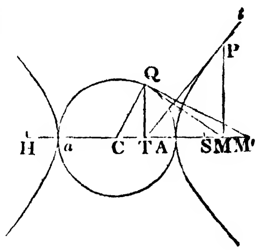
But since PM is drawn from the vertex of triangle HPS perpendicular to HS produced,

$$\begin{array}{lllllll} HM - SM & : & HP + SP & :: & HP - SP & : & HM + SM \\ \text{or,} \quad SH & : & HP + SP & :: & 2\,AC & : & 2\,CM \ldots\ldots\ldots\ldots (2) \end{array}$$

Comparing this with the proportion marked (1), we have

$$\begin{array}{llllllll} & 2\,CT & : & 2\,AC & :: & 2\,AC & : & 2\,CM \\ \text{or,} & CT & : & CA & :: & CA & : & CM. \end{array}$$

PROP. XI.



Let AQ*a* be a circle described on the major axis, from the point T, draw TQ perpendicular to A*a*, meeting the circle in Q, join QM.

Then QM is a tangent to the circle at Q.

Join C, Q.

For if QM be not a tangent, draw QM′ a tangent at Q, cutting AC in M′.

Then CQM′ is a right angle.

$\therefore$ Since QT is drawn from the right angle CQM′ perpendicular to the hypothenuse,

$$\therefore\ CM' : CQ :: CQ : CT.$$

$$\text{or, } CM' : CA :: A : CT, \quad \because\ CQ = CA.$$

But by the last Prop.,

$$CM : CA :: CA : CT$$

$$\therefore\ CM = CM',$$

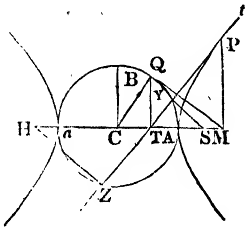
which is absurd; $\therefore$ QM is not a tangent at Q; and in the same manner it may be proved that no line but QM′ can be a tangent at Q.

PROP. XII.

The square of any semi-ordinate to the axis, is to the rectangle under the abscissæ, as the square of the semi-axis minor, is to the square of the semi-axis major.

That is, if P be any point in the curve,

$$PM^2 : AM \cdot Ma :: BC^2 : AC^2.$$



Describe a circle on Aa, and draw PT a tangent to the hyperbola at P, intersecting the circle in the points Y, Z, and the major axis in T.

Draw TQ perpendicular to Aa, meeting the circle in Q; join QM

Then QM is a tangent to the circle at Q by Prop. 11, and $\therefore$ the angle CQM is a right angle.

Join S, Y; H, Z; SY and HZ are perpendicular to Tt, Prop. 7.

Hence the triangles PMT, SYT, HZT, are similar to each other.

$\therefore PM : SY :: MT : TY$

and, $PM : HZ :: MT : TZ$

$\therefore PM^2 : SY \cdot HZ :: MT^2 : TY \cdot TZ$,

or, $PM^2 : BC^2 :: MT^2 : TQ^2$, Prop. 8, and Geom. Theor. 61.

$:: QM^2 : CQ^2$, $\because$ MQT, MCQ are similar triangles.

$:: AM \cdot Ma : AC^2$, Geom. Theor. 61.

$\therefore PM^2 : AM \cdot Ma :: BC^2 : AC^2$.

Cor. 1.

Let $P_1 M_1$, $P_2 M_2$, ... be ordinates to the axis from any point P_1, P_2,.....

Then by Prop.

$$P_1 M_1^2 : AM_1 \cdot M_1 a :: BC^2 : AC^2$$

$$P_2 M_2^2 : AM_2 \cdot M_2 a :: BC^2 : AC^2$$

$$\therefore P_1 M_1^2 : P_2 M_2^2 :: AM_1 \cdot M_1 a : AM_2 \cdot M_2 a.$$

That is, the square of the ordinates to the axis are to each other as the rectangles of their abscissa.

Cor. 2. By Prop.

$$PM^2 : AM \cdot Ma :: BC^2 : AC^2$$

But $AM = CM - CA$, $Ma = CM + CA$,

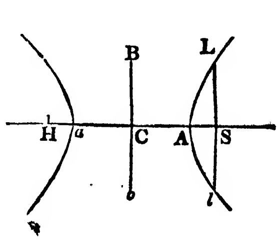
$$\therefore PM^2 : CM^2 - CA^2 :: BC^2 : AC^2.$$

PROP. XIII.

The latus rectum is a third proportional to the axis major and minor.

That is,

$$Aa : Bb :: Bb : Ll.$$



Since LS is a semiordinate to the axis,

$AC^2 : BC^2 :: AS \cdot Sa : LS^2$, Prop. 12.

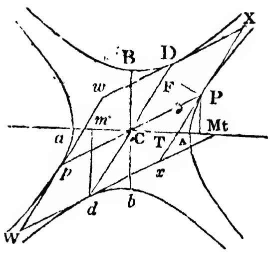
$:: BC^2 : LS^2$, Prop. 3.

$\therefore AC : BC :: BC : LS$

or, $Aa : Bb :: Bb : Ll$

PROP. XIV.

The area of all parallelograms, formed by drawing tangents at the extremities of two conjugate diameters, is constant, each being equal to the rectangle under the axes.



Let Pp, Dd, be any two conjugate diameters, $WwXx$, a parallelogram inscribed between the opposite and conjugate hyperbolas by drawing tangents at P, p, D, d; then Pp, Dd, divide the parallelogram $WxXw$ into four equal parallelograms.

Draw Pm, dm, ordinates to the axis; PF perpendicular to Dd.

Let CA meet PX in T and Wx in t;

$$
\begin{array}{rl}
\text{Then} & CT : CA :: CA : CM \\
 & Ct : CA :: CA : Cm, \quad \text{Prop. 11.} \\
\therefore & CT : Ct :: Cm : CM \\
\text{But,} & CT : Ct :: MT : Cm, \text{ by similar triangles.} \\
\therefore & MT : Cm :: Cm : CM \\
\therefore & CM \cdot MT = Cm^2 \quad \ldots\ldots\ldots\ldots (1)
\end{array}
$$

$$
\begin{array}{rl}
\text{Again,} & CM : CA :: CA : CT \\
\therefore & CM : CA :: MA : AT, \text{ dividendo:} \\
\text{Or,} & CM : Ma :: MA : MT, \text{ componendo:} \\
\therefore & AM \cdot Ma = CM \cdot MT = Cm^2 \quad \ldots\ldots\ldots\ldots (2)
\end{array}
$$

$$
\begin{array}{rl}
\text{But,} & AC^2 : BC^2 :: AM \cdot Ma : PM^2 \\
\text{Or,} & AC^2 : BC^2 :: Cm^2 : PM^2 \\
\therefore & AC : BC :: Cm : PM \\
\text{Similarly,} & AC : BC :: CM : dm \\
\text{Or,} & BC : dm :: CA : CM \\
\text{But,} & CT : CA :: CA : CM \\
 & CT : CA :: BC : dm \\
\text{But,} & PF : CT :: dm : Cd \\
\therefore & PF : CA :: BC : Cd
\end{array}
$$

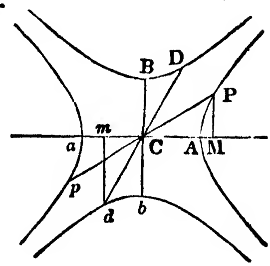
$$
\begin{array}{rl}
\therefore \text{ Rectangle } PF \cdot CD & = \text{rectangle } AC \cdot BC \\
\text{or, Parallelogram } CX & = \text{rectangle } AC \cdot BC \\
\therefore \text{ Parallelogram } WwXx & = 4\, AC \cdot BC \\
 & = Aa \cdot Bb.
\end{array}
$$

Cor. By (2),

$$
\begin{array}{rl}
Cm^2 & = AM \cdot Ma \\
 & = (CM - CA)\,(CM + CA). \\
 & = CM^2 - CA^2 \\
\therefore \quad CA^2 & = CM^2 - Cm^2 \\
\text{And similarly, } CB^2 & = dm^2 - PM^2.
\end{array}
$$

PROP. XV.

The difference of the squares of any two conjugate diameters, is equal to the same constant quantity, namely, the difference of the squares of the two axes.



That is, if Pp, Dd, be any two conjugate diameters,

$$Pp^2 - Dd^2 = Aa^2 - Bb^2.$$

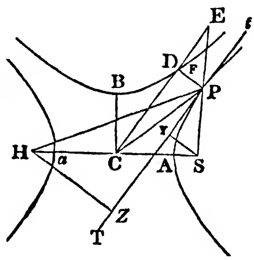
Draw PM, dm, ordinates to the axis.

Then, by Cor. to last Prop.

$$AC^2 - BC^2 = CM^2 + PM^2 - (Cm^2 + dm^2)$$
$$= CP^2 - Cd^2$$
$$\therefore Aa^2 - Bb^2 = Pp^2 - Dd^2.$$

PROP. XVI.

The rectangle under the focal distances of any point, is equal to the square of the semi-conjugate.



That is, if CD be conjugate to CP,

$$SP . HP = CD^2.$$

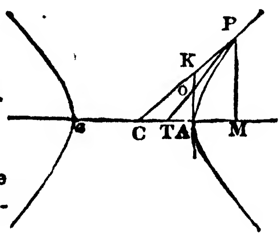
Draw SY, HZ, perpendiculars to the tangent at P, and PF perpendicular to CD;

Then by similar triangles, SPY, PEF,

$$SP : SY :: PE : PF$$
or, $$SP : SY :: AC : PF$$
$$\because PE = AC. \text{ by Prop. 6.}$$
Similarly, $$HP : HZ :: AC : PF$$
$$\therefore SP . HP : SY . HZ :: AC^2 : PF^2$$
$$:: CD^2 : CB^2, \text{ by Prop. 14.}$$
But $$SY . HZ = CB^2, \text{ by Prop. 8.}$$
$$\therefore SP . HP = CD^2.$$

PROP. XVII.

If two tangents be drawn, one at the principal vertex, the other at the vertex of any other diameter, each meeting the other's diameter produced, the two tangential triangles thus formed will be equal.



That is,

$$\text{triangle } CPT = \text{triangle } CAK.$$

Draw the ordinate PM; then

$$CM : CA :: CP : CK, \text{ by similar triangles.}$$
But, $$CM : CA :: CA : CT$$
$$\therefore CA : CT :: CP : CK.$$

The two triangles CPT, CAK, have thus the angle C common and the sides about that angle reciprocally proportional; these triangles are equal.

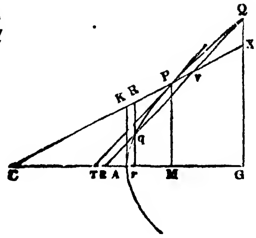
Cor. 1. Take each of the equal triangles CPT, CAK, from the common space CAOP; there remains

$$\text{triangle OAT} = \text{OKP}.$$

Cor. 2. Also take the equal triangles CPT, CAK, from the common triangle CPM; there remains

$$\text{triangle MPT} = \text{trapez. AKPM}.$$

PROP. XVIII.



The same being supposed, as in last proposition, then any straight lines, QG, QE, drawn parallel to the two tangents shall cut off equal spaces.

That is,

$$\text{triangle GQE} = \text{trapez. AKXG}$$
$$\text{triangle } rq\text{E} = \text{trapez. AKR}r$$

Draw the ordinate PM.

The three similar triangles CAK, CMP, CGX, are to each other as CA^2, CM^2, CG^2,

$\therefore$ AKPM : trap. AKXG :: $CM^2 - CA^2$: $CG^2 - CA^2$, dividendo.

But, PM^2 : QG^2 :: $CM^2 - CA^2$: $CG^2 - CA^2$,

$\therefore$ trap. AKPM : trap. AKXG :: PM^2 : QG^2

But, trian. MPT : trian. GQE :: PM^2 : QG^2, $\because$ the triangles are similar.

$\therefore$ trap. AKPM : trian. MPT :: trap. AKXG : trian. GQE,

But, by Prop. XVII., Cor. 2,

$$\text{trap. AKPM} = \text{triangle MPT};$$
$$\therefore \text{trap. AKXG} = \text{triangle GQE}.$$

And similarly, trap. AKRr = triangle rqE.

Cor. 1. The three spaces AKXG, TPXG, GQE, are all equal.

Cor. 2. From the equals, AKXG, EQG, take the equals AKRr, Eqr; there remains,

$$\text{R}r\text{XG} = rq\text{QG}.$$

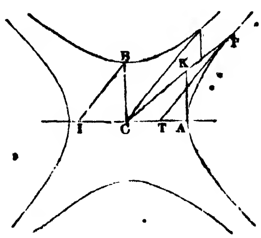
Cor. 3. From the equals RrXG, rqQG, take the common space rqvXG; there remains,

$$\text{triangle } v\text{QX} = \text{triangle } vq\text{R}.$$

Cor. 4. From the equals EQG, TPXG, take the common space EvXG; there remains,

$$\text{TP}v\text{E} = \text{triangle } v\text{QX}.$$

Cor. 5. If we take the particular case in which QG coincides with the minor axis,



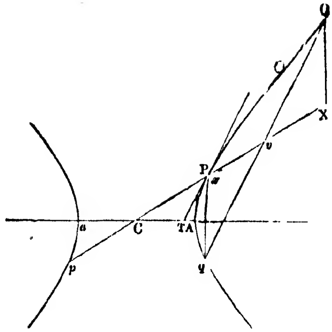
The triangle EQG becomes the triangle IBC,

The figure AKXG becomes the triangle AKC,

$\therefore$ triangle IBC = triangle AKC
= triangle CPT.

PROP. XIX.

Any diameter bisects all its own ordinates.



That is,

If Qq be any ordinate to a diameter CP,

$$Qv = vq.$$

Draw QX, qx, at right angles to the major axis;

Then triangle vQX = triangle vqx; Prop. XVIII., Cor. 3.

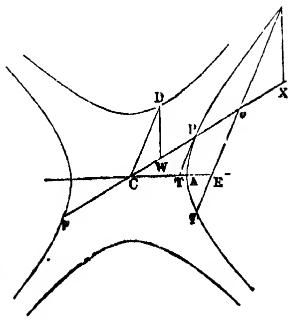
But these triangles are also equiangular;

$$\therefore \quad Qv = vq.$$

Cor. Hence, any diameter divides the hyperbola into two equal parts.

PROP. XX.

The square of the semi-ordinate to any diameter, is to the rectangle under the abscissæ, as the square of the semi-conjugate to the square of the semi-diameter.



That is,

If Qq be an ordinate to any diameter CP,

$$Qv^2 : Pv \cdot vp :: CD^2 . CP^2.$$

Let Qq meet the major axis in E;

Draw QX, DW, perpendicular to the major axis, and meeting PC in X and W.

Then, since the triangles CPT, CvE, are similar,

trian. CPT : trian. CvE :: $CP^2 : Cv^2$

or, trian. CPT : trap. TPvE :: $CP^2 : Cv^2 - CP^2$

Again, since the triangles CDW, vQX, are similar,

	triangle CDW : triangle vQX :: $CD^2 : vQ^2$;
But,	triangle CDW = triangle CPT; Prop. XVIII., Cor. 5,
And	triangle vQX = trapez. TPvE; Prop. XVIII., Cor. 3

$$\therefore \quad CP^2 : CD^2 :: Cv^2 - CP^2 : vQ^2$$

$$\text{Or,} \quad Qv^2 : Pv \cdot vp :: CD^2 : CP^2.$$

Cor. 1. The squares of the ordinates to any diameter, are to each other as the rectangles under their respective abscissæ.

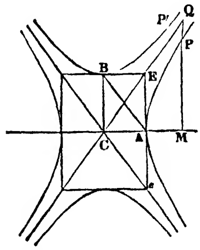
Cor. 2. The above proposition is merely an extension of the property already proved in Prop. 12, with regard to the relation between ordinates to the axis and their abscissæ.

ON THE ASYMPTOTES OF THE HYPERBOLA.

Definition.—An *Asymptote* is a diameter which approaches nearer to meet the curve, the farther it is produced, but which, being produced ever so far, does never actually meet it.

PROP. XXI.

If tangents be drawn at the vertices of the axes, the diagonals of the rectangle so formed are asymptotes to the four curves.



Let MP meet CE in Q;

Then, $MQ^2 : CM^2 :: AE^2 : AC^2$

$:: BC^2 : AC^2$

$:: MP^2 : CM^2 - CA^2$.

Now, as CM increases, the ratio of CM^2 to $CM^2 - CA^2$ continually approaches to a ratio of equality; but $CM^2 - CA^2$ can never become actually equal to CM^2, however much CM may be increased. Hence, MP is always less than MQ, but approaches continually nearer to an equality with it.

In the same manner it may be proved, that CQ is an asymptote to the conjugate hyperbola BP′.

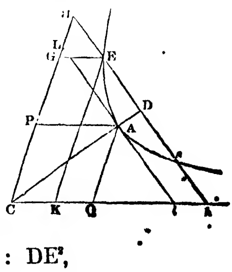
Cor. 1. The two asymptotes make equal angles with the axis major and with the axis minor.

Cor. 2. The line AB joining the vertices of the conjugate axes is bisected by one asymptote and is parallel to the other.

Cor. 3. All lines perpendicular to either axis and terminated by the asymptotes are bisected by the axis.

PROP. XXII.

All the parallelograms are equal, which are formed between the asymptotes and curve, by lines drawn parallel to the asymptotes.



That is, the lines GE, EK, AP, AQ, being parallel to the asymptotes CH, Ch, then the parallelogram CGEK = parallelogram CPAQ.

For, let A be the vertex of the curve, or extremity of the semi-transverse axis AC, perpendicular to which draw AL or Al, which will be equal to the semi-conjugate, by definition XIX. Also, draw $HEDeh$ parallel to Ll.

Then, $CA^2 : AL^2 :: CD^2 - CA^2 : DE^2$,

and by parallels, $CA^2 : AL^2 :: CD^2 : DH^2$;

therefore, by subtract. $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$ or rect. $HE \cdot Eh$;

consequently, the square $AL^2 =$ the rectangle $HE \cdot Eh$.

But, by similar trian. $PA : AL :: GE : EH$,
and, by the same, $QA : Al :: EK : Eh$;
therefore, by comp. $PA . AQ : AL^2 :: GE . EK : HE . Eh$;
and, because $AL^2 = HE . Eh$, therefore $PA . AQ = GE . EK$.

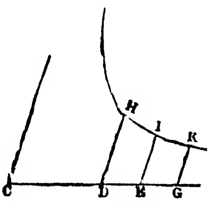
But the parallelograms CGEK, CPAQ, being equiangular, are as the rectangles GE.EK and PA.AQ.

And therefore the parallelogram GK = the parallelogram PQ.

That is, all the inscribed parallelograms are equal to one another. Q. E. D

Corol. 1. Because the rectangle GEK or CGE is constant, therefore GE is reciprocally as CG, or $CG : CP :: PA : GE$. And hence the asymptote continually approaches towards the curve, but never meets it; for GE decreases continually as CG increases; and it is always of *some* magnitude, except when CG is supposed to be infinitely great, for then GE is infinitely small or nothing. So that the asymptote CG may be considered as a tangent to the curve at a point infinitely distant from C.

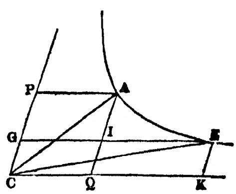
Corol. 2. If the abscissas CD, CE, CG, &c., taken on the one asymptote, be in geometrical progression increasing; then shall the ordinates DH, EI, GK, &c., parallel to the other asymptote, be a decreasing geometrical progression, having the same ratio. For, all the rectangles CDH, CEI, CGK, &c., being equal, the ordinates DH, EI, GK, &c., are reciprocally as the abscissas CD, CE, CG, &c., which are geometricals. And the reciprocals of geometricals are also geometricals, and in the same ratio, but decreasing, or in converse order.



PROP. XXIII.

The three following spaces, between the asymptotes and the curve, are equal; namely, the sector or trilinear space contained by an arc of the curve and two radii, or lines drawn from its extremities to the centre; and each of the two quadrilaterals, contained by the said arc, and two lines drawn from its extremities parallel to one asymptote, and the intercepted part of the other asymptote.

That is,
The sector CAE = PAEG = QAEK, all standing on the same arc AE.



For, as has been already shown, CPAQ = CGEK;
Subtract the common space CGIQ,
So shall the paral. PI = the paral. IK;
To each add the trilineal IAE,
Then is the quadril. PAEG = QAEK.

Again, from the quadrilateral CAEK, take the equal triangle CAQ, CEK, and there remains the sector CAE = QAEK.

Therefore, CAE = QAEK = PAEG. Q. E. D.

APPLICATION OF ALGEBRA

TO

GEOMETRY.

When it is proposed to resolve a geometrical problem algebraically, or by algebra, it is proper, in the first place, to draw a figure that shall represent the several parts or conditions of the problem, and to suppose that figure to be the true one. Then, having considered attentively the nature of the problem, the figure is next to be prepared for a solution, if necessary, by producing or drawing such lines in it as appear most conducive to that end. This done, the usual symbols or letters, for known and unknown quantities, are employed to denote the several parts of the figure, both the known and unknown parts, or as many of them as necessary, as also such unknown line or lines as may be easiest found, whether required or not. Then proceed to the operation, by observing the relations that the several parts of the figure have to each other; from which, and the proper theorems in the foregoing elements of geometry, make out as many equations independent of each other, as there are unknown quantities employed in them: the resolution of which equations, in the same manner as in arithmetical problems, will determine the unknown quantities, and resolve the problem proposed.

As no general rule can be given for drawing the lines, and selecting the fittest quantities to substitute for, so as always to bring out the most simple conclusions, because different problems require different modes of solution; the best way to gain experience, is to try the solution of the same problem in different ways, and then apply that which succeeds best, to other cases of the same kind when they afterwards occur. The following particular directions, however, may be of some use.

1*st*, In preparing the figure, by drawing lines, let them be either parallel or perpendicular to other lines in the figure, or so as to form similar triangles. And if an angle be given, it will be proper to let the perpendicular be opposite to that angle, and to fall from one end of a given line, if possible.

2*d*, In selecting the quantities proper to substitute for, those are to be chosen, whether required or not, which lie nearest the known or given parts of the figure, and by means of which the next adjacent parts may be expressed by addition and subtraction only, without using surds.

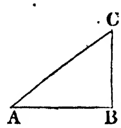
3*d*, When two lines or quantities are alike related to other parts of the figure or problem, the way is, not to make use of either of them separately, but to substitute for their sum, or difference, or rectangle, or the sum of their alternate quotients, or for some line or lines in the figure, to which they have both the same relation.

4*th* When the area, or the perimeter, of a figure, is given, or such parts of it as have only a remote relation to the parts required; it is sometimes of use to assume another figure similar to the proposed one, having one side equal to

unity, or some other known quantity. For, hence the other parts of the figure may be found, by the known proportions of the like sides, or parts, and so an equation be obtained. For examples, take the following problems.

PROBLEM I.

In a right-angled triangle, having given the base (3), *and the sum of the hypothenuse and perpendicular* (9); *to find both these two sides.*



Let ABC represent the proposed triangle right-angled at B. Put the base AB $= 3 = b$, and the sum AC + BC of the hypothenuse and perpendicular $= 9 = s$; also, let x denote the hypothenuse AC, and y the perpendicular BC.

Then by the question $x + y = s$,
and by theorem 34, $x^2 = y^2 + b^2$,
By transposition y in the 1st equation gives, $x = s - y$,
This value of x substituted in the 2d, gives $s^2 - 2sy + y^2 = y^2 + b^2$,
Taking away y^2 on both sides leaves $s^2 - 2sy = b^2$,
By transposing $2sy$ and b^2, gives $s^2 - b^2 = 2sy$,
And dividing by $2s$, gives $\frac{s^2 - b^2}{2s} = y = 4 =$ BC.
Hence $x = s - 5 = y =$ AC.

N. B. In this solution, and the following ones, the notation is made by using as many unknown letters, x and y, as there are unknown sides of the triangle, a separate letter for each; in preference to using only one unknown letter for one side, and expressing the other unknown side in terms of that letter and the given sum or difference of the sides; though this latter way would render the solution shorter and sooner; because the former way gives occasion for more and better practice in reducing equations; which is the very end and reason for which these problems are given at all.

PROBLEM II.

In a right-angled triangle, having given the hypothenuse (5), *and the sum of the base and perpendicular* (7); *to find both these two sides.*

Let ABC represent the proposed triangle right-angled at B. Put the given hypothenuse AC $= 5 = a$, and the sum AB + BC of the base and perpendicular $= 7 = s$; also let x denote the base AB, and y the perpendicular BC.

Then by the question, $x + y = s$,
and by theorem 34 $x^2 + y^2 = a^2$,
By transposing y in the 1st, gives $x = s - y$,
By substituting this value for x, gives $s^2 - 2sy + 2y^2 = a^2$,
By transposing s^2, gives $2y^2 - 2sy = a^2 - s^2$
By dividing by 2, gives $y^2 - sy = \frac{1}{2}a^2 - \frac{1}{2}s^2$,
By completing the square, gives $y^2 - sy + \frac{1}{4}s^2 = \frac{1}{2}a^2 - \frac{1}{4}s^2$,
By extracting the root, gives $y - \frac{1}{2}s = \sqrt{\frac{1}{2}a^2 - \frac{1}{4}s^2}$
By transposing $\frac{1}{2}s$, gives $y = \frac{1}{2}s \pm \sqrt{\frac{1}{2}a^2 - \frac{1}{4}s^2} =$
4 and 3, the values of x and y.

PROBLEM III.

In a rectangle, having given the diagonal (10), and the perimeter, or sum of all the four sides (28); to find each of the sides severally.

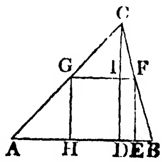
Let ABCD be the proposed rectangle; and put the diagonal AC $= 10 = d$, and half the perimeter AB $+$ BC or AD $+$ DC $= 14 = a$; also put one side AB $= x$, and the other BC $= y$.

Hence, by right-angled triangles,	$x^2 + y^2 = d^2$,
And by the question	$x + y = a$,
Then by transposing y in the 2d, gives	$x = a - y$,
This value substituted in the 1st, gives	$a^2 - 2ay + 2y^2 = d^2$,
Transposing a^2, gives	$2y^2 - 2ay = d^2 - a^2$,
And dividing by 2, gives	$y^2 - ay = \frac{1}{2}d^2 - \frac{1}{2}a^2$,
By completing the square, it is	$y^2 - ay + \frac{1}{4}a^2 = \frac{1}{2}d^2 - \frac{1}{4}a^2$,
And extracting the root, gives	$y - \frac{1}{2}a = \sqrt{\frac{1}{2}d^2 - \frac{1}{4}a^2}$,
And transposing $\frac{1}{2}a$, gives	$y = \frac{1}{2}a \pm \sqrt{\frac{1}{2}d^2 - \frac{1}{4}a^2} = 8$,

or 6, the values of x and y.

PROBLEM IV.

Having given the base and perpendicular of any triangle; to find the side of a square inscribed in the same.

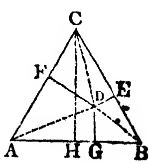


Let ABC represent the given triangle, and EFGH its inscribed square. Put the base AB $= b$, the perpendicular CD $= a$, and the side of the square GF or GH $=$ DI $= x$; then will CI $=$ CD $-$ DI $= a - x$.

Then, because the like lines in the similar triangles ABC, GFC, are proportional (by theorem 84, Geom.), AB : CD :: GF : CI, that is, $b : a :: x : a - x$. Hence, by multiplying extremes and means, $ab - bx = ax$, and transposing bx, gives $ab = ax + bx$; then dividing by $a + b$, gives $x = \frac{ab}{a+b} =$ GF or GH, the side of the inscribed square; which therefore is of the same magnitude, whatever the species or the angles of the triangles may be.

PROBLEM V.

In an equilateral triangle, having given the lengths of the three perpendiculars, drawn from a certain point within, on the three sides; to determine the sides.



Let ABC represent the equilateral triangle, and DE, DF, and DG, the given perpendiculars from the point D. Draw the lines DA, DB, DC, to the three angular points; and let fall the perpendicular CH on the base AB. Put the three given perpendiculars, DE $= a$, DF $= b$, DG $= c$, and put $x =$ AH or BH, half the side of the equilateral triangle. Then is AC or BC $= 2x$, and by right-angled triangles the perpendicular CH $=$ $\sqrt{AC^2 - AH^2} = \sqrt{4x^2 - x^2} = \sqrt{3x^2} = x\sqrt{3}$

Now, since the area or space of a rectangle, is expressed by the product of the base and height (cor. 2, th. 81, Geom.), and since a triangle is equal to half a rectangle of equal base and height (cor. 1, th. 26), it follows that,

the whole triangle	ABC is	$= \frac{1}{2}AB \times CH = x \times a\sqrt{3} = x^2\sqrt{3}$,
the triangle	ABD	$= \frac{1}{2}AB \times DG = x \times c = cx$,
the triangle	BCD	$= \frac{1}{2}BC \times DE = x \times a = ax$,
the triangle	ACD	$= \frac{1}{2}AC \times DF = x \times b = bx$.

But the three last triangles make up, or are equal to, the whole former or great triangle;

that is, $x^2\sqrt{3} = ax + bx + cx$: hence, dividing by x, gives

$x\sqrt{3} = a + b + c$, and dividing by $\sqrt{3}$, gives

$x = \frac{a + b + c}{\sqrt{3}}$, half the side of the triangle sought.

Also, since the whole perpendicular CH is $= x\sqrt{3}$, it is therefore $= a + b + c$. That is, the whole perpendicular CH, is just equal to the sum of all the three smaller perpendiculars DE + DF + DG taken together, wherever the point D is situated.

PROBLEM VI.

In a right-angled triangle, having given the base (3), and the difference between the hypothenuse and perpendicular (1); to find both these two sides.

PROBLEM VII.

In a right-angled triangle, having given the hypothenuse (5), and the difference between the base and perpendicular (1); to determine both these two sides.

PROBLEM VIII.

Having given the area, or measure of the space, of a rectangle, inscribed in a given triangle; to determine the sides of the rectangle.

PROBLEM IX.

In a triangle, having given the ratio of the two sides, together with both the segments of the base, made by a perpendicular from the vertical angle; to determine the sides of the triangle.

PROBLEM X.

In a triangle, having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base; to find the sides of the triangle.

PROBLEM XI.

In a triangle, having given the two sides about the vertical angle, with the line bisecting that angle, and terminating in the base; to find the base.

PROBLEM XII.

To determine a right-angled triangle; having given the lengths of two lines drawn from the acute angles, to the middle of the opposite sides.

PROBLEM XIII.

To determine a right angled-triangle; having given the perimeter, and the radius of its inscribed circle.

PROBLEM XIV.

To determine a triangle; having given the base, the perpendicular, and the ratio of the two sides.

PROBLEM XV.

To determine a right-angled triangle; having given the hypothenuse, and the side of the inscribed square.

PROBLEM XVI.

To determine the radii of three equal circles, described in a given circle, to touch each other and also the circumference of the given circle.

PROBLEM XVII.

In a right-angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle on the hypothenuse; to determine the triangle, that is, its sides.

PROBLEM XVIII.

To determine a right-angled triangle; having given the hypothenuse, and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.

PROBLEM XIX.

To determine a triangle; having given the base, the perpendicular, and the difference of the two other sides.

PROBLEM XX.

To determine a triangle; having given the base, the perpendicular, and the rectangle or product of the two sides.

PROBLEM XXI.

To determine a triangle; having given the lengths of three lines drawn from the three angles, to the middle of the opposite sides.

PROBLEM XXII.

In a triangle, having given all the three sides; to find the radius of the inscribed circle.

PROBLEM XXIII.

To determine a right-angled triangle; having given the side of the inscribed square, and the radius of the inscribed circle.

PROBLEM XXIV.

To determine a triangle, and the radius of the inscribed circle; having given the lengths of three lines drawn from the three angles, to the centre of that circle.

PROBLEM XXV.

To determine a right-angled triangle; having given the hypothenuse, and the radius of the inscribed circle.

PROBLEM XXVI.

To determine a triangle; having given the base, the line bisecting the vertical angle, and the diameter of the circumscribing circle.

PROBLEMS ON MAXIMA AND MINIMA.

TO BE SOLVED GEOMETRICALLY.

1. Divide a right line into two parts so that their rectangle shall be a maximum.

2. Find a point in a given straight line, from which if two straight lines be drawn to two given points on the same side of the given line, and in the same plane with it, their sum shall be a maximum.

3. Let ABC be a right-angled triangle of which AB is the hypothenuse. Draw through the angular point, C, a right line such, that the sum of two perpendiculars let fall upon it from A and B, respectively, shall be a minimum.

4. Through a given point within a circle, which is not the centre, to draw the least chord.

5. Through either of the points of intersection of two given circles that cut each other, to draw the greatest of all straight lines, passing through that point, and terminated both ways by the two circumferences.

6. Two semicircles whose radii are in a known ratio, lie on contrary sides of the same right line, the circumference of one terminating in the centre of the other. Draw the greatest right line perpendicular to the common diametral line, and terminated both ways by the two curves.

7. Through a given point in a given circle, out of the centre, draw a chord which shall cut off the least segment.

8. To find a point in the circumference of a given circle, at which any given straight line drawn from the centre, but less than the radius of the circle, shall subtend the greatest angle.

9. Given the base and the ratio of the sides, to determine the triangle when its area is a maximum.

10. In a given triangle to inscribe the greatest rectangle.

11. To divide a given right line into two parts, such that the sum of the squares of the two parts may be a minimum.

12. In a given plane triangle to inscribe another, having its angular points in the three sides of the given one, and its perimeter a minimum.

13. Given the hypothenuse of a right-angled triangle, to construct it when the sum of one leg and the diameter of the inscribed circle is a maximum.

PLANE TRIGONOMETRY.

DEFINITIONS.

1 PLANE TRIGONOMETRY treats of the relations and calculations of the sides and angles of plane triangles.

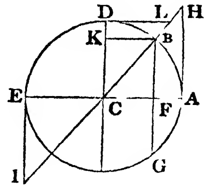
2. The circumference of every circle (as before observed in Geom. def. 56) is supposed to be divided into 360 equal parts, called Degrees; also each degree into 60 Minutes, each minute into 60 seconds, and so on.

Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

3. The Measure of any angle (def. 57, Geom.) is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc.

Hence, a right angle being measured by a quadrant, or quarter of the circle, is an angle of 90 degrees; and the sum of the three angles of every triangle, or two right angles, is equal to 180 degrees. Therefore, in a right-angled triangle, taking one of the acute angles from 90 degrees, leaves the other acute angle; and the sum of two angles, in any triangle, taken from 180 degrees, leaves the third angle; or one angle being taken from 180 degrees, leaves the sum of the other two angles.

4. Degrees are marked at the top of the figure with a small °, minutes with ′, seconds with ″, and so on. Thus, 57° 30′ 12″, denote 57 degrees 30 minutes and 12 seconds.



5. The Complement of an arc, is what it wants of a quadrant or 90°. Thus, if AD be a quadrant, then BD is the complement of the arc AB; and, reciprocally, AB is the complement of BD. So that, if AB be an arc of 50°, then its complement BD will be 40°.

6. The Supplement of an arc, is what it wants of a semicircle, or 180° Thus, if ADE be a semicircle, then BDE is the supplement of the arc AB; and, reciprocally, AB is the supplement of the arc BDE. So that, if AB be an arc of 50°, then its supplement BDE will be 130°.

7. The Sine, or Right Sine, of an arc, is the line drawn from one extremity of the arc, perpendicular to the diameter passing through the other extremity. Thus, BF is the sine of the arc AB, or of the arc BDE.

Corol. Hence the sine (BF) is half the chord (BG) of the double arc (BAG).

8. The Versed Sine of an arc, is the part of the diameter intercepted between the arc and its sine. So, AF is the versed sine of the arc AB, and EF the versed sine of the arc EDB.

9. The Tangent of an arc, is a line touching the circle in one extremity of that arc, continued from thence to meet a line drawn from the centre through the other extremity: which last line is called the Secant of the same arc. Thus, AH is the tangent, and CH the secant, of the arc AB. Also, EI is the tangent, and CI the secant, of the supplemental arc BDE. And this latter tan-

gent and secant are equal to the former, but are accounted negative, as being drawn in an opposite or contrary direction to the former.

10. The Cosine, Cotangent, and Cosecant, of an arc, are the sine, tangent, and secant of the complement of that arc, the Co being only a contraction of the word complement. Thus, the arcs AB, BD being the complements of each other, the sine, tangent or secant of the one of these, is the cosine, cotangent or cosecant of the other. So, BF, the sine of AB, is the cosine of BD; and BK, the sine of BD, is the cosine of AB: in like manner, AH, the tangent of AB, is the cotangent of BD; and DL, the tangent of DB, is the cotangent of AB. also, CH, the secant of AB, is the cosecant of BD; and CL, the secant of BD, is the cosecant of AB.

Corol. Hence several remarkable properties easily follow from these definitions; as,

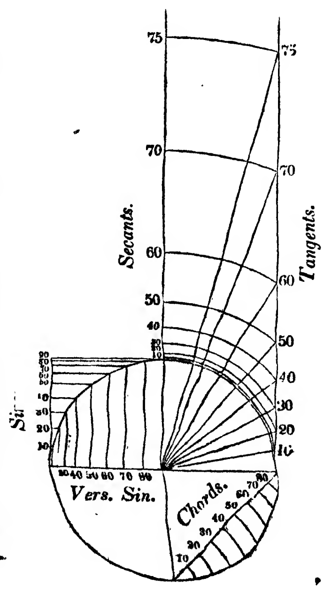
1st, That an arc and its supplement have the same sine, tangent, and secant; but the two latter, the tangent and secant, are accounted negative when the arc is greater than a quadrant or 90 degrees.

2d, When the arc is 0, or nothing, the sine and tangent are nothing, but the secant is then the radius CA.—But when the arc is a quadrant AD, then the sine is the greatest it can be, being the radius CD of the circle; and both the tangent and secant are infinite.

3d, Of any arc AB, the versed sine AF, and cosine BK, or CF, together make up the radius CA of the circle.—The radius CA, tangent AH, and secant CH, form a right-angled triangle CAH. So also do the radius, sine, and cosine, form another right-angled triangle CBF or CBK. As also the radius, cotangent, and cosecant, another right-angled triangle CDL. And all these right-angled triangles are similar to each other.

11. The sine, tangent, or secant of an angle, is the sine, tangent, or secant of the arc by which the angle is measured, or of the degrees, &c. in the same arc or angle.

12. The method of constructing the scales of chords, sines, tangents, and secants, usually engraven on instruments, for practice, is exhibited in the annexed figure.



13. A Trigonometrical Canon, is a table exhibiting the length of the sine, tangent, and secant, to every degree and minute of the quadrant, with respect to the radius, which is expressed by unity, or 1, and conceived to be divided into 10000000 or more decimal parts. And farther, the logarithms of these sines, tangents, and secants are also ranged in the tables; which are most commonly used, as they perform the calculations by only addition and subtraction, instead of the multiplication and division by the natural sines, &c. according to the nature of logarithms.

Upon this table depends the numeral solution of the several cases in trigono-

metry. It will therefore be proper to begin with the mode of constructing it, which may be done in the following manner:

PROBLEM I.

To find the sine and cosine of a given arc.

This problem is resolved after various ways. One of these is as follows, viz. by means of the ratio between the diameter and circumference of a circle, together with the known series for the sine and cosine, hereafter demonstrated. Thus, the semicircumference of the circle, whose radius is 1, being 3·141592653589793, &c., the proportion will therefore be,

as the number of degrees or minutes in the semicircle,
is to the degrees or minutes in the proposed arc,
so is 3.14159265, &c. to the length of the said arc.

This length of the arc being denoted by the letter a; also its sine and cosine by s and c; then will these two be expressed by the two following series, viz.

$$s = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7} + \&c.$$
$$= a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040} + \&c.$$
$$c = 1 - \frac{a^2}{2} + \frac{a^4}{2.3.4.} - \frac{a^6}{2.3.4.5.6.} + \&c.$$
$$= 1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720} + \&c.$$

EXAMPLE I.—If it be required to find the sine and cosine of one minute. Then, the number of minutes in 180° being 10800, it will be first, as 10800 : 1 :: 3·14159265, &c. : ·000290888208665 = the length of an arc of one minute. Therefore, in this case,

$$\begin{aligned} a &= \cdot 0002908882 \\ \text{and } \tfrac{1}{6}a^3 &= \cdot 000000000004, \&c. \\ \text{the difference is } s &= \cdot 0002908882 \text{ the sine of 1 minute.} \\ \text{Also, from } & \ 1\cdot \\ \text{take } \tfrac{1}{2}a^2 &= 0\cdot 0000000423079, \&c. \\ \text{leaves } c &= \cdot 9999999577 \text{ the cosine of 1 minute.} \end{aligned}$$

EXAMPLE II.—For the sine and cosine of 5 degrees.

Here, as 180° : 5° :: 3·14159265, &c. : ·08726646 = a the length of 5 degrees.

$$\begin{aligned} \text{Hence, } \quad a &= \cdot 08726646 \\ -\tfrac{1}{6}a^3 &= -\cdot 00011076 \\ +\tfrac{1}{120}a^5 &= \cdot 00000004 \\ \text{these collected, give } s &= \cdot 08715574 \text{ the sine of } 5^\circ. \end{aligned}$$

$$\begin{aligned} \text{And, for the cosine, } \quad 1 &= 1\cdot \\ -\tfrac{1}{2}a^2 &= -\cdot 00380771 \\ +\tfrac{1}{24}a^4 &= \cdot 00000241 \\ \text{these collected, give } c &= \cdot 99619470 \text{ the cosine of } 5^\circ. \end{aligned}$$

After the same manner, the sine and cosine of any other arc may be computed. But the greater the arc is, the slower the series will converge, in which case a greater number of terms must be taken to bring out the conclusion to the same degree of exactness.

Or, having found the sine, the cosine will be found from it, by the property of the right-angled triangle CBF, viz. the cosine $CF = \sqrt{CB^2 - BF^2}$, or $c = \sqrt{1 - s^2}$.

There are also other methods of constructing the canon of sines and cosines, which, for brevity's sake, are here omitted.

PROBLEM II.

To compute the tangents and secants.

The sines and cosines being known, or found by the foregoing problem; the tangents and secants will be easily found, from the principle of similar triangles, in the following manner:—

In the first figure, where, of the arc AB, BF is the sine, CF or BK the cosine, AH the tangent, CH the secant, DL the cotangent, and CL the cosecant, the radius being CA, or CB, or CD; the three similar triangles CFB, CAH, CDL, give the following proportions:

1*st*, CF : FB :: CA : AH; whence the tangent is known, being a fourth proportional to the cosine, sine, and radius.

2*d*, CF : CB :: CA : CH; whence the secant is known, being a third proportional to the cosine and radius.

3*d*, BF : FC :: CD : DL; whence the cotangent is known, being a fourth proportional to the sine, cosine, and radius.

4*th*, BF : BC :: CD : CL; whence the cosecant is known, being a third proportional to the sine and radius.

Having given an idea of the calculation of sines, tangents, and secants, we may now proceed to resolve the several cases of Trigonometry; previous to which, however, it may be proper to add a few preparatory notes and observations, as below.

Note 1.—There are usually three methods of resolving triangles, or the cases of trigonometry; namely, Geometrical Construction, Arithmetical Computation, and Instrumental Operation.

In the First Method.—The triangle is constructed by making the parts of the given magnitudes, namely, the sides from a scale of equal parts, and the angles from a scale of chords, or by some other instrument. Then, measuring the unknown parts, by the same scales or instruments, the solution will be obtained near the truth.

In the Second Method.—Having stated the terms of the proportion according to the proper rule or theorem, resolve it like any other proportion, in which a fourth term is to be found from three given terms, by multiplying the second and third together, and dividing the product by the first, in working with the natural numbers; or, in working with the logarithms, add the logs. of the second and third terms together, and from the sum take the log. of the first term; then the natural number answering to the remainder is the fourth term sought.

In the Third Method.—Or Instrumentally, as suppose by the log. lines on one side of the common two foot scales; Extend the compasses from the first term, to the second or third, which happens to be of the same kind with it; then that extent will reach from the other term to the fourth term, as required, taking both extents towards the same end of the scale.

Note 2.—In every triangle, or case in trigonometry, there must be given three parts, to find the other three. And, of the three parts that are given, one of

them at least must be a side; because the same angles are common to an infinite number of triangles.

Note 3.—All the cases in trigonometry may be comprised in three varieties only; viz.

1st, When a side and its opposite angle are given.

2d, When two sides and the contained angle are given.

3d, When the three sides are given.

For there cannot possibly be more than these three varieties of cases; for each of which it will therefore be proper to give a separate theorem, as follows:

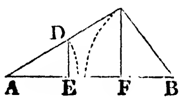
THEOREM I.

When a side and its opposite angle are two of the given parts.

Then the sides of the triangle have the same proportion to each other, as the sines of their opposite angles have.

That is, As any one side,

Is to the sine of its opposite angle;
So is any other side,
To the sine of its opposite angle.

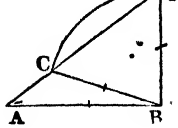


Demonstr.—For, let ABC be the proposed triangle, having AB the greatest side, and BC the least. Take AD = BC, considering it as a radius; and let fall the perpendiculars DE, CF, which will evidently be the sines of the angles A and B, to the radius AD or BC. But the triangles ADE, ACF, are equiangular, and therefore AC : CF :: AD or BC : DE; that is, AC is to the sine of its opposite angle B, as BC to the sine of its opposite angle A.

Note 1.—In practice, to find an angle, begin the proportion with a side opposite a given angle. And to find a side, begin with an angle opposite a given side.

Note 2.—An angle found by this rule is ambiguous, or uncertain whether it be acute or obtuse, unless it be a right angle, or unless its magnitude be such as to prevent the ambiguity; because the sine answers to two angles, which are supplements to each other; and accordingly the geometrical construction forms two triangles with the same parts that are given, as in the example below; and when there is no restriction or limitation included in the question, either of them may be taken. The degrees in the table, answering to the sine, is the acute angle; but if the angle be obtuse, subtract those degrees from 180°, and the remainder will be the obtuse angle. When a given angle is obtuse, or a right one, there can be no ambiguity; for then neither of the other angles can be obtuse, and the geometrical construction will form only one triangle.

EXAMPLE I.



In the plane triangle ABC,

Given, $\begin{cases} \text{AB } 345 \text{ yards} \\ \text{BC } 232 \text{ yards} \\ \text{angle A } 37^\circ\ 20' \end{cases}$

Required the other parts.

1. *Geometrically.*

Draw an indefinite line, upon which set off AB = 345, from some convenient scale of equal parts.—Make the angle A = 37°⅓.—With a radius of 232, taken from the same scale of equal parts, and centre B, cross AC in the two points C, C.—Lastly, join BC, BC, and the figure is constructed, which gives two triangles, showing that the case is ambiguous.

Then, the sides AC measured by the scale of equal parts, and the angles B and C measured by the line of chords, or other instrument, will be found to be nearly as below; viz.

AC	174	angle B	27°	angle C	115°½
or	374½	or	78¼	or	64 ½

2. *Arithmetically.*

First, to find the angles at C:

As side	BC 232	log.	2·3654880
To sin. opp. angle	A 37° 20′		9·7827958
So side	AB 345		2·5378191
To sin. opp. angle	C 115° 36′ or 64° 24′		9·9551269
Add angle	A 37 20 — 37 20		
The sum	152 56 or 101 44		
Taken from	180 00 — 180 00		
Leaves angle B	27 04 or 78 16		

Then, to find the side AC:

As sine angle A	37° 20′	log.	9·7827958
To opposite side BC	232		2·3654880
So sine angle B	27° 04′		9·6580371
	78 16		9·9908291
To opposite side AC	174·07		2·2407293
or,	374·56		2·5735213

3. *Instrumentally.*

In the first proportion.—Extend the compasses from 232 to 345 upon the line of numbers; then that extent will reach, on the sines, from 37°⅓ to 64°½, the angle C.

In the second proportion.—Extend the compasses from 37°⅓ to 27° or 78°¼, on the sines; then that extent will reach, on the line of numbers, from 232 to 174 or 374½, the two values of the side AC.

EXAMPLE II.

In the plane triangle ABC,

Given, AB 365 poles; angle A 57° 12′; angle B 24 45

Required the other parts.

Ans. angle C 98° 3′; AC 154·33; BC 309·86

EXAMPLE III.

In the plane triangle ABC,

Given, { AC 120 feet, BC 112 feet, angle A 57° 27′ }

Required the other parts.

Ans. { angle B 64° 34′ 21″, or, 115 25 39; angle C 57 58 39, or, 7 7 21; AB 112·65 feet, or, 16·47 feet }

THEOREM II.

When two sides and their contained angle are given.

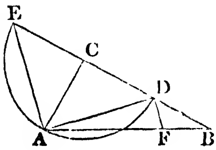
Then it will be,

As the sum of those two sides,
Is to the difference of the same sides;
So is the tang. of half the sum of their opposite angles,
To the tang. of half the difference of the same angles.

Hence, because it has been shown under Algebra, that the half sum of any two quantities increased by their half difference, gives the greater, and diminished by it gives the less, if the half difference of the angles, so found, be added to their half sum, it will give the greater angle, and subtracting it will leave the less angle.

Then, all the angles being now known, the unknown side will be found by the former theorem.

Demonstr.—Let ABC be the proposed triangle, having the two given sides AC, BC, including the given angle C. With the centre C, and radius CA, the less of these two sides, describe a semicircle, meeting the other side BC produced in D and E. Join AE, AD, and draw DF parallel to AE.



Then, BE is the sum, and BD the difference of the two given sides CB, CA. Also, the sum of the two angles CAB, CBA, is equal to the sum of the two CAD, CDA, these sums being each the supplement of the vertical angle C to two right angles: but the two latter CAD, CDA, are equal to each other, being opposite to the two equal sides CA, CD: hence, either of them, as CDA, is equal to half the sum of the two unknown angles CAB, CBA. Again, the exterior angle CDA is equal to the two interior angles B and DAB; therefore the angle DAB is equal to the difference between CDA and B, or between CAD and B; consequently the same angle DAB is equal to half the difference of the unknown angles B and CAB; of which it has been shown that CDA is the half sum.

Now the angle DAE, in a semicircle, is a right angle, or AE is perpendicular to AD; and DF, parallel to AE, is also perpendicular to AD: consequently, AE is the tangent of CDA the half sum, and DF the tangent of DAB the half difference of the angles, to the same radius AD, by the definition of a tangent. But, the tangents AE, DF, being parallel, it will be as BE : BD :: AE : DF; that is, as the sum of the sides is to the difference of the sides, so is the tangent of half the sum of the opposite angles, to the tangent of half their difference.

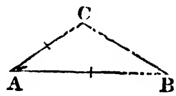
Note.—The sum of the unknown angles is found, by taking the given angle from 180°.

EXAMPLE I.

In the plane triangle ABC,

Given, { AB 345 yards; AC 174·07 yards; angle A 37° 20′ }

Required the other parts.



1. *Geometrically.*

Draw AB = 345 from a scale of equal parts. Make the angle A = 37° 20 Set off AC = 174 by the scale of equal parts. Join BC, and it is done.

Then the other parts being measured, they are found to be nearly as follows; viz. the side BC 232 yards, the angle B 27°, and the angle C 115°½.

2. *Arithmetically.*

As sum of sides AB, AC, ...	519·07	log.	2·7152259
To difference of sides AB, AC, ...	170·93	...	2·2328183
So tangent half sum angles C and B, ...	71° 20′	...	10·4712979
To tangent half difference angles C and B	44 16	...	9·9888903
Their sum gives angle C	115 36		
Their diff. gives angle B	27 4		

Then, by the former theorem,

As sine angle C 115° 36′ or 64° 24′	log.	9·9551259
To its opposite side AB 345		2·5378191
So sine angle A 37° 20′		9·7827958
To its opposite side BC 232		2·3654890

3. *Instrumentally.*

In the first proportion.—Extend the compasses from 519 to 171, on the line of numbers; and that extent will reach, on the tangents, from 71°⅓ (the contrary way, because the tangents are set back again from 45°), a little beyond 45, which being set so far back from 45, falls upon 44°¼, the fourth term.

In the second proportion.—Extend from 64°¼ to 37°⅓, on the sines; and that extent will reach, on the numbers, from 345 to 232, the fourth term sought.

EXAMPLE II.

In the plane triangle ABC,

Given, { AB 365 poles; AC 154·33; angle A 57° 12′ }

Required the other parts.

Ans. { BC 309·86; angle B 24° 45′; angle C 98° 3′ }

EXAMPLE III.

In the plane triangle ABC,

Given, { AC 120 yards; BC 112 yards; angle C 57° 58′ 39″ }

Required the other parts.

Ans. { AB 112·65; angle A 57° 27′ 0; angle B 64 34 21 }

THEOREM III.

When the three sides of the triangle are given.

Then, having let fall a perpendicular from the greatest angle upon the opposite side, or base, dividing it into two segments, and the whole triangle into two right-angled triangles; it will be,

As the base, or sum of the segments,
Is to the sum of the other two sides;
So is the difference of those sides,
To the difference of the segments of the base.

Then half the difference of the segments being added to the half sum, or the half base, gives the greater segment; and the same subtracted gives the less segment.

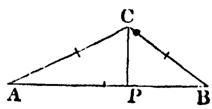
Hence, in each of the two right-angled triangles, there will be known two sides, and the angle opposite to one of them; consequently, the other angles will be found by the first problem.

Demonstr.—By Cor. to Theorem 35, Geometry, the rectangle under the sum and difference of the two sides, is equal to the rectangle under the sum and difference of the two segments. Therefore, by forming the sides of these rectangles into a proportion, it will appear that the sums and differences are proportional as in this theorem, by Theor. 76, Geometry.

EXAMPLE I.

In the plane triangle ABC,

Given, the sides $\begin{cases} \text{AB} & 345 \text{ yards} \\ \text{AC} & 232 \\ \text{BC} & 174{\cdot}07 \end{cases}$



To find the angles.

1. *Geometrically.*

Draw the base AB = 345 by a scale of equal parts. With radius 232, and centre A, describe an arc; and with radius 174, and centre B, describe another arc, cutting the former in C. Join AC, BC, and it is done.

Then, by measuring the angles, they will be found to be nearly as follow; viz. angle A 27°, angle B 37°¼, and angle C 115°¾.

2. *Arithmetically.*

Having let fall the perpendicular CP, it will be,

As the base AB : AC + BC :: AC — BC : AP — BP,
that is, as 345 : 406·07 :: 57·93 : 68·18 = AP — BP,

its half is	34·09	
the half base is..................	172·50	
the sum of these is	206·59	= AP
and their difference	138·41	= BP

Then, in the triangle APC, right-angled at P,

As the side AC	232	....	log. 2·3654880
To sine opposite angle	90°	....	10·0000000
So is side AP	206·59	....	2·3151093
To sine opposite angle ACP	62° 56′	....	9·9496213
Which taken from	90 00		
Leaves the angle A	27 04		

Again, in the triangle BPC, right-angled at P,

As the side BC	174·07	log.	2·2407239
To sine opposite angle P	90°		10·0000000
So is side BP....................	138·41		2·1411675
To sin. opposite angle BCP ...	52° 40′		9·9004436
Which taken from.........	90 00		
Leaves the angle B	37 20		
Also, the angle ACP	62° 56′		
Added to angle BCP	52 40		
Gives the whole angle ACB	115 36		

So that all the three angles are as follow; viz.

the angle A 27° 4′; the angle B 37° 20′; the angle C 115° 36′.

3. *Instrumentally.*

In the first proportion.—Extend the compasses from 345 to 406, on the line of numbers; then that extent will reach, on the same line, from 58 to 68·2 nearly, which is the difference of the segments of the base.

In the second proportion.—Extend from 232 to 206½, on the line of numbers; then that extent will reach, on the sines, from 90° to 63°.

In the third proportion.—Extend from 174 to 138½; then that extent will reach from 90° to 52°⅔ on the sines.

EXAMPLE II.

In the plane triangle ABC,

Given the sides, $\begin{cases} \text{AB } 365 \text{ poles} \\ \text{AC } 154{\cdot}33 \\ \text{BC } 309{\cdot}86 \end{cases}$

To find the angles. Ans. $\begin{cases} \text{angle A } 57^\circ\ 12' \\ \text{angle B } 24\ \ 45 \\ \text{angle C } 98\ \ \ 3 \end{cases}$

In the plane triangle ABC,

Given the sides, $\begin{cases} \text{AB } 120 \\ \text{AC } 112{\cdot}65 \\ \text{BC } 112 \end{cases}$

To find the angles. Ans. $\begin{cases} \text{angle A } 57^\circ\ 27'\ 00'' \\ \text{angle B } 57\ \ 58\ \ 39 \\ \text{angle C } 64\ \ 34\ \ 21 \end{cases}$

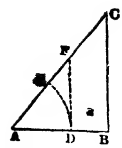
The three foregoing theorems include all the cases of plane triangles, both right-angled and oblique; besides which, there are other theorems suited to some particular forms of triangles, which are sometimes more expeditious in their use than the general ones; one of which, as the case for which it serves so frequently occurs, may be here taken, as follows:

THEOREM IV.

When, in a right-angled triangle, there are given one leg and the angles; to find the other leg or the hypothenuse; it will be,

As radius, *i. e.* sine of 90° or tangent of 45°
Is to the given leg,
So is the tangent of its adjacent angle
To the other leg;
And so is the secant of the same angle
To the hypothenuse.

Demonstr. AB being the given leg, in the right-angled triangle ABC; with the centre A, and any assumed radius AD, describe an arc DE, and draw DF perpendicular to AB, or parallel to BC. Now it is evident, from the definitions, that DF is the tangent, and AF the secant, of the arc DE, or of the angle A which is measured by that arc, to the radius AD. Then, because of the parallels BC, DF, it will be as AD : AB :: DF : BC :: AF : AC, which is the same as the theorem is in words.



EXAMPLE I.

In the right-angled triangle ABC,

Given $\left\{ \begin{array}{l} \text{the leg AB } 162 \\ \text{angle A } 53^\circ\, 7'\, 48'' \end{array} \right\}$ To find AC and BC.

1. *Geometrically.*

Make AB = 162 equal parts, and the angle A = $53^\circ\, 7'\, 48''$; then raise the perpendicular BC, meeting AC in C. So shall AC measure 270, and BC 216.

2. *Arithmetically.*

As radius	tang. 45°	....	log. 10·0000000
To leg AB	162	....	2·2095150
So tang. angle A	$53^\circ\, 7'\, 48''$	....	10·1249371
To leg BC	216	....	2·3344521
So secant angle A	$53^\circ\, 7'\, 48''$	....	10·2218477
To hyp. AC	270	....	2·4313627

3. *Instrumentally.*

Extend the compasses from 45° to $53^\circ\frac{1}{8}$, on the tangents. Then that extent will reach from 162 to 216 on the line of numbers.

EXAMPLE II.

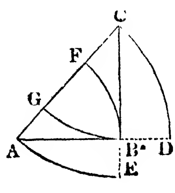
In the right-angled triangle ABC,

Given $\left\{ \begin{array}{l} \text{the leg AB } 180 \\ \text{the angle A } 62^\circ\, 40' \end{array} \right.$

To find the other two sides. Ans. $\left\{ \begin{array}{l} \text{AC } 392\cdot0147 \\ \text{BC } 348\cdot2464 \end{array} \right.$

Note. There is sometimes given another method for right-angled triangles, which is this:

ABC being such a triangle, make one leg AB radius, that is, with centre A, and distance AB, describe an arc BF. Then it is evident that the other leg BC represents the tangent, and the hypothenuse AC the secant, of the arc BF, or of the angle A.



In like manner, if the leg BC be made radius; then the other leg AB will represent the tangent, and the hypothenuse AC the secant, of the arc BG or angle C.

But if the hypothenuse be made radius; then each leg will represent the sine of its opposite angle; namely, the leg AB the sine of the arc AE or angle C, and the leg BC the sine of the arc CD or angle A.

And then the general rule for all these cases, is this, namely, that the sides of the triangle bear to each other the same proportion as the parts which they represent.

And this is called, Making every side radius.

OF HEIGHTS AND DISTANCES, &c.

By the mensuration and protraction of lines and angles, are determined the lengths, heights, depths, and distances of bodies or objects.

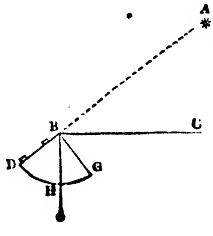
Accessible lines are measured by applying to them some certain measure a number of times, as an inch, or foot, or yard. But inaccessible lines must be measured by taking angles, or by some such method, drawn from the principles of geometry.

When instruments are used for taking the magnitude of the angles in degrees, the lines are then calculated by trigonometry: in the other methods, the lines are calculated from the principle of similar triangles, without regard to the measure of the angles.

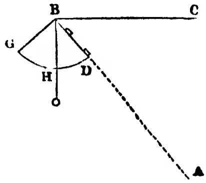
Angles of elevation, or of depression, are usually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furnished with a plummet suspended from the centre, and two sides fixed on one of the radii, or else with telescopic sights.

To take an angle of altitude and depression with the quadrant.

Let A be any object, as the sun, moon, or a star, or the top of a tower, or hill, or other eminence: and let it be required to find the measure of the angle ABC, which a line drawn from the object makes with the horizontal line BC.



Fix the centre of the quadrant in the angular point, and move it round there as a centre, till with one eye at D, the other being shut, you perceive the object A through the sights: then will the arc GH of the quadrant, cut off by the plumb line BH, be the measure of the angle ABC as required.



The angle ABC of depression of any object A, is taken in the same manner; except that here the eye is applied to the centre, and the measure of the angle is the arc GH, on the other side of the plumb line.

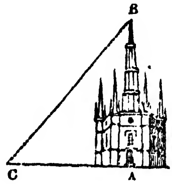
The following examples are to be constructed and calculated by the foregoing methods, treated of in Trigonometry.

EXAMPLE I.

Having measured a distance of 200 feet, in a direct horizontal line, from the bottom of a steeple, the angle of elevation of its top, taken at that distance, was found to be 47° 30′: from hence it is required to find the height of the steeple.

Construction.

Draw an indefinite line, upon which set off AC = 200 equal parts, for the measured distance. Erect the indefinite perpendicular AB; and draw CB so as to make the angle C = 47° 30′, the angle of elevation; and it is done. Then AB, measured on the scale of equal parts, is nearly 218¼.



Calculation.

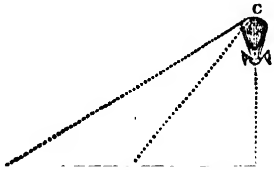
As radius	10·0000000
To AC 200	2·3010300
So tang. angle C 47° 30	10·0379475
To AB 218·26 required	2·3389775

EXAMPLE II.

What was the perpendicular height of a cloud, or of a balloon, when its angles of elevation were 35° and 64°, as taken by two observers, at the same time, both on the same side of it, and in the same vertical plane; their distance as under being half a mile or 880 yards. And what was its distance from the said two observers?

Construction.

Draw an indefinite ground line, upon which set off the given distance AB = 880; then A and B are the places of the observers. Make the angle A = 35°, and the angle B = 64°; and the intersection of the lines at C will be the place of the balloon; from whence the perpendicular CD, being let fall, will be its perpendicular height. Then, by measurement, are found the distances and height nearly as follows; viz. AC 1631, BC 1041, DC 936.



Calculation.

First, from angle B 64°
Take angle A 35
Leaves angle ACB 29

Then, in the triangle ABC,

As sine	angle ACB	29°	....	9·6855712
To op. side	AB	880	....	2·9444827
So sine	angle A	35°	....	9·7585913
To opposite side	BC	1041·125	....	3·0175028
As sine	angle ACB	29°	....	9·6855712
To opposite side	AB	880	....	2·9444827
So sine	angle B	116° or 64°	....	9·9536602
To opposite side	AC	1631·442	....	3·2125717

And, in the triangle BCD,

As sine	angle D	90°	....	10·0000000
To opposite side	BC	1041·125	....	3·0175028
So sine	angle B	64°	....	9·9536602
To opposite side	CD	935·757	....	2·9711600

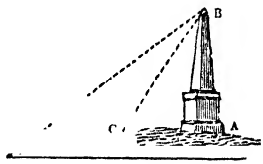
EXAMPLE III.

Having to find the height of an obelisk standing on the top of a declivity, I first measured from its bottom a distance of 40 feet, and there found the angle, formed by the oblique plane and a line imagined to go to the top of the obelisk, 41°; but, after measuring on in the same direction 60 feet further, the like angle was only 23° 45′. What then was the height of the obelisk ?

Construction.

Draw an indefinite line for the sloping plane or declivity, in which assume any point A for the bottom of the obelisk, from whence set off the distance AC = 40, and again CD = 60 equal parts. Then make the angle C = 41°, and the angle D = 23° 45′; and the point B, where the two lines meet, will be the top of the obelisk. Therefore AB, joined, will be its height.

Calculation.



From the angle C	41°	00′
Take the angle D	23	45
Leaves the angle DBC	17	15

Then, in the triangle DBC,

As sine	angle DBC	17° 15′	....	9·4720856
To opposite side	DC	60	....	1·7781513
So sine	angle D	23 45	....	9·6050320
To opposite side	CB	81·488	....	1·9110977

And, in the triangle ABC,

As sum of sides	CB, CA	121·488	....	2·0845333
To difference of sides	CB, CA	41·488	....	1·6179225
So tang. half sum angles	A, B	69° 30′	....	10·4272623
To tang. half diff. angles	A, B	42 24½	....	9.9606516
The diff. of these is angle CBA		27 5½		

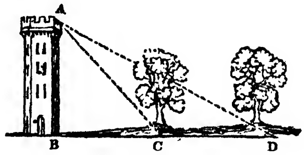
Lastly, as sine angle CBA		27° 5½′	....	9·6582842
To opposite side	CA	40	....	1·6020600
So sine	angle C	41° 0′	....	9.8169429
To opposite side	AB	57·623	....	1·7607187

EXAMPLE IV.

Wanting to know the distance between two inaccessible trees, or other objects, from the top of a tower, 120 feet high, which lay in the same right line with the two objects, I took the angles formed by the perpendicular wall and lines conceived to be drawn from the top of the tower to the bottom of each

tree, and found them to be 33° and 64°½. What then may be the distance between the two objects?

Construction.



Draw the indefinite ground line BD, and perpendicular to it BA = 120 equal parts. Then draw the two lines AC, AD, making the two angles BAC, BAD, equal to the given angles 33° and 64½°. So shall C and D be the places of the two objects.

Calculation.

First, In the right-angled triangle ABC,

As radius		10·0000000
To AB	120	2·0791812
So tang. angle BAC	33°	9·8125174
To BC	77·929	1·8916986

And, in the right-angled triangle ABD,

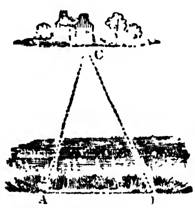
As radius		10·0000000
To AB	120	2·0791812
So tang. angle BAD	64°½	10·3215039
To BD	251·585	2·4006851
From which take BC	77·929	
Leaves the dist. CD	173·656 as required.	

EXAMPLE V.

Being on the side of a river, and wanting to know the distance to a house which was seen on the other side, I measured 200 yards in a straight line by the side of the river; and then at each end of this line of distance, took the horizontal angle formed between the house and the other end of the line; which angles were, the one of them 68° 2′, and the other 73° 15′. What then were the distances from each end to the house?

Construction.

Draw the line AB = 200 equal parts. Then draw AC so as to make the angle A = 68° 2′, and BC to make the angle B = 73° 15′. So shall the point C be the place of the house required.



Calculation.

To the given angle A	68°	2′
Add the given angle B	73	15
Then their sum	141	17
Being taken from	180	0
Leaves the third angle C	38	43

Hence,	As sin. angle C	38° 43′	9·7962062
	To op. side AB	200	2·3010300
	So sin. angle A	68° 2′	9·9672679
	To op. side BC	296·54	2·4720917
And,	As sin. angle C	38° 43′	9·7962062
	To op. side AB	200	2·3010300
	So sin. angle B	73° 15′	9·9811711
	To op. side AC	306·19	2·4859949

EXAM. VI.—From the edge of a ditch of 36 feet wide, surrounding a fort, having taken the angle of elevation of the top of the wall, it was found to be 62° 40′: required the height of the wall, and the length of a ladder to reach from my station to the top of it?

Ans. { height of wall 69·64, ladder 78·4 feet.

EXAM. VII.—Required the length of a shoar, which, being to strut 11 feet from the upright of a building, will support a jamb 23 feet 10 inches from the ground?

Ans. 26 feet 3 inches.

EXAM. VIII.—A ladder, 40 feet long, can be so planted, that it shall reach a window 33 feet from the ground, on one side of the street; and by turning it over, without moving the foot out of its place, it will do the same by a window 21 feet high, on the other side: required the breadth of the street?

Ans. 56·649 feet.

EXAM. IX.—A Maypole, whose top was broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole: what was the height of the whole maypole, supposing the broken piece to measure 39 feet in length?

Ans. 75 feet.

EXAM. X.—At 170 feet distance from the bottom of a tower, the angle of its elevation was found to be 52° 30′: required the altitude of the tower.

Ans. 221 feet.

EXAM. XI.—From the top of a tower, by the sea-side, of 143 feet height, it was observed that the angle of depression of a ship's bottom, then at anchor, measured 35°: what then was the ship's distance from the bottom of the wall?

Ans. 204·22 feet.

EXAM. XII.—What is the perpendicular height of a hill; its angle of elevation, taken at the bottom of it, being 46°, and 200 yards farther off, on a level with the bottom of it, the angle was 31°?

Ans. 286·28 yards.

EXAM. XIII.—Wanting to know the height of an inaccessible tower; at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°: required its height, and my distance from it at the first station?

Ans. { Height, 307·53; Distance, 192·15

EXAM. XIV.—Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inaccessible hill; I took the angle of elevation of the top of the hill equal 40°, and of the top of the tower equal 51°; then measuring in a line directly from it to the distance of 200 feet farther, I found the angle to the top of the tower to be 33° 45′: what then is the height of the tower?

Ans. 93·33148 feet.

EXAM. XV.—From a window near the bottom of a house, which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′: what then is the height and distance of the steeple?

Ans. { Height, 210·44; Distance, 250·79

EXAM. XVI.—Wanting to know the height of, and my distance from, an object on the other side of a river, which seemed to be on a level with the place where I stood, close by the side of the river; and not having room to measure backwards, on the same plane, because of the immediate rise of the bank, I placed a mark where I stood, and measured in a direction from the object up

the ascending ground to the distance of 264 feet, where it was evident that I was above the level of the top of the object; there the angles of depression were found to be, viz. of the mark left at the river's side 42°, of the bottom of the object 27°, and of its top 19°. Required then the height of the object, and the distance of the mark from its bottom? Ans. {Height, 57·26; Distance, 150·50}

EXAM. XVII.—If the height of the mountain called the Peak of Teneriffe be 4 miles, and the angle taken at the top of it, as formed between a plumb line and a line conceived to touch the earth in the horizon, or farthest visible point, be 87° 25′ 55″; it is required from hence to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly round? Ans. {Dist. 178·458 miles. Diam. 7957·818}

EXAM. XVIII.—Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it with effect. In order, therefore, to measure the distance, they separate from each other a quarter of a mile, or 440 yards; then each ship observes and measures the angles which the other ship and the fort subtends, which angles were 83° 45′ and 85° 15′. What then was the distance between each ship and the fort? Ans. {2292·26 yards. 2298·05}

EXAM. XIX.—Being on the side of a river, and wanting to know the distance to a house which was seen at a distance on the other side; I measured out for a base 400 yards in a right line by the side of the river, and found that the two angles, one at each end of this line, subtended by the other end and the house. were 68° 2′ and 73° 15′. What then was the distance between each station and the house? Ans. {593·08 yards. 612 38}

EXAM. XX.—Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree close on the bank on the other side of the river, to be 53° and 79° 12′. What then was the perpendicular breadth of the river? Ans. 529·48 yards.

EXAM. XXI.—Wanting to know the extent of a piece of water, or distance between two headlands; I measured from each of them to a certain point inland, and found the two distances to be 735 yards and 840 yards; also, the horizontal angle subtended between these two lines was 55° 40′. What then was the distance required? Ans. 741·2 yards.

EXAM. XXII.—A point of land was observed, by a ship at sea, to bear east-by-south; and after sailing north-east 12 miles, it was found to bear south-east-by-east. It is required to determine the place of that headland, and the ship's distance from it at the last observation? Ans. 26·0728 miles.

EXAM. XXIII.—Wanting to know the distance between a house and a mill, which were seen at a distance on the other side of a river, I measured a base line along the side where I was of 600 yards, and at each end of it took the angles subtended by the other end and the house and mill, which were as follow; viz. at one end the angles were 58° 20′ and 95° 20′, and at the other end the like angles were 53° 30′ and 98° 45′. What then was the distance between the house and mill? Ans. 959·5866 yards.

EXAM. XXIV.—Wanting to know my distance from an inaccessible object O, on the other side of a river; and having no instrument for taking angles, but only a chain or cord for measuring distances; from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct line from the object O 100 yards, viz. AC and BD each equal to 100 yards; also the diagonal AD measured 550 yards, and the diagonal BC 560. What then was the distance of the object O from each station A and B?

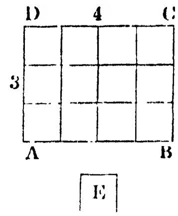
Ans. $\begin{cases} AO\ 536{\cdot}25 \\ BO\ 500{\cdot}09 \end{cases}$

MENSURATION OF PLANES.

THE area of any plane figure, is the measure of the space contained within its extremes or bounds; without any regard to thickness.

This area, or the content of the plane figure, is estimated by the number of little squares that may be contained in it; the side of those little measuring squares being an inch, a foot, a yard, or any other fixed quantity. And hence, the area or content is said to be so many square inches, or square feet, or square yards, &c.

Thus, if the figure to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit proposed: then, as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain, which in the present case is 12.



PROBLEM I.

To find the area of any parallelogram, whether it be a square, a rectangle, a rhombus, or a rhomboid.

Multiply the length by the perpendicular breadth, or height, and the product will be the area.*

* The truth of this rule is proved in the Geometry, Theor. 81, Cor. 2.

The same is otherwise proved thus: Let the foregoing rectangle be the figure proposed; and let the length and breadth be divided into equal parts, each equal to the lineal measuring unit, being here 4 for the length, and 3 for the breadth; and let the opposite points of division be connected by right lines. Then, it is evident that these lines divide the rectangle into a number of little squares, each equal to the square measuring unit E; and farther, that the number of these little squares, or the area of the figure, is equal to the number of lineal measuring units in the length, repeated as often as there are lineal measuring units in the breadth, or height; that is, equal to the length drawn into the height; which here is 4 × 3 or 12.

And it is proved (Geometry, Theor. 25, Cor. 2), that a rectangle is equal to any oblique parallelogram, of equal length and perpendicular breadth. Therefore, the rule is general for all parallelograms whatever.

EXAMPLES.

Ex. 1.—To find the area of a parallelogram, whose length is 12·25, and height 8·5.

12·25	length
8·5	breadth
6125	
9800	
104·125	area

Ex. 2.—To find the area of a square, whose side is 35·25 chains.

Ans. 124 acres, 1 rood, 1 perch.

Ex. 3.—To find the area of a rectangular board, whose length is $12\frac{1}{2}$ feet, and breadth 9 inches. Ans. $9\frac{3}{8}$ feet.

Ex. 4.—To find the content of a piece of land, in form of a rhombus, its length being 6.20 chains, and perpendicular height 5·45.

Ans. 3 acres, 1 rood, 20 perches.

Ex. 5.—To find the number of square yards of painting in a rhomboid, whose length is 37 feet, and breadth 5 feet 3 inches.

Ans. $21\frac{7}{12}$ square yards.

PROBLEM II.

To find the area of a triangle.

Rule i.—Multiply the base by the perpendicular height, and half the product will be the area.* Or, multiply the one of these dimensions by half the other.

EXAMPLES.

Ex. 1.—To find the area of a triangle, whose base is 625, and perpendicular height 520 links?

Here $625 \times 260 = 162500$ square links,

or equal 1 acre, 2 roods, 20 perches, the answer.

Ex. 2.—How many square yards contains the triangle, whose base is 40, and perpendicular 30 feet? Ans. $66\frac{2}{3}$ square yards.

Ex. 3.—To find the number of square yards in a triangle, whose base is 49 feet, and height $25\frac{1}{4}$ feet. Ans. $68\frac{53}{72}$, or 68·7361.

Ex. 4.—To find the area of a triangle, whose base is 18 feet 4 inches, and height 11 feet 10 inches. Ans. 108 feet, $5\frac{2}{3}$ inches.

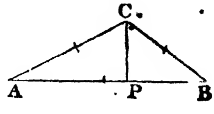
Rule ii.—When two sides and their contained angle are given: Multiply the two given sides together, and take half their product: Then say, as radius is to the sine of the given angle, so is that half product, to the area of the triangle.

Or, multiply that half product by the natural sine of the said angle. †

* The truth of this rule is evident, because any triangle is the half of a parallelogram of equal base and altitude, by Geometry, Theor. 26.

† For, let AB, AC, be the two given sides, including the given angle A. Now $\frac{1}{2}AB \times CP$ is the area, by the first rule, CP being perpendicular. But, by Trigonometry, as sine angle P, or radius, is to sine angle A :: AC : CP = sine angle A × AC, taking radius = 1. Therefore, the area $\frac{1}{2}AB \times CP$ is $= \frac{1}{2}AB \times AC \times$ sin. angle A, to radius 1; or,

as radius : sin. angle A :: $\frac{1}{2}AB \times AC$: the area.



Ex. 1.—What is the area of a triangle, whose two sides are 30 and 40, and their contained angle 28° 57′ 18″ ?

Here $\frac{1}{2} \times 40 \times 30 = 600$,
Therefore, 1 : ·4841226 nat. sin. 28° 57′ 18″
600

290·47356, the answer.

Ex. 2.—How many square yards contains the triangle, of which one angle is 45°, and its containing sides 25 and $21\frac{1}{4}$ feet? Ans. 20·86947.

Rule iii.—When the three sides are given: Add all the three sides together, and take half that sum. Next, subtract each side severally from the said half sum, obtaining three remainders. Lastly, multiply the said half sum and those three remainders all together, and extract the square root of the last product, for the area of the triangle.*

Ex. 1.—To find the area of the triangle whose three sides are 20, 30, 40.

20	45	45	45
30	20	30	40
40	25, first rem.	15, second rem.	5, third rem.
2) 90			
45, half sum.			

Then $45 \times 25 \times 15 \times 5 = 84375$.
The root of which is 290·4737, the area.

Ex. 2.—How many square yards of plastering are in a triangle, whose sides are 30, 40, 50? Ans. $66\frac{2}{3}$.

Ex. 3.—How many acres, &c. contains the triangle, whose sides are 2569, 4900, 5025 links? Ans. 61 acres, 1 rood, 39 perches.

PROBLEM III.

To find the area of a trapezoid.

Add together the two parallel sides; then multiply their sum by the perpendicular breadth or distance between them; and half the product will be the area; by Geometry, theorem 29.

Ex. 1.—In a trapezoid, the parallel sides are 750 and 1225, and the perpendicular distance between them 1540 links: to find the area.

1225
750

$1975 \times 770 = 152075$ square links $=$ 15 acres, 33 perches.

* For, let a, b, c, denote the sides opposite respectively to A, B, C, the angles of the triangle A B C (see last fig.); then by Theor. 37, Geom. we have $BC^2 = AB^2 + AC^2 - 2AB.AP$, or $a^2 = b^2 + c^2 - 2c.AP$ $\therefore AP = \frac{b^2+c^2-a^2}{2c}$; hence we have

$$CP^2 = b^2 - \frac{(b^2+c^2-a^2)^2}{4c^2} = \frac{4b^2c^2-(b^2+c^2-a^2)^2}{4c^2} = \frac{(2bc+b^2+c^2-a^2).(2bc-b^2-c^2+a^2)}{4c^2}$$

$$\therefore 4c^2.CP^2 = \left\{(b+c)^2-a^2\right\} \cdot \left\{a^2-(c-b)^2\right\} = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$

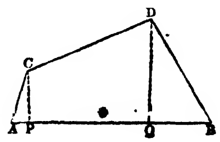
$$\therefore \frac{1}{2}AB.CP = \frac{1}{2}c.CP = \sqrt{\left\{\frac{a+b+c}{2} \cdot \frac{-a+b+c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2}\right\}} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$ = half the sum of the three sides.

Ex. 2.—How many square feet are contained in the plank, whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches? Ans. $13\frac{13}{24}$ feet.

Ex. 3.—In measuring along one side AB of a quadrangular field, that side and the two perpendiculars let fall on it from the two opposite corners, measured as below: required the content.

$$\begin{aligned} AP &= 110 \text{ links.} \\ AQ &= 745 \\ AB &= 1110 \\ CP &= 352 \\ DQ &= 595 \end{aligned}$$



Ans. 4 acres, 1 rood, 5·792 perches.

PROBLEM IV

To find the area of any trapezium.

Divide the trapezium into two triangles by a diagonal: then find the areas of these triangles, and add them together.

Note.—If two perpendiculars be let fall on the diagonal, from the other two opposite angles, the sum of these perpendiculars being multiplied by the diagonal, half the product will be the area of the trapezium.

Ex. 1.—To find the area of the trapezium, whose diagonal is 42, and the two perpendiculars on it 16 and 18.

$$\begin{aligned} \text{Here } 16 + 18 &= 34, \text{ its half is } 17. \\ \text{Then } 42 \times 17 &= 714, \text{ the area.} \end{aligned}$$

Ex. 2.—How many square yards of paving are in the trapezium, whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet? Ans. $222\frac{1}{12}$ yards.

Ex. 3.—In the quadrangular field ABCD, on account of obstructions there could only be taken the following measures, viz. the two sides BC 265, and AD 220 yards, the diagonal AC 378, and the two distances of the perpendiculars from the ends of the diagonal, namely, AE 100, and CF 70 yards. Required the area in acres, when 4840 square yards make an acre? Ans. 17 acres, 2 roods, 21 perches.

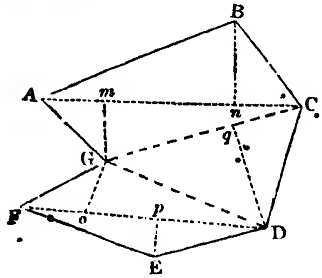
PROBLEM V.

To find the area of an irregular polygon.

Draw diagonals dividing the proposed polygon into trapeziums and triangles. Then find the areas of all these separately, and add them together for the content of the whole polygon.

Exam.—To find the content of the irregular figure ABCDEFGA, in which are given the following diagonals and perpendiculars; namely,

AC 55
FD 52
GC 44
Gm 13
Bn 18
Go 12
Ep 8
Dq 23



Ans. 1878·5.

PROBLEM VI.

To find the area of a regular polygon.

RULE I.—Multiply the perimeter of the polygon, or sum of its sides, by the perpendicular drawn from its centre on one of its sides, and take half the product for the area.*

Ex. 1.—To find the area of the regular pentagon, each side being 25 feet, and the perpendicular from the centre on each side is 17.2047737.

Here $25 \times 5 = 125$ is the perimeter.
And $17{\cdot}2047737 \times 125 = 2150{\cdot}5967125$.
Its half 1075·298356 is the area sought.

RULE II.—Square the side of the polygon; then multiply that square by the area or multiplier set against its name in the following table, and the product will be the area. †

No. of Sides.	Names.	Areas, or Multipliers.
3	Trigon, or triangle	0·4330127
4	Tetragon, or square	1·0000000
5	Pentagon	1·7204774
6	Hexagon	2·5980762
7	Heptagon	3·6339124
8	Octagon	4·8284271
9	Nonagon	6·1818242
10	Decagon	7·6942088
11	Undecagon	9·3656399
12	Dodecagon	11·1961524

EXAM.—Taking here the same example as before, namely, a pentagon, whose side is 25 feet.

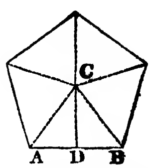
Then, 25^2 being $= 625$,
And the tabular area 1·7204774;
Therefore, $1{\cdot}7204774 \times 625 = 1075{\cdot}298375$, as before.

Ex. 2.—To find the area of the trigon, or equilateral triangle, whose side is 20, Ans. 173·20508.

* This is only in effect resolving the polygon into as many equal triangles as it has sides, by drawing lines from the centre to all the angles; then finding their areas, and adding them all together.

† This rule is founded on the property, that like polygons, being similar figures, are to one another as the squares of their like sides; which is proved in the Geometry, Theorem 89. Now, the multipliers in the table, are the areas of the respective polygons to the side 1. Whence the rule is manifest.

Note.—The areas in the table, to each side 1, may be computed in the following manner: From the centre C of the polygon draw lines to every angle, dividing the whole figure into as many equal triangles as the polygon has sides; and let ABC be one of those triangles, the perpendicular of which is CD. Divide 360 degrees by the number of sides in the polygon, the quotient gives the angle at the centre ACB. The half of this gives the angle ACD; and this taken from 90°, leaves the angle CAD. Then, as radius is to AD, so is tangent angle CAD to the perpendicular CD. This multiplied by AD, gives the area of the triangle ABC; which, being multiplied by the number of the triangles or of the sides of the polygon, gives its whole area, as in the table.



Ex. 3. To find the area of a hexagon, whose side is 20 Ans. 1039·23048.

Ex. 4. To find the area of an octagon, whose side is 20. Ans. 1931·37084.

Ex. 5 To find the area of a decagon, whose side is 20. Ans. 3077·68352.

PROB. VII.

To find the diameter and circumference of any circle, the one from the other.

THIS may be done nearly by either of the two following proportions, viz.

As 7 is to 22, so is the diameter to the circumference.

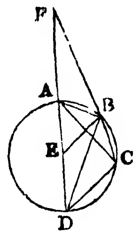
Or, As 1 is to 3·1416, so is the diameter to the circumference.*

Ex. 1. To find the circumference of the circle whose diameter is 20.

By the first rule, as 7 : 22 : : 20 : $62\frac{6}{7}$, the answer.

Ex. 2. If the circumference of the earth be 25000 miles, what is its diameter?

By the 2d rule, as 3·1416 : 1 : : 25000 : $7957\frac{3}{4}$, nearly the diameter.



* For, let ABCD be any circle, whose centre is E, and let AB, BC, be any two equal arcs. Draw the several chords as in the figure, and join BE; also draw the diameter DA, which produce to F, till BF be equal to the chord BD.

Then the two isosceles triangles DEB, DBF, are equiangular, because they have the angle at D common; consequently DE : DB : : DB : DF. But the two triangles AFB, DCB are identical, or equal in all respects, because they have the angle F = the angle BDC, being each equal the angle ADB, these being subtended by the equal arcs AB, BC; also the exterior angle FAB of the quadrangle ABCD, is equal the opposite interior angle at C; and the two triangles have also the side BF = the side BD; therefore the side AF is also equal the side DC. Hence the proportion above, viz. DE : DB : : DB : DF = DA+AF becomes DE : DB : : DB : 2 DE+DC. Then, by taking the rectangles of the extremes and means, it is $DB^2 = 2\,DE^2 + DE \cdot DC$.

Now, if the radius DE be taken=1, this expression becomes $DB^2 = 2+DC$, and hence $DB = \sqrt{2+DC}$. That is, if the measure of the supplemental chord of any arc be increased by the number 2, the square root of the sum will be the supplemental chord of half that arc.

Now, to apply this to the calculation of the circumference of the circle, let the arc AC be taken equal to $\frac{1}{6}$ of the circumference, and be successively bisected by the above theorem: thus, the chord AC, of $\frac{1}{6}$ of the circumference, is the side of the inscribed regular hexagon, and is therefore equal the radius AE or 1; hence, in the right-angle triangle ACD, it will be $DC = \sqrt{AD^2 - AC^2} = \sqrt{2^2 - 1^2} = \sqrt{3} = 1{\cdot}7320508076$, the supplemental chord of $\frac{1}{6}$ of the periphery.

Then, by the foregoing theorem, by always bisecting the arcs, and adding 2 to the last square root, there will be found the supplemental chords of the 12th, the 24th, the 48th, the 96th, &c. parts of the periphery; thus,

$\sqrt{3{\cdot}7320508076}$	=	1·9318516525	for the supplemental chord of	$\frac{1}{12}$	of the periphery
$\sqrt{3{\cdot}9318516525}$	=	1·9828897227		$\frac{1}{24}$	
$\sqrt{3{\cdot}9828897227}$	=	1·9957178465		$\frac{1}{48}$	
$\sqrt{3{\cdot}9957178465}$	=	1·9989291743		$\frac{1}{96}$	
$\sqrt{3{\cdot}9989291743}$	=	1·9997322757		$\frac{1}{192}$	
$\sqrt{3{\cdot}9997322757}$	=	1·9999330678		$\frac{1}{384}$	
$\sqrt{3{\cdot}9999330678}$	=	1·9999832669		$\frac{1}{768}$	
$\sqrt{3{\cdot}9999832669}$	=	. . . .		$\frac{1}{1536}$	

Since then it is found that 3·9999832669 is the square of the supplemental chord of the 1536th part of the periphery, let this number be taken from 4, the square of the diameter, and the remainder 0·0000167331 will be the square of the chord of the said 1536th part of the periphery, and consequently the root $\sqrt{0{\cdot}0000167331} = 0{\cdot}0040906112$ is the length of that chord; this number then being multiplied by 1536, gives 6·2831788 for the perimeter of a regular polygon of 1536 sides inscribed in the circle; which, as the sides of the polygon nearly coincide with the circumference of the circle, must also express the length of the circumference itself, very nearly.

PROBLEM VIII.

To find the length of any arc of a circle.

Multiply the degrees in the given arc by the radius of the circle, and the product again by the decimal ·01745, for the length of the arc *

Ex. 1.—To find the length of an arc of 30 degrees, the radius being 9 feet.
Ans. 4·7115.

Ex. 2.—To find the length of an arc of 12° 10′, or $12°\frac{1}{6}$, the radius being 10 feet.
Ans. 2·1231.

PROBLEM IX.

To find the area of a circle.

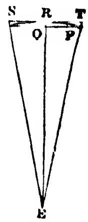
† RULE I.—Multiply half the circumference by half the diameter. Or multiply the whole circumference by the whole diameter, and take $\frac{1}{4}$ of the product.

RULE II.—Square the diameter, and multiply that square by the decimal ·7854, for the area.

Ex. 1.—To find the area of a circle whose diameter is 10, and its circumference 31·416.

By Rule 1.		By Rule 2.
31·416		·7854
10		100 = 10^2
4)314·16		
78·54	the area	78·54

But now, to show how near this determination is to the truth, let AQP = 0·0040906112 represent one side of such a regular polygon of 1536 sides, and SRT a side of another similar polygon described about the circle; and from the centre E let the perpendicular EQR be drawn, bisecting AP and ST in Q and R. Then, since AQ is = $\frac{1}{2}$AP = 0·0020453056, and EA = 1, therefore $EQ^2 = EA^2 - AQ^2$ = ·9999958167, and consequently its root gives EQ = ·9999979084; then, because of the parallels AP, ST, it is EQ : ER :: AP : ST :: the whole inscribed perimeter : the circumscribed one; that is, as ·9999979084 : 1 :: 6·2831788 : 6·2831920 the perimeter of the circumscribed polygon. But the circumference of the circle being greater than the perimeter of the inner polygon, and less than that of the outer, it must consequently be greater than 6·2831788,
but less than 6·2831920,
and must therefore be nearly equal $\frac{1}{2}$ their sum, or 6.2831854, which in fact is true to the last figure, which should be a 3 instead of the 4.



Hence, the circumference being 6·2831854 when the diameter is 2, it will be the half of that, or 3·1415927, when the diameter is 1, to which the ratio in the rule, viz. 1 to 3·1416 is very near. Also the other ratio in the rule 7 to 22 or 1 to $3\frac{1}{7}$ = 3·1428, &c., is another near approximation.

* It having been found, in the demonstration of the foregoing problem, that when the radius of a circle is 1, the length of the whole circumference is 6·2831854, which consists of 360 degrees; therefore, as 360° : 6·2831854 : 1° : ·01745, &c., the length of the arc of 1 degree. Hence, the number ·01745, multiplied by any number of degrees, will give the length of the arc of those degrees. And, because the circumferences, and arcs, are as the diameters, or radii of the circles; therefore, as the radius 1 is to any other radius r, so is the length of the arc above mentioned to $r \times$ ·01745 $\times$ degrees in the arc, which is the length of that arc as in the rule.

† This first rule is proved in the Geometry, Theor. 94.

And the second rule is deduced from the first in this manner: It appears by the demonstration of Problem 7, that when the diameter of a circle is 1, its circumference is 3·1415927, or nearly 3·1416; then, by the first rule, 1 × 3·1416 ÷ 4 = ·7854, which is therefore the area of the circle whose diameter is 1. But the areas of different circles are to each other as the square of their diameters, by Geometry, Theor. 93; therefore, as 1^2 : d^2 :: ·7854 : ·7854 d^2, the area of the circle whose diameter is d, as in the second rule.

Ex. 2.—To find the area of a circle, whose diameter is 7, and circumference 22. Ans. $38\frac{1}{2}$.

Ex. 3.—How many square yards are in a circle, whose diameter is $3\frac{1}{2}$ feet? Ans. 1·069.

PROBLEM X.

To find the area of a circular ring, or space included between two concentric circles.

Take the difference between the areas of the two circles, as found by the last problem.—Or, which is the same thing, subtract the square of the less diameter from the square of the greater, and multiply their difference by ·7854.—Or, lastly, multiply the sum of the diameters by the difference of the same, and that product by ·7854; which is still the same thing, because the product of the sum and difference of any two quantities, is equal to the difference of their squares.

Ex. 1.—The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences.

Here $10 + 6 = 16$ the sum, and $10 - 6 = 4$ the difference,

Therefore, $·7854 \times 16 \times 4 = ·7854 \times 64 = 50·2656$, the area.

Ex. 2.—What is the area of the ring, the diameters of whose bounding circles are 10 and 20? Ans. 235·62.

PROBLEM XI.

To find the area of the sector of a circle.

Rule i.—Multiply the radius, or half the diameter, by half the arc of the sector, for the area. Or, multiply the whole diameter by the whole arc of the sector, and take $\frac{1}{4}$ of the product. The reason of which is the same as for the first rule to problem 9.

Rule ii.—As 360 is to the degrees in the arc of the sector, so is the area of the whole circle, to the area of the sector.

This is evident, because the sector is proportional to the length of the arc, or to the degrees contained in it.

Ex. 1.—To find the area of a circular sector, whose arc contains 18 degrees· the diameter being 3 feet.

1.—By the 1st Rule.

First, $3·1416 \times 3 = 9·4248$, the circumference.

And $360 : 18 :: 9·4248 : ·47124$, the length of the arc.

Then, $·47124 \times 3 \div 4 = ·11781 \times 3 = ·35343$, the area.

2.—By the 2d Rule.

First, $·7854 \times 3^2 = 7·0686$, the area of the whole circle.

Then, as $360 : 18 :: 7·0686 : ·35343$, the area of the sector.

Ex. 2.—To find the area of a sector, whose radius is 10, and arc 20. Ans. 100.

Ex. 3.—Required the area of a sector, whose radius is 25, and its arc containing 147° 29′. Ans. 804·4017.

PROBLEM XII.

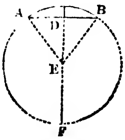
To find the area of a segment of a circle.

Rule i.—Find the area of the sector having the same arc with the segment by the last problem.

Find also the area of the triangle, formed by the chord of the segment and the two radii of the sector.

Then take the sum of these two for the answer, when the segment is greater than a semicircle: or take their difference for the answer, when it is less than a semicircle.—As is evident by inspection.

Ex. 1.—To find the area of the segment ACBDA, its chord AB being 12, and the radius AE or CE 10.



First, As AE : AD : : sin. angle D 90°: sin. 36° 52′¼ = 36·87 degrees, the degrees in the angle AEC or arc AC. Their double, 73·74, are the degrees in the whole arc ACB.

Now, ·7854 × 400 = 314·16, the area of the whole circle,

Therefore, 360° : 73·74 : : 314·16 : 64·3504, area of the whole sector ACBE.

Again, $\sqrt{AE^2 - AD^2} = \sqrt{100 - 36} = \sqrt{64} = 8 = DE$.

Therefore, AD × DE = 6 × 8 = 48, the area of the triangle AEB.

Hence, sector ACBE — triangle AEB = 16·3504, area of seg. ACBDA.

Rule ii.—Divide the height of the segment by the diameter, and find the quotient in the column of heights in the following tablet:—Take out the corresponding area in the next column on the right hand; and multiply it by the square of the circle's diameter, for the area of the segment.*

Note.—When the quotient is not found exactly in the table, proportion may be made between the next less and greater area, in the same manner as is done for logarithms, or any other table.

TABLE OF THE AREAS OF CIRCULAR SEGMENTS.

Height.	Area of the segment.	Height.	Area of the segment.	Height.	Area of the segment.	Height.	Area of the segment.	Height.	Area of the segment.
·01	·00133	·11	·04701	·21	·11990	·31	·20738	·41	30319
·02	·00375	·12	·05339	·22	·12811	·32	·21667	·42	·31304
·03	·00687	·13	·06000	·23	·13646	·33	·22603	·43	·32293
·04	·01054	·14	·06683	24	·14494	34	·23547	·44	·33284
·05	·01468	·15	·07387	·25	·15354	·35	·24498	·45	·34278
·06	·01924	·16	·08111	·26	16226	·36	·25455	·46	·35274
·07	·02417	·17	·08853	·27	·17109	37	·26418	·47	·36272
·08	·02944	·18	·09613	·28	·18002	·38	·27386	·48	·37270
·09	·03502	·19	·10390	·29	·18905	·39	·28359	·49	·38270
·10	·04088	·20	·11182	·30	19817	·40	29337	·50	·39270

Ex. 2.—Taking the same example as before, in which are given the chord AB 12, and the radius 10, or diameter 20.

And having found, as above, DE = 8; then CE — DE = CD = 10 — 8 = 2. Hence, by the rule, CD ÷ CF = 2 ÷ 20 = ·1, the tabular height.

* The truth of this rule depends on the principle of similar plane figures, which are to one another as the square of their like linear dimensions. The segments in the table are those of a circle whose diameter is 1; and the first column contains the corresponding heights or versed sines divided by the diameter. Thus then, the area of the similar segment, taken from the table, and multiplied by the square of the diameter, gives the area of the segment to this diameter.

This being found in the first column of the table, the corresponding tabular area is ·04088. Then ·04088 × 20² = ·04088 × 400 = 16·352, the area, nearly the same as before.

Ex. 3.—What is the area of the segment, whose height is 18, and diameter of the circle 50? Ans. 636·375.

Ex. 4.—Required the area of the segment whose chord is 16, the diameter being 20? Ans. 44·7292.

PROBLEM XIII.

To measure long irregular figures.

Take or measure the breadth in several places at equal distances; then add all these breadths together, and divide the sum by the number of them, for the mean breadth; which multiply by the length for the area. *

Note 1.—Take half the sum of the extreme breadths for one of the said breadths.

Note 2.—If the perpendiculars or breadths be not at equal distances, compute all the parts separately, as so many trapezoids, and add them all together for the whole area.

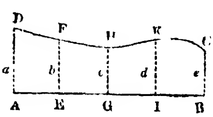
Or else, add all the perpendicular breadths together, and divide their sum by the number of them for the mean breadth, to multiply by the length; which will give the whole area, not far from the truth.

Ex. 1.—The breadths of an irregular figure, at five equidistant places, being 8.2, 7.4, 9.2, 10.2, 8.6; and the whole length 39: required the area?

First, (8·2 + 8·6) ÷ 2 = 8·4, the mean of the two extremes.
Then, 8·4 + 7·4 + 9·2 + 10·2 = 35·2, sum of breadths.
And, 35·2 ÷ 4 = 8·8, the mean breadth.
Hence, 8·8 × 39 = 343·2, the answer.

Ex. 2.—The length of an irregular figure being 84, and the breadths at six equidistant places 17·4, 20·6, 14·2, 16·5, 20·1, 24·4; what is the area? Ans. 1550·64.

* This rule is made out as follows: Let ABCD be the irregular piece; having the several breadths AD, EF, GH, IK, BC, at the equal distances AE, EG, GI, IB. Let the several breadths in order be denoted by the corresponding letters a, b, c, d, e, and the whole length AB by l; then compute the areas of the parts into which the figure is divided by the perpendiculars, as so many trapezoids by Problem 3, and add them all together. Thus, the sum of the parts is,



$$\frac{a+b}{2} \times AE + \frac{b+c}{2} \times EG + \frac{c+d}{2} \times GI + \frac{d+e}{2} \times IB$$
$$= \frac{a+b}{2} \times \tfrac{1}{4}l + \frac{b+c}{2} \times \tfrac{1}{4}l + \frac{c+d}{2} \times \tfrac{1}{4}l + \frac{d+e}{2} \times \tfrac{1}{4}l$$
$$= (\tfrac{1}{2}a + b + c + d + \tfrac{1}{2}e) \times \tfrac{1}{4}l = (m + b + c + d)\,\tfrac{1}{4}l,$$

which is the whole area, agreeing with the rule; m being the arithmetic mean between the extremes and 4 the number of the parts. And the same for any other number of parts.

MENSURATION OF SOLIDS.*

By the Mensuration of Solids are determined the spaces included by contiguous surfaces, and the sum of the measures of these including surfaces, is the whole surface or superficies of the body.

The measure of a solid, is called its solidity, capacity, or content.

Solids are measured by cubes, whose sides are inches, or feet, or yards, &c. And hence the solidity of a body is said to be so many cubic inches, feet, yards, &c., as will fill its capacity or space, or another of equal magnitude.

The least solid measure is the cubic inch, other cubes being taken from it according to the proportion in the following table:

Table of Cubic or Solid Measures.

1728	cubic inches make	1 cubic foot
27	cubic feet make	1 cubic yard
$166\frac{3}{8}$	cubic yards make..................	1 cubic pole
64000	cubic poles make	1 cubic furlong
512	cubic furlongs make	1 cubic mile.

PROBLEM I.

To find the superficies of a prism.

Multiply the perimeter of one end of the prism by the length or height of the solid, and the product will be the surface of all its sides. To which, add also the area of the two ends of the prism, when required.†

Or, compute the areas of all the sides and ends separately, and add them all together.

Ex. 1.—To find the surface of a cube, the length of each side being 20 feet.
Ans. 2400 feet.

Ex. 2.—To find the whole surface of a triangular prism, whose length is 20 feet, and each side of its end or base 18 inches. Ans. 91·948 feet.

Ex. 3.—To find the convex surface of a round prism, or cylinder, whose length is 20 feet, and diameter of its base is 2 feet. Ans. 125·664.

Ex. 4.—What must be paid for lining a rectangular cistern with lead, at 2*d.* a pound weight, the thickness of the lead being such as to weigh 7 lb. for each square foot of surface; the inside dimensions of the cistern being as follow, viz. the length 3 feet 2 inches, the breadth 2 feet 8 inches, and depth 2 feet 6 inches? Ans. £2. 3*s.* $10\frac{1}{2}$*d.*

* Before perusing this chapter the student must make himself master of the treatise on the "Geometry of Solids," which immediately follows the "Geometry of Planes." The principle upon which the rules are founded are explained in the Differential Calculus.

† The truth of this will easily appear, by considering that the sides of any prism are parallelograms, whose common length is the same as the length of the solid, and their breadths taken all together make up the perimeter of the ends of the same.

And the rule is evidently the same for the surface of a cylinder.

PROBLEM II.

To find the surface of a pyramid or cone.

Multiply the perimeter of the base by the slant height, or length of the side, and half the product will evidently be the surface of the sides, or the sum of the areas of all the triangles which form it. To which, add the area of the end or base, if requisite.

Ex. 1.—What is the upright surface of a triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet? Ans. 90 feet.

Ex. 2.—Required the convex surface of a cone, or circular pyramid, the slant height being 50 feet, and the diameter of its base $8\frac{1}{2}$ feet. Ans. 667.59.

PROBLEM III.

To find the surface of the frustum of a pyramid or cone; being the lower part, when the top is cut off by a plane parallel to the base.

Add together the perimeters of the two ends, and multiply their sum by the slant height, taking half the product for the answer.—As is evident, because the sides of the solid are trapezoids, having the opposite sides parallel.

Ex. 1.—How many square feet are in the surface of the frustum of a square pyramid, whose slant height is 10 feet; also, each side of the base or greater end being 3 feet 4 inches, and each side of the less end 2 feet 2 inches?
Ans. 110 feet.

Ex. 2.—To find the convex surface of the frustum of a cone, the slant height of the frustum being $12\frac{1}{2}$ feet, and the circumferences of the two ends 6 and 8·4. Ans. 90 feet.

PROBLEM IV

To find the solid content of any prism or cylinder.

Find the area of the base, or end, whatever the figure of it may be; and multiply it by the length of the prism or cylinder, for the solid content.

Ex. 1.—To find the solid content of a cube, whose side is 24 inches.
Ans. 13824.

Ex. 2.—How many cubic feet are in a block of marble, its length being 3 feet 2 inches, breadth 2 feet 8 inches, and thickness 2 feet 6 inches?
Ans. $21\frac{1}{9}$

Ex. 3.—How many gallons of water will the cistern contain, whose dimensions are the same as in the last example, when 277·274 cubic inches are contained in one gallon? Ans. 131·566.

Ex. 4.—Required the solidity of a triangular prism, whose length is 10 feet, and the three sides of its triangular end or base, are 3, 4, 5 feet. Ans. 60.

Ex. 5.—Required the content of a round pillar, or cylinder, whose length is 20 feet, and circumference 5 feet 6 inches. Ans. 48·1459.

PROBLEM V.

To find the content of any pyramid or cone.

Find the area of the base, and multiply that area by the perpendicular height; then take $\frac{1}{3}$ of the product for the content.

Ex. 1.—Required the solidity of the square pyramid, each side of its base being 30, and its perpendicular height 25. Ans. 7500.

Ex. 2.—To find the content of a triangular pyramid, whose perpendicular height is 30, and each side of the base 3. Ans. 38·97117.

Ex. 3.—To find the content of a triangular pyramid, its height being 14 feet 6 inches, and the three sides of its base 5, 6, 7. Ans. 71·0352.

Ex. 4.—What is the content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet? Ans. 27·5276.

Ex. 5.—What is the content of the hexagonal pyramid, whose height is 6·4, and each side of its base 6 inches? Ans. 1·38564 feet.

Ex. 6.—Required the content of a cone, its height being $10\frac{1}{2}$ feet, and the circumference of its base 9 feet. Ans. 22·56093.

PROBLEM VI.

To find the solidity of the frustum of a cone or pyramid.

Add into one sum, the areas of the two ends, and the mean proportional between them, or the square root of their product; and $\frac{1}{3}$ of that sum will be a mean area; which, being multiplied by the perpendicular height or length of the frustum, will give its content.

Ex. 1.—To find the number of solid feet in a piece of timber, whose bases are squares, each side of the greater end being 15 inches, and each side of the less end 6 inches; also, the length or perpendicular altitude 24 feet?

Ans. $19\frac{1}{2}$.

Ex. 2.—Required the content of a pentagonal frustum, whose height is 5 feet, each side of the base 18 inches, and each side of the top or less end 6 inches. Ans. 9·31925 feet.

Ex. 3.—To find the content of a conic frustum, the altitude being 18, the greatest diameter 8, and the least diameter 4. Ans. 527·7888.

Ex. 4.—What is the solidity of the frustum of a cone, the altitude being 25, also the circumference at the greater end being 20, and at the less end 10?

Ans. 464·216.

Ex. 5.—If a cask, which is two equal conic frustums joined together at the bases, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold? Ans. 79·0613.

PROBLEM VII.

To find the surface of a sphere, or any segment.

Rule i.—Multiply the circumference of the sphere by its diameter, and the product will be the whole surface of it.

Rule ii.—Multiply the square of the diameter by 3·1416, and the product will be the surface.

Note.—For the surface of a segment or frustum, multiply the whole circumference by the height of the part required.

Ex. 1.—Required the convex superficies of a sphere, whose diameter is 7, and circumference 22. Ans. 154.

Ex. 2.—Required the superficies of a globe, whose diameter is 24 inches.

Ans. 1809·5616.

Ex. 3.—Required the area of the whole surface of the earth, its diameter being $7957\frac{3}{4}$ miles, and its circumference 25000 miles.

Ans. 198943750 sq. miles.

Ex. 4.—The axis of a sphere being 42 inches, what is the convex superficies of the segment, whose height is 9 inches? Ans. 1187·5248 inches.

Ex. 5.—Required the convex surface of a spherical zone, whose breadth or height is 2 feet, and cut from a sphere of $12\frac{1}{2}$ feet diameter. Ans. 78·54 feet.

PROBLEM VIII.

To find the solidity of a sphere or Globe.

Rule i.—Multiply the surface by the diameter, and take $\frac{1}{6}$ of the product for the content.

Rule ii.—Multiply the cube of the diameter by the decimal ·5236, for the content.

Ex. 1.—To find the content of a sphere whose axis is 12. Ans. 904·7808.

Ex. 2.—To find the solid content of the globe of the earth, supposing its circumference to be 25000 miles. Ans. 263,857,437,760 miles.

PROBLEM IX.

To find the solid content of a spherical segment.

Rule i.—From three times the diameter of the sphere take double the height of the segment; then multiply the remainder by the square of the height and the product by the decimal ·5236. for the content.

Rule ii.—To three times the square of the radius of the segment's base, add the square of its height; then multiply the sum by the height, and the product by ·5236, for the content.

Ex. 1.—To find the content of a spherical segment, of 2 feet in height, cut from a sphere of 8 feet in diameter. Ans. 41·888.

Ex. 2.—What is the solidity of the segment of a sphere, its height being 9, and the diameter of its base 20? Ans. 1795·4244.

Note.—The general rules for measuring all sorts of figures having been now delivered, we may next proceed to apply them to the several practical uses in life, as follows.

LAND SURVEYING.

SECTION I.

DESCRIPTION AND USE OF THE INSTRUMENTS.

1.—OF THE CHAIN.

Land is measured with a chain, called Gunter's Chain, from its inventor, of 4 poles or 22 yards, or 66 feet in length. It consists of 100 equal links; and the length of each link is therefore $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or 7·92 inches.

Land is estimated in acres, roods, and perches. An acre is equal to 10 square chains, that is, 10 chains in length and one chain in breadth. Or it is $220 \times 22 = 4840$ square yards. Or it is $40 \times 4 = 160$ square poles. Or it is $1000 \times 100 = 1{,}000{,}000$ square links. These being all the same quantity.

Also, an acre is divided into four parts called roods, and a rood into 40 parts called perches, which are square poles, or the square of a pole of $5\frac{1}{2}$ yards long, or the square of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 square links. So that the divisions of land measure will be thus:

625 square links	=	1 pole or perch,
40 perches	=	1 rood,
4 roods	=	1 acre.

The length of lines, measured with a chain, are best set down in links as integers, every chain in length being 100 links; and not in chains and decimals. Therefore, after the content is found, it will be in square links; then cut off five of the figures on the right hand for decimals, and the rest will be acres. These decimals are then multiplied by 4 for roods, and the decimals of these again by 40 for perches.

Exam.—Suppose the length of a rectangular piece of ground be 792 links, and its breadth 385: to find the area in acres, roods, and perches.

```
   792          3·04920
   385                4
  ----          -------
  3960           ·19680
 6336                40
2376            -------
-------         7·87200
3·04920         -------
-------
```

Ans. 3 acres, 0 roods, 7 perches.

2.—OF THE PLAIN TABLE.

This instrument consists of a plain rectangular board, of any convenient size the centre of which, when used, is fixed by means of screws to a three-legged stand, having a ball and socket, or other joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belong various parts, as follow:

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a sheet of paper upon the table. The one side of this frame is usually divided into equal parts, for drawing lines across the table, parallel or perpendicular to the sides; and the other side of the frame is divided into 360 degrees from a centre which is in the middle of the table; by means of which the table is to be used as a theodolite, &c.

2. A needle and compass screwed into the side of the table, to point out the directions, and to be a check upon the sights.

3. An index, which is a brass two-foot scale, with either a small telescope, or open sights erected perpendicularly upon the ends. These sights and one edge of the index, are in the same plane, and that edge is called the fiducial edge of the index.

To use this instrument, take a sheet of paper which will cover it, and wet it to make it expand; then spread it flat upon the table, pressing down the frame upon the edges, to stretch it and keep it fixed there; and when the paper is become dry, it will, by contracting again, stretch itself smooth and flat from any cramps and unevenness. On this paper is to be drawn the plan or form of the thing measured.

Then, begin at any part of the ground the most proper, and make a point on a convenient part of the paper or table, to represent that point of the ground; then fix in that point one leg of the compasses, or a fine steel pin, and apply to it the fiducial edge of the index, moving it round till through the sights you perceive some remarkable object, as the corner of a field, &c.; and from the station point draw a line with the point of the compasses along the fiducial edge of the index; then set another object or corner, and draw its line; do the same by another, and so on, till as many objects are set as may be thought fit. Then measure from the station, towards as many of the objects as may be necessary, and no more, taking the requisite offsets to corners or crooks in the hedges, laying the measures down on their respective lines on the table. Then, at any convenient place, measured to, fix the table in the same position, and set the objects which appear from thence, &c. as before; and thus continue till the work is finished, measuring such lines as are necessary, and determining as many as may be, by intersecting lines of direction drawn from different stations.

OF SHIFTING THE PAPER ON THE PLAIN TABLE.

When one paper is full, and you have occasion for more; draw a line in any manner through the farthest point of the last station line, to which the work can be conveniently laid down; then take the sheet off the table, and fix another on, drawing a line upon it, in a part the most convenient for the rest of the work; then fold or cut the old sheet by the line drawn on it, applying the edge to the line on the new sheet, and as they lie in that position, continue the last station line on the new paper, placing on it the rest of the measure, beginning at where the old sheet left off. And so on from sheet to sheet.

When the work is done, and you would fasten all the sheets together into one piece, or rough plan, the aforesaid lines are to be accurately joined together, in the same manner as when the lines were transferred from the old sheets to the new ones.

But it is to be noted, that if the said joining lines, on the old and new sheets have not the same inclination to the side of the table, the needle will not point

to the original degree when the table is rectified; and if the needle be required to respect still the same degree of compass, the easiest way of drawing the lines in the same position, is to draw them both parallel to the same sides of the table, by means of the equal divisions marked on the other two sides.

3. OF THE THEODOLITE.

The theodolite is a brazen circular ring, divided into 360 degrees, and having an index with sights, or a telescope, placed on the centre, about which the index is moveable; also a compass fixed to the centre, to point out courses and check the sights; the whole being fixed by the centre on a stand of a convenient height for use.

In using this instrument, an exact account, or field-book, of all measures and things necessary to be remarked in the plan, must be kept, from which to make out the plan on returning home from the ground.

Begin at such part of the ground, and measure in such directions, as you judge most convenient; taking angles or directions to objects, and measuring such distances as appear necessary, under the same restrictions as in the use of the plain table. And it is safest to fix the theodolite in the original position at every station, by means of fore and back objects, and the compass, exactly as in using the plain table; registering the number of degrees cut off by the index when directed to each object; and, at any station, placing the index at the same degree as when the direction towards that station was taken from the last preceding one, to fix the theodolite there in the original position.

The best method of laying down the aforesaid lines of direction, is to describe a pretty large circle; then quarter it, and lay on it the several numbers of degrees cut off by the index in each direction, and drawing lines from the centre to all these marked points in the circle. Then, by means of a parallel ruler, draw, from station to station, lines parallel to the aforesaid lines drawn from the centre to the respective points in the circumference.

4.—OF THE CROSS.

The cross consists of two pair of sights set at right angles to each other, upon a staff having a sharp point at the bottom to stick in the ground.

The cross is very useful to measure small and crooked pieces of ground. The method is to measure a base or chief line, usually in the longest direction of the piece, from corner to corner; and while measuring it, finding the places where perpendiculars would fall on this line, from the several corners and bends in the boundary of the piece, with the cross, by fixing it, by trials, on such parts of the line, so that through one pair of the sights both ends of the line may appear, and through the other pair you can perceive the corresponding bends or corners: and then measuring the lengths of the said perpendiculars.

REMARKS.

Besides the fore-mentioned instruments, which are most commonly used, there are some others; as the circumferentor, which resembles the theodolite in shape and use; and the semicircle, for taking angles, &c.

The perambulator is used for measuring roads, and other great distances on

level ground, and by the sides of rivers. It has a wheel of $8\frac{1}{4}$ feet, or half a pole in circumference, upon which the machine turns; and the distance measured, is pointed out by an index, which is moved round by clock-work.

Levels, with telescopic or other sights, are used to find the level between place and place, or how much one place is higher or lower than another. And in measuring any sloping or oblique line, either ascending or descending, a small pocket level is useful for showing how many links for each chain are to be deducted, to reduce the line to the true horizontal length.

An offset staff is a very useful and necessary instrument for measuring the offsets and other short distances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten small arrows, or rods of iron or wood, are used to mark the end of every chain length in measuring lines. And sometimes pickets, or staves with flags, are set up as marks or objects of direction.

Various scales are also used in protracting and measuring on the plan or paper; such as plane scales, line of chords, protractor, compasses, reducing scale, parallel and perpendicular rules, &c. Of plane scales, there should be several sizes, as a chain in 1 inch, a chain in $\frac{3}{4}$ of an inch, a chain in $\frac{1}{2}$ an inch, &c. And of these, the best for use are those that are laid on the very edges of the ivory scale, to prick off distances by, without compasses.

5.—OF THE FIELD-BOOK.

In surveying with the plane table, a field-book is not used, as every thing is drawn on the table immediately when it is measured. But in surveying with the theodolite, or any other instrument, some sort of a field-book must be used, to write down in it a register or account of all that is done and occurs relative to the survey in hand.

This book every one contrives and rules as he thinks fittest for himself. The following is a specimen of a form which has been formerly used. It is ruled into 3 columns: the middle, or principal column, is for the stations, angles, bearings, distances measured, &c.; and those on the right and left are for the offsets on the right and left, which are set against their corresponding distances in the middle column; as also for such remarks as may occur, and may be proper to note in drawing the plan, &c.

Here ⊙ 1 is the first station, where the angle or bearing is 105° 25′. On the left, at 73 links in the distances or principal line, is an offset of 92; and at 610 an offset of 24 to a cross hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the first line, the 0 denotes its terminating in the hedge. And so on for the other stations.

A line is drawn under the work, at the end of every station line, to prevent confusion.

FORM OF THIS FIELD-BOOK.

Offsets and Remarks on the left.	Stations, Bearings, and Distances.	Offsets and Remarks on the right.
	⊙ 1	
	105° 25′	
	00	25, corner.
92	73	
	248	Brown's hedge.
Cross a hedge, 24	610	35
	954	00
	⊙ 2	
	53° 10′	
	00	00
House corner, 51	25	21
	120	29, a tree.
34	734	40, a style.
	⊙ 3	
	67° 20′	
	61	35
A brook, 30	248	
	639	16, a spring.
Foot path, 16	810	
Cross hedge, 18	973	20, a pond.

But some skilful surveyors now make use of a different method for the field book, namely beginning at the bottom of the page and writing upwards; by which they sketch a neat boundary on either hand, as they pass along; an example of which will be given further on, in the method of surveying a large estate.

In smaller surveys and measurement, a good way of setting down the work, is, to draw by the eye, on a piece of paper, a figure resembling that which is to be measured; and so writing the dimensions, as they are found, against the corresponding parts of the figure. And this method may be practised to a considerable extent, even in the larger surveys.

SECTION II.

THE PRACTICE OF SURVEYING.

This part contains the several works proper to be done in the field, or the ways of measuring by all the instruments, and in all situations.

PROBLEM I.

To measure a line or distance.

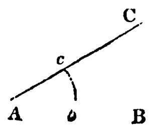
To measure a line on the ground with the chain: Having provided a chain, with ten small arrows, or rods, to stick one into the ground, as a mark, at the end of every chain; two persons take hold of the chain, one at each end of it; and all the ten arrows are taken by one of them, who goes foremost, and is called the leader; the other being called the follower, for distinction's sake.

A picket, or station-staff, being set up in the direction of the line to be measured, if there do not appear some marks naturally in that direction; they measure straight towards it, the leader fixing down an arrow at the end of every chain, which the follower always takes up, till all the ten arrows are used. They are then all returned to the leader, to use over again. And thus the arrows are changed from the one to the other at every ten chains' length, till the whole line is finished; then the number of changes of the arrows shows the number of tens, to which the follower adds the arrows he holds in his hand, and the number of links of another chain over to the mark or end of the line. So, if there have been three changes of the arrows, and the follower hold six arrows, and the end of the line cut off 45 links more, the whole length of the line is set down in links thus, 3645.

When the ground is on a declivity, ascending or descending; at every chain length, lay the offset staff, or link-staff down in the slope of the chain, upon which lay the small pocket level, to show how many links or parts the slope line is longer than the true level one; then draw the chain forward so many links or parts, which reduces the line to the horizontal direction.

PROBLEM II.

To take angles and bearings.



Let B and C be two objects, or two pickets set up perpendicular, and let it be required to take their bearings, or the angle formed between them at any station A.

1.—WITH THE PLAIN TABLE.

The table being covered with a paper, and fixed on its stand; plant it at the station A, and fix a fine pin, or a point of the compasses, in a proper point of the paper, to represent the point A: close by the side of this pin lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights: then by the fiducial edge of the index draw a line; in the very same manner draw another line in the direction of the other object C. And it is done.

2.—WITH THE THEODOLITE, &c.

Direct the fixed sights along one of the lines, as AB, by turning the instrument about till the mark B is seen through these sights; and there screw the instrument fast. Then turn the moveable index about, till through its sights you see the other mark C. Then the degrees cut by the index, upon the graduated limb or ring of the instrument, show the quantity of the angle.

3.—WITH THE MAGNETIC NEEDLE AND COMPASS.

Turn the instrument, or compass so, that the north end of the needle point to the flower-de-luce. Then direct the sights to one mark, as B, and note the degrees cut by the needle. Next direct the sights to the other mark C, and note again the degrees cut by the needle. Then their sum or difference, as the case is, will give the quantity of the angle BAC.

4.—BY MEASUREMENT WITH THE CHAIN, &c.

Measure one chain length, or any other length, along both directions, as to B and C; then measure the distance B, C, and it is done. This is easily trans-

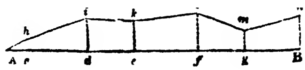
ferred to paper, by making a triangle ABC with these three lengths, and then measuring the angle A.

PROBLEM III.

To measure the offsets.

Ahiklmn being a crooked hedge, or river, &c.: From A measure in a straight direction along the side of it to B. And in measuring along this line AB, observe when you are opposite any bends or corners of the hedge, as at *c*, *d*, *e*, &c.; and from thence measure the perpendicular offsets *ch*, *di*, &c., with the offset-staff, if they are not very large, otherwise with the chain itself. And the work is done. The register, or field-book, may be as follows:

Offset, left.		Base line AB.	
	0	⊙ A	
ch	62	45	A*c*
di	84	220	A*d*
ek	70	340	A*e*
fl	93	510	A*f*
gm	57	634	A*g*
B*n*	91	785	AB

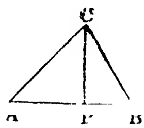


PROBLEM IV.

To survey a triangular field ABC.

1.—BY THE CHAIN.

AP	794
AB	1321
PC	826



Having set up marks at the corners, which is to be done in all cases where there are not marks naturally; measure with the chain from A to P, where a perpendicular would fall from the angle C, and set up a mark at P, noting down the distance AP. Then complete the distance AB by measuring from P to B. Having set down this measure, return to P, and measure the perpendicular PC. And thus, having the base and perpendicular, the area from them is easily found. Or, having the place P of the perpendicular, the triangle is easily constructed.

Or, measure all the three sides with the chain, and note them down. From which the content is easily found, or the figure constructed.

2.—BY TAKING ONE OR MORE OF THE ANGLES.

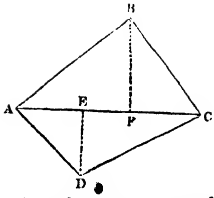
Measure two sides, AB, AC, and the angle A between them. Or measure one side AB, and the two adjacent angles A and B. From either of these ways the figure is easily planned; then by measuring the perpendicular CP on the plan, and multiplying it by half AB, you have the content.

PROBLEM V.

To measure a four-sided field

1.—BY THE CHAIN.

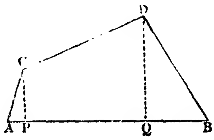
AE	214	210	DE
AF	362	306	BF
AC	592		



Measure along either of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the last problem; or else the sides AB, BC, CD, DA. From either of which the figure may be planned and computed as before directed.

OTHERWISE BY THE CHAIN.

AP	110	352	PC
AQ	745	595	QD
AB	1110		



Measure on the longest side, the distances AP, AQ, AB; and the perpendiculars PC, QD.

2.—BY TAKING ONE OR MORE OF THE ANGLES.

Measure the diagonal AC (see the last fig. but one), and the angles DAB, CAD, ACD. Or, measure the four sides, and any one of the angles as ABC.

Thus,

AC	591	
CAB	37°	20
CAD	41	20
ACB	72	25
ACD	54	40

Or thus,

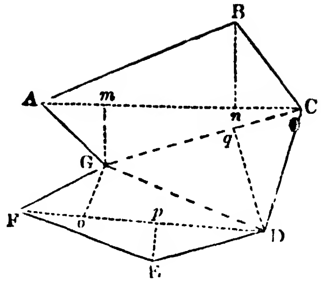
AB	486
BC	394
CD	410
DA	462
BAD	78° 35′

PROBLEM VI.

To survey any field by the chain only.

Having set up marks at the corners, where necessary, of the proposed field ABCDEFG, walk over the ground, and consider how it can best be divided in triangles and trapeziums; and measure them separately as in the last two problems. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the first trapezium, beginning at A, measure the diagonal AC, and the two perpendiculars G*m*, B*n*. Then, the base GC, and the perpendicular D*q*. Lastly, the diagonal DF, and the two perpendiculars *p*E, *o*G. All which measures write against the corresponding parts of a rough figure drawn to resemble the figure to be surveyed, or set them down in any other form you choose.

ius,		Thus,	
A*m*	135	130	*m*G
A*n*	410	180	*n*B
AC	550		
C*q*	152	230	*q*D
CG	440		
F*o*	206	120	*o*G
F*p*	288	80	*p*E
FD	520		



OR THUS.

Measure all the sides AB, BC, CD, DE, EF, FG, and GA; and the diagonals AC, CG, GD, DF

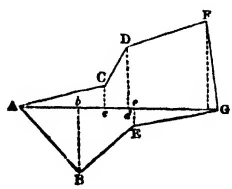
OTHERWISE.

Many pieces of land may be very well surveyed, by measuring any base line, either within or without them, together with the perpendiculars let fall upon it from every corner of them. For they are by those means divided into several triangles and trapezoids, all whose parallel sides are perpendicular to the base line; and the sum of these triangles and trapeziums will be equal to the figure proposed if the base line fall within it; if not, the sum of the parts which are without being taken from the sum of the whole, which are both within and without, will leave the area of the figure proposed.

In pieces that are not very large, it will be sufficiently exact to find the points, in the base line, where the several perpendiculars will fall, by means of the *cross*, and from thence measuring to the corners for the lengths of the perpendiculars.—And it will be most convenient to draw the line so as that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and measuring along the line AG, the distances and perpendiculars, on the right and left, are as below.

A*b*	315	350	*b*B
A*c*	440	70	*c*C
A*d*	585	320	*d*D
A*e*	610	·50	*e*E
A*f*	990	470	*f*F
AG	1020	0	

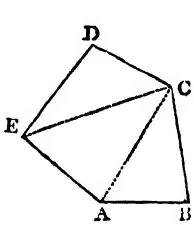


PROBLEM VII.

To survey any field with the plain table.

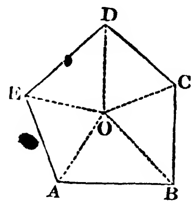
1.—FROM ONE STATION.

Plant the table at any angle, as C, from whence all the other angles, or marks set up, can be seen; turn the table about till the needle point to the flower-de-luce; and there screw it fast. Make a point for C on the paper on the table, and lay the edge of the index to C, turning it about C till through the sights you see the mark D; and by the edge of the index draw a dry or obscure line: then measure the distance CD, and lay that distance down on the line CD.



Then turn the index about the point C, till the mark E be seen through the sights, by which draw a line, and measure the distance to E, laying it on the line from C to E. In like manner determine the positions of CA and CB, by turning the sights successively to A and B; and lay the lengths of those lines down. Then connect the points with the boundaries of the field, by drawing the black lines CD, DE, EA, AB, BC.

2.—FROM A STATION WITHIN THE FIELD.



When all the other parts cannot be seen from one angle, choose some place O within; or even without, if more convenient, from whence the other parts can be seen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index successively to O, turning it round with the sights to each angle A, B, C, D, E, drawing dry lines to them by the edge of the index; then measuring the distances OA, OB, &c., and laying them down upon those lines. Lastly, draw the boundaries AB, BC, CD, DE, EA.

3.—BY GOING ROUND THE FIGURE.

When the figure is a wood, or water, or from some other obstruction you cannot measure lines across it; begin at any point A, and measure round it, either within or without the figure, and draw the directions of all the sides thus: Plant the table at A, and turn it with the needle to the north or flower-de-luce; fix it, and mark the point A. Apply the index to A, turning it till you can see the point E, there draw a line; then the point B, and there draw a line: then measure these lines, and lay them down from A to E and B. Next, move the table to B, lay the index along the line AB, and turn the table about till you can see the mark A, and screw fast the table; in which position also the needle will again point to the flower-de-luce, as it will do indeed at every station when the table is in the right position. Here turn the index about B till through the sights you see the mark C; there draw a line, measure BC, and lay the distance upon that line after you have set down the table at C. Turn it then again into its proper position, and in like manner find the next line CD. And so on quite round by E to A again. Then the proof of the work will be the joining at A: for if the work is all right, the last direction EA on the ground, will pass exactly through the point A on the paper; and the measured distance will also reach exactly to A. If these do not coincide, or nearly so, some error has been committed, and the work must be examined over agair

PROBLEM VIII.

To survey a field with the theodolite, &c.

1.—FROM ONE POINT OR STATION.

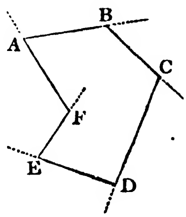
When all the angles can be seen from one point, as the angle C (first fig. to last problem), place the instrument at C, and turn it about till, through the fixed sights, you see the mark B, and there fix it. Then turn the moveable index about till the mark A is seen through the sights, and note the degrees cut on the instrument. Next turn the index successively to E and D, noting the degrees cut off at each; which gives all the angles BCA, BCE, BCD.

Lastly, measure the lines CB, CA, CE, CD; and enter the measures in a field-book, or rather against the corresponding parts of a rough figure, drawn by guess to resemble the field.

2.—FROM A POINT WITHIN OR WITHOUT.

Plant the instrument at O (last fig.), and turn it about till the fixed sights point to any object, as A; and there screw it fast. Then turn the moveable index round till the sights point successively to the other points E, D, C, B noting the degrees cut off at each of them; which gives all the angles round the point O. Lastly, measure the distances OA, OB, OC, OD, OE, noting them down as before, and the work is done.

3.—BY GOING ROUND THE FIELD.



By measuring round, either within or without the field, proceed thus: Having set up marks at B, C, &c. near the corners as usual, plant the instrument at any point A, and turn it till the fixed index be in the direction AB, and there screw it fast: then turn the moveable index to the direction AF; and the degrees cut off will be the angle A. Measure the line AB, and plant the instrument at B, and there in the same manner observe the angle B. Then measure BC, and observe the angle C. Then measure the distance CD, and take the angle D. Then measure DE, and take the angle E. Then measure EF, and take the angle F. And, lastly, measure the distance FA.

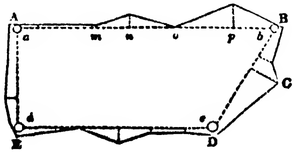
To prove the work: add all the inward angles A, B, C, &c., together; for when the work is right, their sum will be equal to twice as many right angles as the figure has sides, wanting four right angles. But when there is an angle, as F, that bends inwards, and you measure the external angle, which is less than two right angles, subtract it from four right angles, or 360 degrees, to give the internal angle greater than a semicircle, or 180 degrees.

OTHERWISE.

Instead of observing the internal angles, you may take the external angles, formed without the figure by producing the sides further out. And in this case when the work is right, their sum altogether will be equal to 360 degrees. But when one of them, as F, runs inwards, subtract it from the sum of the rest, to leave 360 degrees.

PROBLEM IX.

To survey a field with crooked hedges; &c.

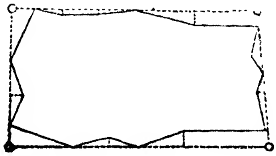


With any of the instruments, measure the lengths and positions of imaginary lines running as near the sides of the field as you can; and, in going along them, measure the offsets in the manner before taught; then you will have the plan on the paper in using the plain table, drawing the crooked hedges through the ends of the offsets; but in surveying with the theodolite, or other instrument, set down the measures properly in a field-book, or memorandum-book, and plan them after returning from the field, by laying down all the lines and angles.

So in surveying the piece ABCDE, set up marks *a*, *b*, *c*, *d*, dividing it into as few sides as may be. Then begin at any station *a*, and measure the lines *ab*, *bc*, *cd*, *da*, taking their positions, or the angles *a*, *b*. *c*. *d*; and, in going

along the lines, measure all the offsets, as at m, n, o, p, &c., along every station line.

And this is done either within the field, or without, as may be most convenient. When there are obstructions within, as wood, water, hills, &c., then measure without, as in the figure here given.

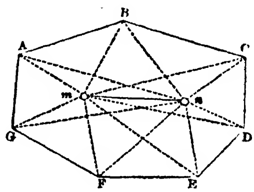


PROBLEM X.

To survey a field, or any other thing, by two stations.

This is performed by choosing two stations, from whence all the marks and objects can be seen; then measuring the distance between the stations, and at each station taking the angles formed by every object, from the station line or distance.

The two stations may be taken either within the bounds, or in one of the sides, or in the direction of two of the objects, or quite at a distance and without the bounds of the objects or part to be surveyed.



In this manner, not only grounds may be surveyed, without even entering them, but a map may be taken of the principal parts of a county, or the chief places of a town, or any part of a river or coast surveyed, or any other inaccessible objects; by taking two stations, on two towers, or two hills, or such like.

PROBLEM XI.

To survey a large estate.

If the estate be very large, and contain a great number of fields, it cannot well be done by surveying all the fields singly, and then putting them together; nor can it be done by taking all the angles and boundaries that inclose it. For in these cases, any small errors will be so multiplied, as to render it very much distorted.

1. Walk over the estate two or three times, in order to get a perfect idea of it, and till you can carry the map of it tolerably well in your head. And to help your memory, draw an eye draught of it on paper or at least of the principal parts of it, to guide you; setting the names within the fields in that draught.

2. Choose two or more eminent places in the estate, for stations, from whence all the principal parts of it can be seen; and let these stations be as far distant from one another as possible.

3. Take such angles, between the stations, as you think necessary, and measure the distances from station to station always in a right line: these things must be done till you get as many angles and lines as are sufficient for determining all the points of station. And in measuring any of these station distances, mark accurately where these lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c.; and where any remarkable object is placed, by measuring its distance from the station line; and where a perpendicular from it cuts that line. And thus as you go along any main station line, take offsets to the ends of all hedges, and to any pond, house, mill, bridge, &c., omitting nothing that is remarkable, and noting every thing down.

4. As to the inner parts of the estate, they must be determined in like manner, by new station lines: for after the main stations are determined, and every

thing adjoining to them, then the estate must be subdivided into two or three parts by new station lines; taking inner stations at proper places, where you can have the best view. Measure these station lines as you did the first, and all their intersections with hedges, and offsets to such objects as appear. Then proceed to survey the adjoining fields, by taking the angles that the sides make with the station line, at the intersections, and measuring the distances to each corner, from the intersections. For the station lines will be the bases to all the future operations; the situation of all parts being entirely dependent upon them; and therefore they should be taken of as great length as possible; and it is best for them to run along some of the hedges or boundaries of one or more fields, or to pass through some of their angles. All things being determined for these stations, you must take more inner stations, and continue to divide and subdivide till at last you come to single fields; repeating the same work for the inner stations, as for the outer ones, till all is done; and close the work as often as you can, and in as few lines as possible.

5. An estate may be so situated, that the whole cannot be surveyed together; because one part of the estate cannot be seen from another. In this case, you may divide it into three or four parts, and survey the parts separately, as if they were lands belonging to different persons; and at last join them together.

6. As it is necessary to protract or lay down the work as you proceed in it, you must have a scale of a due length to do it by. To get such a scale, measure the whole length of the estate in chains; then consider how many inches long the map is to be; and from these will be known how many chains you must have in an inch; then make the scale accordingly, or choose one already made.

THE NEW METHOD OF SURVEYING.

In the former method of measuring a large estate, the accuracy of it depends on the correctness of the instruments used in taking the angles. To avoid the errors incident to such a multitude of angles, other methods have of late years been used by some few skilful surveyors. The most practical, expeditious, and correct, seems to be the following:

Choose two or more eminences, as grand stations, and measure a principal base line from one station to the other, noting every hedge, brook, or other remarkable object as you pass by it; measuring also such short perpendicular lines to the bends of hedges as may be near at hand. From the extremities of this base line, or from any convenient parts of the same, go off with other lines to some remarkable object situated towards the sides of the estate, without regarding the angles they make with the base line or with one another; still remembering to note every hedge, brook, or other object that you pass by. These lines, when laid down by intersections, will, with the base line, form a grand triangle on the estate; several of which, if need be, being thus laid down, you may proceed to form other smaller triangles and trapezoids on the sides of the former: and so on, until you finish with the enclosures individually.

This grand triangle being completed, and laid down on the rough plan-paper, the parts, exterior as well as interior, are to be completed by smaller triangles and trapezoids.

In countries where the lands are enclosed with high hedges, and where many lanes pass through an estate, a theodolite may be used to advantage, in measuring the angles of such lands; by which means, a kind of skeleton of the estate may be obtained, and the lane lines serve as the bases of such triangles and trapezoids as are necessary to fill up the interior parts.

The field-book is ruled into three columns. In the middle one are set down the distances on the chain line at which any mark, offset, or other observation is made; and in the right and left hand columns are entered the offsets and observations made on the right and left hand respectively of the chain line.

It is of great advantage, both for brevity and perspicuity, to begin at the bottom of the leaf and write upwards; denoting the crossing of fences by lines drawn across the middle column, or only a part of such a line on the right and left opposite the figures, to avoid confusion; and the corners of fields, and other remarkable turns in the fences where offsets are taken to, by lines joining in the manner the fences do, as will be best seen by comparing the book with the plan annexed to the field-book following, page 462.

The letter in the left hand corner at the beginning of every line, is the mark or place measured *from;* and, that at the right hand corner at the end, is the mark measured *to.* But when it is not convenient to go exactly from a mark, the place measured from, is described *such a distance* from *one mark* towards *another;* and where a mark is not measured to, the exact place is ascertained by saying, turn to the right or left hand, *such a distance* to *such a mark,* it being always understood that those distances are taken in the chain line.

The characters used are, ⌈ for *turn to the right hand,* ⌉ for *turn to the left hand,* and a ∧ placed over an offset, to show that it is not taken at right angles with the chain line, but in the line with some straight fence; being chiefly used when crossing their directions, and it is a better way of obtaining their true places than by offsets at right angles.

When a line is measured whose position is determined, either by former work, (as in the case of producing a given line, or measuring from one known place or mark to another,) or by itself (as in the third side of a triangle) it is called a *fast line,* and a double line across the book is drawn at the conclusion of it; but if its position is not determined (as in the second side of a triangle), it is called a *loose line,* and a single line is drawn across the book. When a line becomes determined in position, and is afterwards continued, a double line half through the book is drawn.

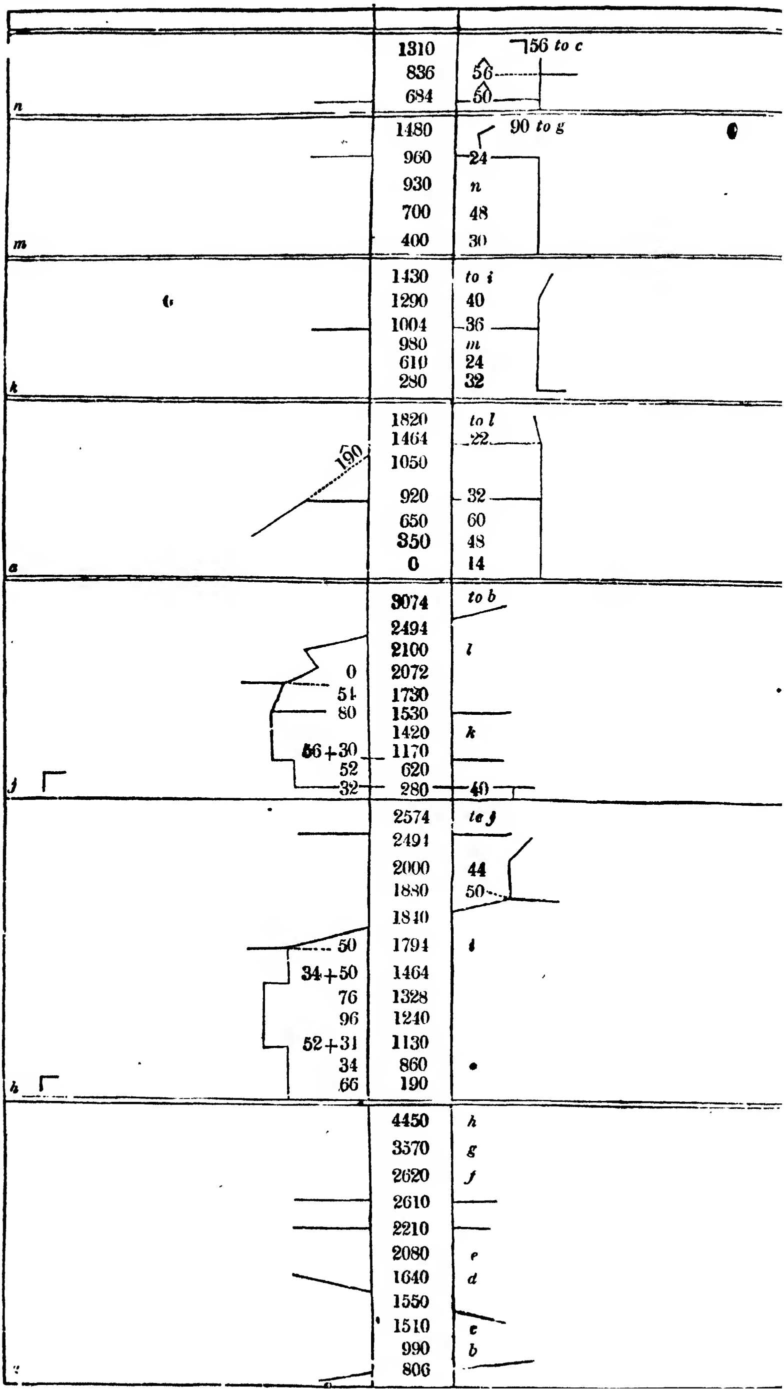
When a loose line is measured, it becomes absolutely necessary to measure some line that will determine its position. Thus, the first line *ah,* being the base of a triangle, is always determined; but the position of the second side *hj,* does not become determined, till the third side *jb* is measured; then the triangle may be constructed, and the position of both is determined.

At the beginning of a line, to fix a loose line to the mark or place measured from, the sign of turning to the right or left hand must be added (as at *j* in the third line); otherwise a stranger, when laying down the work, may as easily construct the triangle *hjb* on the wrong side of the line *ah,* as on the right one: but this error cannot be fallen into, if the sign above named be carefully observed.

In choosing a line to fix a loose one, care must be taken that it does not make a very acute or obtuse angle; as in the triangle *pBr,* by the angle at B being very obtuse, a small deviation from truth, even the breadth of a point at *p* or *r,* would make the error at B, when constructed, very considerable; but by constructing the triangle *pBq,* such a deviation is of no consequence.

Where the words *leave off* are written in the field-book, it is to signify that the taking of offsets is from thence discontinued; and of course something is wanting between that and the next offset.

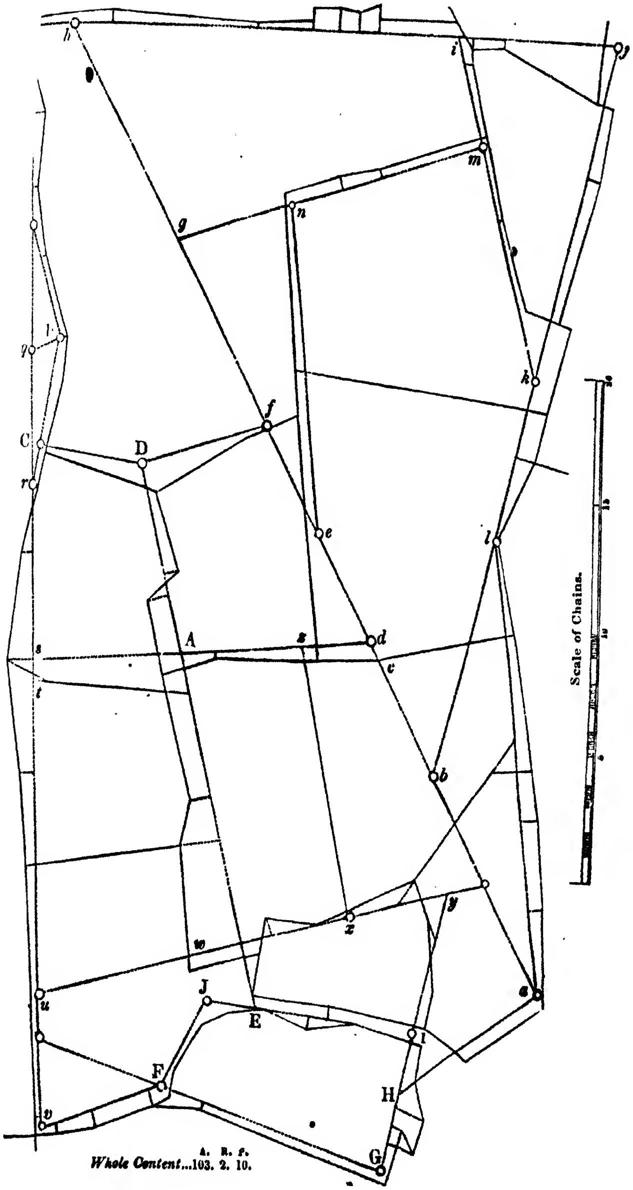
The field-book for this method, and the plan drawn from it, are contained in the four following pages.



1310
836 56
684 50
56 to c
n
1480
960 24
930 n
700 48
400 30
90 to g
m
1430 to i
1290 40
1004 36
980 m
610 24
280 32
k
1820 to l
1464 22
1050
920 32
650 60
350 48
0 14
190
a
3074 to b
2494
2100 l
0 2072
54 1730
80 1530
1420 k
56+30 1170
52 620
32 280 40
j
2574 to j
2494
2000 44
1880 50
1840
50 1794 i
34+50 1464
76 1328
96 1240
52+31 1130
34 860
66 190
h
4450 h
3570 g
2620 f
2610
2210
2080 e
1640 d
1550
1510 c
990 b
806

	768	*to* A
	526	70
	496	
70	460	
40	124	
	100	
	445	D
	400	76
	48	10
	600	*to r*
	432	C
56	160	
14	36	
	152	*to q*
	480	B
24	160	
0	1700	
	1560	44 *to s*
	980	
	885	A
44	668	
79	310	*z*
60	236	
	2148	480 *to b*
	1950	*y*
	1836	
128	1724	
60	1600	
30	1480	*x*
0	1320	
50	1110	
	1080	
	840	*w*
	750	50
	4440	36
	4420	*v*
	3884	*u*
	3380	60
	2992	90
	2692	*t*
120	2624	
	2592	
	2500	*s* *leave off*
	2070	56
	1900	
	1840	*r*
60	1770	
	1320	*q*
	808	*p*
leave off 40	650	
80	360	
20	170	
	220	0
h produced from l	190	46

		580	*to v*
	40	500	
	76	300	
F	76	100	
		420	*to* F
J	20	150	
		954	J
	15	850	
		740	*to* E
	30	490	
		340	60
	0	280	
I	20	170	50
		725	*to* H
		672	0
	70	450	0
u	50	15	
		1160	*to y*
	32	1000	
		890	
		780	32
		590	40
		570	I
		530	40
		376	H
		256	150
		190	64
G		111	130
		1676	G
		1676	30
		896	24
		632	
		620	50
180 *from u towards v*		588	F
		620	*to j*
D		488	32
		2260	
		2250	E
	80	2210	
	56	2050	
		2030	
		1990	130 *to w*
		1552	180
		1380	96
		950	110
		860	54



A. R. P.
Whole Content...103. 2. 10.

PROBLEM XII.

To survey a county, or large tract of land.

1. Choose two, three, or four eminent places, for stations; such as the tops of high hills or mountains, towers, or church steeples, which may be seen from one another; from which most of the towns and other places of note may also be seen; and so as to be as far distant from one another as possible. Upon these places raise beacons, or long poles, with flags of different colours flying at them, so as to be visible from all the other stations.

2. At all the places, which you would set down in the map, plant long poles with flags at them, of several colours, to distinguish the places from one another; fixing them on the tops of church steeples, or the tops of houses, or in the centres of lesser towns.

These marks being then set up at a convenient number of places, and such as may be seen from both stations; go to one of these stations, and, with an instrument to take angles, standing at that station, take all the angles between the other station and each of these marks. Then go to the other station, and take all the angles between the first station and each of the former marks, setting them down with the others, each against his fellow with the same colour. You may, if you can, also take the angles at some third station, which may serve to prove the work, if the three lines intersect in that point where any mark stands. The marks must stand till the observations are finished at both stations; and then they must be taken down, and set up at fresh places. The same operations must be performed, at both stations, for these fresh places; and the like for others. The instrument for taking angles must be an exceeding good one, made on purpose with telescopic sights; and of a good length of radius. A circumferentor is reckoned a good instrument for this purpose.

3. And though it be not absolutely necessary to measure any distance, because a stationary line being laid down from any scale, all the other lines will be proportional to it; yet it is better to measure some of the lines, to ascertain the distances of places in miles, and to know how many geometrical miles there are in any length; as also from thence to make a scale to measure any distance in miles. In measuring any distance, it will not be exact enough to go along the high roads; by reason of their turnings and windings, hardly ever lying in a right line between the stations, which must cause infinite reductions, and create endless trouble to make it a right line; for which reason it can never be exact. But a better way is to measure in a right line with a chain, between station and station, over hills and dales or level fields, and all obstacles. Only in case of water, woods, towns, rocks, banks, &c., where one cannot pass, such parts of the lines must be measured by the methods of inaccessible distances; and besides, allowing for ascents and descents, when they are met with. A good compass that shows the bearing of the two stations, will always direct you to go straight, when you do not see the two stations; and in the progress, if you can go straight, offsets may be taken to any remarkable places, likewise noting the intersection of the station line with all roads, rivers, &c.

4. From all the stations, and in the whole progress, be very particular in observing sea-coasts, river mouths, towns, castles, houses, churches, mills, trees, rocks, sands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, sluices, floodgates, locks, &c., and in general every thing that is remarkable.

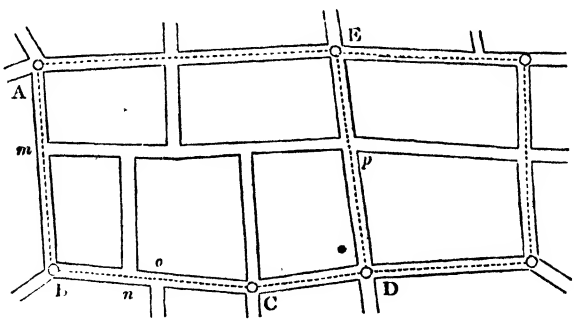
5. After you have done with the first and main station lines, which command the whole county; you must then take inner stations, at some places already determined; which will divide the whole into several partitions: and from these stations you must determine the places of as many of the remaining towns as you can. And if any remain in that part, you must take more stations, at some places already determined; from which you may determine the rest. And thus go through all the parts of the county, taking station after station, till we have determined all we want. And in general the station distances must always pass through such remarkable points as have been determined before by the former stations.

PROBLEM XIII.

To survey a town or city.

This may be done with any of the instruments for taking angles, but best of all with the plain table, where every minute part is drawn while in sight. It is best also to have a chain of fifty feet long, divided into fifty links of one foot each, and an offset staff of ten feet long.

Begin at the meeting of two or more of the principal streets, through which you can have the longest prospects, to get the longest station lines: there having fixed the instrument, draw lines of direction along those streets, using two men as marks, or poles set in wooden pedestals, or perhaps some remarkable places in the houses at the further ends, as windows, doors, corners, &c. Measure these lines with the chain, taking offsets with the staff at all corners of streets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houses, &c. Then remove the instrument to another station, along one of the lines; and there repeat the same process as before. And so on till the whole is finished.



Thus, fix the instrument at A, and draw lines in the direction of all the streets meeting there; then measure AB, noting the street on the left at *m*. At the second station B, draw the directions of the streets meeting there; and measure from B to C, noting the places of the streets at *n* and *o* as you pass by them. At the third station C, take the direction of all the streets meeting there, and measure CD. At D do the same, and measure DE, noting the place of the cross streets at *p*. And in this manner go through all the principal streets. This done, proceed to the smaller and intermediate streets; and lastly, to the lanes, alleys, courts, yards, and every part that it may be thought proper to represent in the plan.

SECTION III.

OF PLANNING, COMPUTING, AND DIVIDING.

PROBLEM I.

To lay down the plan of any survey.

If the survey was taken with a plain table, you have a rough plan of it already on the paper which covered the table. But if the survey was with any other instrument, a plan of it is to be drawn from the measures that were taken in the survey, and first of all a rough plan on paper.

To do this, you must have a set of proper instruments, for laying down both lines, angles, &c., as scales of various sizes, the more of them, and the more accurate, the better; scales of chords, protractors, perpendicular and parallel rulers, &c. Diagonal scales are best for the lines, because they extend to three figures, or chains and links, which are hundredth parts of chains. But in using the diagonal scale, a pair of compasses must be employed to take off the lengths of the principal lines very accurately. But a scale with a thin edge divided, is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of those offsets upon the station line; which is done at only one application of the edge of the scale to that line, and then pricking off all at once the distances along it. Angles are to be laid down either with a good scale of chords, which is perhaps the most accurate way; or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

In general, all lines and angles must be laid down on the plan in the same order in which they were measured in the field, and in which they are written in the field-book; laying down first the angles for the position of lines, next the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themselves, all with dry or obscure lines; then a black line drawn through the extremities of all the offsets, will be the hedge or bounding line of the field, &c. After the principal bounds and lines are laid down, and made to fit or close properly, proceed next to the smaller objects, till you have entered every thing that ought to appear in the plan, as houses, brooks, trees, hills, gates, stiles, roads, lanes, mills, bridges, woodlands, &c.

The north side of a map or plan is commonly placed uppermost, and a meridian somewhere drawn, with the compass or flower-de-luce pointing north. Also, in a vacant part, a scale of equal parts or chains is drawn, with the title of the map in conspicuous characters, and embellished with a compartment. Hills are shadowed, to distinguish them in the map. Colour the hedges with different colours; represent hilly grounds by broken hills and valleys; draw single dotted lines for foot-paths, and double ones for horse or carriage roads. Write the name of each field and remarkable place within it, and, if you choose, its content in acres, roods, and perches.

In a very large estate, or a county, draw vertical and horizontal lines through the map, denoting the spaces between them by letters placed at the top, and bottom, and sides, for readily finding any field or other object mentioned in a table.

In mapping counties, and large estates that have uneven grounds of hills and valleys, reduce all oblique lines, measured up-hill and down hill, to horizontal

straight lines, if that was not done during the survey, before they were entered in the field-book, by making a proper allowance to shorten them. For which purpose there is commonly a small table engraven on some of the instruments for surveying.

PROBLEM II.

To compute the contents of fields.

1. Compute the contents of the figures, whether triangles or trapeziums, &c., by the proper rules for the several figures laid down in measuring; multiply the lengths by the breadths, both in links, and divide by 2; the quotient is acres, after you have cut off five figures on the right for decimals. Then bring these decimals to roods and perches, by multiplying first by 4, and then by 40. An example of which has been already given in the description of the chain.

2. In small and separate pieces, it is usual to cast up their contents from the measures of the lines taken in surveying them, without making a correct plan of them.

3. In pieces bounded by very crooked and winding hedges, measured by offsets, all the parts between the offsets are most accurately measured separately as small trapezoids.

4. Sometimes such pieces as that last mentioned, are computed by finding a mean breadth, by dividing the sum of the offsets by the number of them, accounting that for one of them where the boundary meets the station line; then multiply the length by that mean breadth. — But this method is commonly in some degree erroneous.

5. But in larger pieces, and whole estates, consisting of many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents quite independent of the measures of the lines and angles that were taken in surveying. For, then, new lines are drawn in the fields in the plan, so as to divide them into trapeziums and triangles, the bases and perpendiculars of which are measured on the plan by means of the scale from which it was drawn, and so multiplied together for the contents. In this way, the work is very expeditiously done, and sufficiently correct; for such dimensions are taken as afford the most easy method of calculation; and, among a number of parts, thus taken and applied to a scale, it is likely that some of the parts will be taken a small matter too little, and others too great; so that they will, upon the whole, in all probability, very nearly balance one another. After all the fields and particular parts are thus computed separately, and added all together into one sum, calculate the whole estate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add these also together. Then if this sum be equal to the former, or nearly so, the work is right; but if the sums have any considerable difference, it is wrong, and they must be examined and recomputed, till they nearly agree.

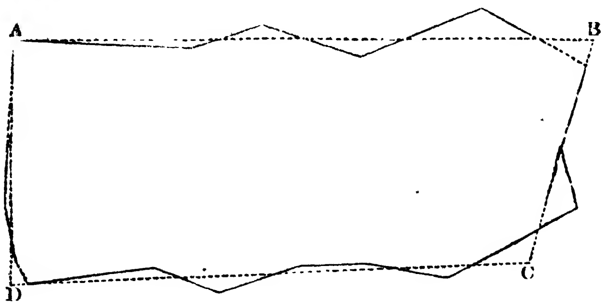
6. But the chief secret in computing consists in finding the contents of pieces bounded by curved or very irregular lines, or in reducing such crooked sides of fields or boundaries to straight lines, that shall inclose the same or equal area with those crooked sides, and so obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now, this reducing the crooked sides to straight ones, is very easily and accurately performed in this manner: Apply the straight edge of a thin, clear piece of lanthorn-horn to the crooked line which is to be reduced, in such a manner, that the small parts

cut off from the crooked figure by it, may be equal to those which are taken in: which equality of the parts included and excluded you will presently be able to judge of very nicely by a little practice; then with a pencil or point of a tracer, draw a line by the straight edge of the horn. Do the same by the other sides of the field or figure. So shall you have a straight-sided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the curved figure proposed.

Or, instead of the straight edge of the horn, a horse-hair may be applied across the crooked sides in the same manner; and the easiest way of using the hair, is to string a small slender bow with it, either of wire, or cane, or whalebone, or such like slender or elastic matter; for, the bow keeping it always stretched, it can be easily and neatly applied with one hand, while the other is at liberty to make two marks by the side of it, to draw the straight line by.

EXAMPLE.

Thus, let it be required to find the contents of the same figure as in problem IX. of the last section, to a scale of 4 chains to an inch.



Draw the four dotted straight lines AB, BC, CD, DA, cutting off equal quantities on both sides of them, which they do as near as the eye can judge: so is the crooked figure reduced to an equivalent right-lined one of four sides ABCD. Then draw the diagonal BD, which, by applying a proper scale to it, measures 1256. Also the perpendicular, or nearest distance, from A to this diagonal, measures 456; and the distance of C from it, is 428.

Then, half the sum of 456 and 428, multiplied by the diagonal 1256, gives 555,152 square links, or 5 acres, 2 roods, 8 perches, the content of the trapezium, or of the irregular crooked piece.

PROBLEM III.

To transfer a plan to another paper, &c.

After the rough plan is completed, and a fair one is wanted, this may be done by any of the following methods:

First Method.—Lay the rough plan on the clean paper, keeping them always pressed flat and close together, by weights laid on them. Then, with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them asunder, and connect the pricked points, on the clean paper, with lines; and it is done. This method is only to be practised in plans of such figures as are small and tolerably regular, or bounded by right lines.

Second Method.—Rub the back of the rough plan over with black-lead powder; and lay the said black part on the clean paper on which the plan is to

be copied, and in the proper position. Then with the blunt point of some hard substance, as brass, or such like, trace over the lines of the whole plan; pressing the tracer so much as that the black-lead under the lines may be transferred to the clean paper: after which, take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink. — Or, instead of blacking the rough plan, you may keep constantly a blacked paper to lay between the plans.

Third Method.—Another method of copying plans, is by means of squares. This is performed by dividing both ends and sides of the plan which is to be copied, into any convenient number of equal parts, and connecting the corresponding points of division with lines; which will divide the plan into a number of small squares. Then divide the paper, upon which the plan is to be copied, into the same number of squares, each equal to the former when the plan is to be copied of the same size, but greater or less than the others, in the proportion in which the plan is to be increased or diminished, when of a different size. Lastly, copy into the clean squares the parts contained in the corresponding squares of the old plan; and you will have the copy, either of the same size, or greater or less in any proportion.

Fourth Method.—A fourth method is by the instrument called a pentagraph, which also copies the plan in any size required.

Fifth Method.—But the neatest method of any is this. Procure a copying frame or glass, made in this manner: namely, a large square of the best window glass, set in a broad frame of wood, which can be raised up to any angle, when the lower side of it rests on a table. Set this frame up to any angle before you, facing a strong light; fix the old plan and clean paper together with several pins quite around, to keep them together, the clean paper being laid uppermost, and over the face of the plan to be copied. Lay them, with the back of the old plan, on the glass, namely, that part which you intend to begin at to copy first; and, by means of the light shining through the papers, you will very distinctly perceive every line of the plan through the clean paper. In this state then trace all the lines on the paper with a pencil. Having drawn that part which covers the glass, slide another part over the glass, and copy it in the same manner. Then another part. And so on, till the whole is copied.

Then take them asunder, and trace all the pencil lines over with a fine pen and Indian ink, or with common ink.

And thus you may copy the finest plan, without injuring it in the least.

When the lines are copied on the clean paper, the next business is to write such names, remarks, or explanations as may be judged necessary; laying down the scale for taking the lengths of any parts, a flower-de-luce to point out the direction, and the proper title ornamented with a compartment; illustrating or colouring every part in the manner that shall seem most natural, such as shading rivers or brooks with crooked lines; drawing the representations of trees, bushes, hills, woods, hedges, houses, gates, roads, &c., in their proper places; running a single dotted line for a footpath, and a double one for a carriage road; and either representing the bases or the elevations of buildings, &c.

OF ARTIFICERS' WORKS

AND

TIMBER MEASURING.

I.—OF THE CARPENTER'S OR SLIDING RULE.

THE Carpenter's or Sliding Rule, is an instrument much used in measuring of timber and artificers' works, both for taking the dimensions, and computing the contents.

The instrument consists of two equal pieces, each a foot in length, which are connected together by a folding joint.

One side or face of the rule is divided into inches, and eighths, or half quarters. On the same face also are several plane scales, divided into 12th parts by diagonal lines; which are used in planning dimensions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely, each foot into ten equal parts, and each of these into ten parts again; so that by means of this last scale, dimensions are taken in feet, tenths and hundredths, and multiplied as common decimal numbers, which is the best way.

On the one part of the other face are four lines, marked A, B, C, D; the two middle ones, B and C, being on a slider, which runs in a groove made in the stock. The same numbers serve for both these two middle lines,—the one being above the numbers, and the other below.

These four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from one to ten. The other or lowest line D, is a single one, proceeding from four to forty. It is also called the girt line, from its use in computing the contents of trees and timber; and upon it are marked WG at 17·15, and AG at 18·95, the wine and ale gauge points, to make this instrument serve the purpose of a gauging rule.

On the other part of this face, there is a table of the value of a load, 50 cubic feet of timber, at all prices, from sixpence to two shillings a foot.

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; but when 1 at the beginning is accounted 10, then the 1 in the middle is 100, and the 10 at the end 1000; and so on. And all the smaller divisions are altered proportionally.

II.—ARTIFICERS' WORK.

ARTIFICERS compute the contents of their works by several different measures; as,

Glazing and masonry by the foot;
Painting, plastering, paving, &c., by the yard, of 9 square feet·
Flooring, partitioning, roofing, tiling, &c., by the square of 100 square feet.

And brickwork, either by the yard of 9 square feet, or by the perch, or square rod or pole, containing $272\frac{1}{4}$ square feet, or $30\frac{1}{4}$ square yards, being the square of the rod or pole of $16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards long.

As this number $272\frac{1}{4}$ is troublesome to divide by, the $\frac{1}{4}$ is often omitted in practice, and the content in feet divided only by the 272. But when the exact divisor $272\frac{1}{4}$ is to be used, it will be easier to multiply the feet by 4, and then divide successively by 9, 11, and 11. Also to divide square yards by $30\frac{1}{4}$, first multiply them by 4, and then divide twice by 11.

All works, whether superficial or solid, are computed by the rules proper to the figure of them, whether it be a triangle or rectangle, a parallelopiped or any other figure.

III.—BRICKLAYERS' WORK.

Brickwork is estimated at the rate of a brick and a half thick. So that, if a wall be more or less than this standard thickness, it must be reduced to it, as follows:

Multiply the superficial content of the wall by the number of half bricks in the thickness, and divide the product by 3.

The dimensions of a building are usually taken by measuring half round on the outside and half round on the inside; the sum of these two gives the compass of the wall,—to be multiplied by the height, for the content of the materials.

Chimneys are by some measured as if they were solid, deducting only the vacuity from the hearth to the mantle, on account of the trouble of them.

And by others they are girt or measured round for their breadth, and the height of the story is their height, taking the depth of the jambs for their thickness. And in this case, no deduction is made for the vacuity from the floor to the mantle-tree, because of the gathering of the breast and wings, to make room for the hearth in the next story.

To measure the chimney shafts, which appear above the building, girt them about with a line for the breadth, to multiply by their height. And account their thickness half a brick more than it really is, in consideration of the plastering and scaffolding,

All windows, doors, &c., are to be deducted out of the contents of the walls in which they are placed. But this deduction is made only with regard to materials; for the whole measure is taken for workmanship, and that all outside measure too, namely, measuring quite round the outside of the building, being in consideration of the trouble of the returns or angles. There are also some other allowances, such as double measure for feathered gable ends, &c.

EXAMPLES.

Ex. I.—How many yards and rods of standard brick-work are in a wall whose length or compass is 57 feet 3 inches, and height 24 feet 6 inches; the walls being $2\frac{1}{2}$ bricks or 5 half bricks thick? Ans. 8 rods, $17\frac{3}{8}$ yards.

Ex. II.—Required the content of a wall 62 feet 6 inches long, and 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick? Ans. 169·753 yards.

Ex. III.—A triangular gable is raised $17\frac{1}{2}$ feet high, on an end wall whose length is 24 feet 9 inches, the thickness being two bricks; required the reduced content? Ans. $32{\cdot}08\frac{1}{3}$ yards.

Ex. iv.—The end wall of a house is 28 feet 10 inches long, and 55 feet 8 inches high, to the eaves; 20 feet high is $2\frac{1}{2}$ bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is $1\frac{1}{2}$ brick thick; above which is a triangular gable, 1 brick thick, and which rises 42 courses of bricks, of which every 4 courses make a foot. What is the whole content in standard measure? Ans. 253·626 yards.

IV.—MASONS' WORK.

To masonry belong all sorts of stone-work; and the measure made use of is a foot, either superficial or solid.

Walls, columns, blocks of stone or marble, &c., are measured by the cubic foot; and pavements, slabs, chimney-pieces, &c., by the superficial or square foot.

Cubic or solid measure is used for the materials, and square measure for the workmanship.

In the solid measure, the true length, breadth, and thickness, are taken, and multiplied continually together. In the superficial, there must be taken the length and breadth of every part of the projection, which is seen without the general upright face of the building.

EXAMPLES.

Ex. i.—Required the solid content of a wall, 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick? Ans. $1310\frac{3}{4}$ feet.

Ex. ii.—What is the solid content of a wall, the length being 24 feet 3 inches, height 10 feet 9 inches, and 2 feet thick? Ans. 521·375 feet.

Ex. iii.—Required the value of a marble slab, at 8s. per foot; the length being 5 feet 7 inches, and breadth 1 foot 10 inches? Ans. £4, 1*s*. $10\frac{1}{2}$*d*.

Ex. iv.—In a chimney-piece, suppose the

Length of the mantle and slab, each	4 feet	6 inches;
Breadth of both together, ...	3	2
Length of each jamb,..................	4	4
Breadth of both together, ...	1	9

Required the superficial content? Ans. 21 feet, 10 inches.

V.—CARPENTERS' AND JOINERS' WORK.

To this branch belongs all the wood-work of a house, such as flooring, partitioning; roofing, &c.

Large and plain articles are usually measured by the square foot or yard, &c., but enriched mouldings, and some other articles, are often estimated by running or lineal measures, and some things are rated by the piece.

In measuring of joists, it is to be observed, that only one of their dimensions is the same with that of the floor; for the other exceeds the length of the room by the thickness of the wall and $\frac{1}{3}$ of the same, because each end is let into the wall about $\frac{2}{3}$ of its thickness.

No deductions are made for hearths, on account of the additional trouble and waste of materials.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

No deduction is made for door-ways, on account of the trouble of framing them.

In measuring of joiners' work, the string is made to ply close to every part of the work over which it passes.

The measure for centering for cellars is found by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length; but in groin centering, it is usual to allow double measure, on account of their extraordinary trouble.

In *roofing*, the length of the house in the inside, together with $\frac{2}{3}$ of the thickness of one gable, is to be considered as the length; and the breadth is equal to double the length of a string which is stretched from the ridge down the rafter, and along the eaves-board, till it meets with the top of the wall.

For stair-cases, take the breadth of all the steps, by making a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step, for the whole area.—By the length of a step is meant the length of the front and the returns at the two ends; and by the breadth, is to be understood the girt of its two outer surfaces, or the tread and riser.

For the balustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel post, for the length; and twice the length of the baluster upon the landing, with the girt of the hand-rail, for the breadth.

For wainscotting, take the compass of the room for the length; and the height from the floor to the ceiling, making the string ply close into all the mouldings, for the breadth.—Out of this must be made deductions for windows, doors, and chimneys, &c., but workmanship is counted for the whole, on account of the extraordinary trouble.

For doors, it is usual to allow for their thickness, by adding it into both the dimensions of length and breadth, and then multiply them together for the area. If the door be pannelled on both sides, take double its measure for the workmanship; but if the one side only be pannelled, take the area and its half for the workmanship.—*For the surrounding architrave*, gird it about the outermost parts for its length; and measure over it, as far as it can be seen when the door is open for the breadth.

Window-shutters, *bases*, &c., are measured in the same manner.

In the measuring of roofing for workmanship alone, holes for chimney shafts and skylights are generally deducted.

But in measuring for work and materials, they commonly measure in all skylights, luthern-lights, and holes for the chimney shafts, on account of their trouble and waste of materials.

EXAMPLES.

Ex. I.—Required the content of a floor 48 feet 6 inches long, and 24 feet 3 inches broad? Ans. 11 squares, $76\frac{1}{2}$ feet.

Ex. II.—A floor being 36 feet 3 inches long, and 16 feet 6 inches broad, how many squares are in it? Ans. 5 squares, $98\frac{1}{8}$ feet.

Ex. III.—How many squares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning? Ans. 18·3972 squares.

Ex. IV.—What cost the roofing of a house at 10*s.* 6*d.* a square; the length within the walls being 52 feet 8 inches, and the breadth 30 feet 6 inches,—reckoning the roof $\frac{3}{2}$ of the flat? Ans. £12, 12*s.* $11\frac{3}{4}$*d.*

Ex. v.—To how much, at 6*s.* per square yard, amounts the wainscotting of a room; the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compass 83 feet 8 inches; also the three window-shutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the door and shutters, being worked on both sides, are reckoned work and half work? Ans. £36, 12s. 2½*d.*

IV.—SLATERS' AND TILERS' WORK.

In these articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of slates at the bottom, or for how much one row of slates or tiles is laid over another.

When the roof is of a true pitch, that is, forming a right angle at top, then the breadth of the building with its half added, is the girt over both sides.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip.

Deductions are made for chimney shafts or window holes.

EXAMPLES.

Ex. I.—Required the content of a slated roof, the length being 45 feet 9 inches, and the whole girt 34 feet 3 inches? Ans. 174·104 yards.

Ex. II.—To how much amounts the tiling of a house, at 25*s.* 6*d.* per square; the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches, also the eaves projecting 16 inches on each side, and the roof of a true pitch? Ans. £24, 9*s.* 5½*d.*

VII.—PLASTERERS' WORK.

Plasterers' work is of two kinds, namely, ceiling—which is plastering upon laths—and rendering, which is plastering upon walls; which are measured separately

The contents are estimated either by the foot or yard, or square of 100 feet. Enriched mouldings, &c., are rated by running or lineal measure.

Deductions are to be made for chimneys, doors, windows, &c. But the windows are seldom deducted, as the plastered returns at the top and sides are allowed to compensate for the window opening.

EXAMPLES.

Ex. I.—How many yards contains the ceiling, which is 43 feet 3 inches long, and 25 feet 6 inches broad? Ans. 122·541.

Ex. II.—To how much amounts the ceiling of a room, at 10*d.* per yard; the length being 21 feet 8 inches, and the breadth 14 feet 10 inches?

£1, 9*s.* 8¾*d.*

Ex. III.—The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering,—the former at 8*d.* and the latter at 3*d.* per yard,—allowing for the door of 7 feet by 3 feet 8 inches, and a fire-place of 5 feet square?

Ans. £1. 13*s.* 3¼*d.*

Ex. IV.—Required the quantity of plastering in a room, the length being 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 3 inches to the under side of the cornice, which girts $8\frac{1}{2}$ inches, and projects 5 inches from the wall on the upper part next the ceiling—deducting only for a door 7 feet by 4?

Ans. 53 yards	5 feet	3 inches	of rendering,
18	5	6	of ceiling,
	39	$0\frac{1}{12}$	of cornice.

VIII.—PAINTERS' WORK.

PAINTERS' work is computed in square yards. Every part is measured where the colour lies; and the measuring line is forced into all the mouldings and corners.

Windows are done at so much a piece. And it is usual to allow double measure for carved mouldings, &c.

EXAMPLES.

Ex. I.—How many yards of painting contains the room which is 65 feet 6 inches in compass, and 12 feet 4 inches high? Ans. $89\frac{3}{4}$ yards.

Ex. II.—The length of a room being 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows each 6 feet by 3 feet 2 inches? Ans. $73\frac{2}{27}$ yards.

Ex. III.—What costs the painting of a room at 6*d.* per yard; its length being 24 feet 6 inches, its breadth 16 feet 3 inches, and height 12 feet 9 inches; also the door is 7 feet by 3 feet 6 inches, and the window-shutters to two windows each 7 feet 9 inches by 3 feet 6 inches, but the breaks of the windows themselves are 8 feet 6 inches high, and 1 foot 3 inches deep—deducting the fire-place of 5 feet by 5 feet 6 inches? Ans. £3, 3*s.* $10\frac{1}{2}$*d.*

IX.—GLAZIERS' WORK.

GLAZIERS take their dimensions either in feet, inches, and parts, or feet, tenths, and hundredths. And they compute their work in square feet.

In taking the length and breadth of a window, the cross bars between the squares are included. Also, windows of round or oval forms are measured as square, measuring them to their greatest length and breadth, on account of the waste in cutting the glass.

EXAMPLES.

Ex. I.—How many square feet contains the window which is 4·25 feet long, and 2·75 feet broad? Ans. $11\frac{11}{16}$.

Ex. II.—What will the glazing a triangular skylight come to, at 10*d.* a foot; the base being 12 feet 6 inches and the perpendicular height 6 feet 9 inches? Ans. £1. 15*s.* $1\frac{1}{4}$*d.*

Ex. III.—There is a house with three tier of windows, three windows in each tier, their common breadth 3 feet 11 inches; now,

The height of the first tier is	7 feet	10 inches,
.... of the second	6	8
.... of the third	5	4

Required the expense of glazing, at 14*d.* per foot? Ans. £13, 11*s.* $10\frac{1}{4}$*d.*

Ex. IV.—Required the expense of glazing the windows of a house at 13*d.* a foot; there being three stories, and three windows in each story;

The height of the lower tier is	7 feet	9 inches,
.... of the middle	6	6
.... of the upper	5	$3\frac{1}{4}$
and of an oval window over the door	1	$10\frac{1}{2}$

the common breadth of all the windows being 3 feet 9 inches?

Ans. £12, 5*s.* 6*d.*

X.—PAVERS' WORK.

Pavers' work is done by the square yard. And the content is found by multiplying the length by the breadth.

EXAMPLES.

Ex. I.—What cost the paving a footpath at 3*s.* 4*d.* a yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches? Ans. £5, 7*s.* $11\frac{1}{4}$*d.*

Ex. II.—What cost the paving a court, at 3*s.* 2*d.* per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches? Ans. £7, 4*s.* $5\frac{1}{4}$*d.*

Ex. III.—What will be the expense of paving a rectangular court-yard, whose length is 63 feet, and breadth 45 feet; in which there is laid a footpath of 5 feet 3 inches broad, running the whole length, with broad stones, at 3*s.* a yard—the rest being paved with pebbles, at 2*s.* 6*d.* a yard? Ans. £40, 5*s.* $10\frac{1}{2}$*d.*

XI.—PLUMBERS' WORK.

Plumbers' work is rated at so much a pound, or else by the hundred weight, of 112 pounds.

Sheet lead used in roofing, guttering, &c., is from 7 to 12 lb. to the square foot. And a pipe of an inch bore is commonly 13 to 14 lb. to the yard in length.

EXAMPLES.

Ex. I.—How much weighs the lead which is 39 feet 6 inches long, and 3 feet 3 inches broad, at $8\frac{1}{2}$ lb. to the square foot? Ans. $1091\frac{2}{11}$ lb.

Ex. II.—What cost the covering and guttering a roof with lead, at 18*s.* the cwt.; the length of the roof being 43 feet, and breadth or girt over it 32 feet—the guttering 57 feet long, and 2 feet wide; the former 9·831 lb., and the latter 7·373 lb. to the square foot? Ans. £115, 9*s.* $1\frac{1}{2}$*d.*

XII.—TIMBER MEASURING.

PROBLEM I.

To find the area or superficial content of a board or plank.

Multiply the length by the mean breadth.

Note.—When the board is tapering, add the breadths at the two ends together, and take half the sum for the mean breadth.

BY THE SLIDING RULE.

Set 12 on B on the breadth in inches on A; then against the length in feet on B is the content on A, in feet and fractional parts.

EXAMPLES.

Ex. I.—What is the value of a plank, at $1\frac{1}{2}d.$ per foot, whose length is 12 feet 6 inches, and mean breadth 11 inches? Ans. 1*s.* 5*d.*

Ex. II.—Required the content of a board, whose length is 11 feet 2 inches, and breadth 1 foot 10 inches. Ans. 20 feet, 5 inches, 8″.

Ex. III.—What is the value of a plank, which is 12 feet 9 inches long, and 1 foot 3 inches broad, at $2\frac{1}{2}d.$ a foot? Ans. 3*s.* $3\frac{3}{4}d.$

Ex. IV.—Required the value of 5 oaken planks at 3*d.* per foot, each of them being $17\frac{1}{2}$ feet long, and their several breadths as follows; namely, two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and $11\frac{1}{4}$ at the narrower. Ans. £1, 5*s.* $9\frac{1}{4}d.$

PROBLEM II.

To find the solid content of squared or four-sided timber

Multiply the mean breadth by the mean thickness, and the product again by the length, and the last product will give the content.

BY THE SLIDING RULE.

C	D	D	C
As length :	12 or 10 ::	quarter girt :	solidity.

That is, as the length in feet on C, is to 12 on D when the quarter girt is in inches, or to 10 on D when it is in tenths of feet; so is the quarter girt on D, to the content on C.

Note 1.—If the tree taper regularly from the one end to the other; either take the mean breadth and thickness in the middle, or take the dimensions at the two ends, and half their sum will be the mean dimensions.

Note 2.—If the piece do not taper regularly, but is unequally thick in some parts and small in others, take several different dimensions, add them all together, and divide their sum by the number of them, for the mean dimensions.

Note 3.—The quarter girt, is a geometrical mean proportional between the mean breadth and thickness, that is the square root of their product. Sometimes unskilful measurers use the arithmetical mean instead of it, that is, half their sum; but this is always attended with error, and the more so as the breadth and depth differ the more from each other.

EXAMPLES.

Ex. I.—The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and less end 1 foot 6 inches and 1 foot 3 inches, and the thickness at the greater and less end 1 foot 3 inches and 1 foot: required the solid content? Ans. 28 feet 7 inches.

Ex. II.—What is the content of the piece of timber whose length is $24\frac{1}{2}$ feet, and the mean breadth and thickness each 1·04 feet? Ans. $26\frac{1}{2}$ feet.

Ex. III.—Required the content of a piece of timber, whose length is 20·38 feet, and its ends unequal squares, the side of the greater being $19\frac{1}{8}$, and the side of the less $9\frac{7}{8}$? Ans. 29·7562 feet.

Ex. IV.—Required the content of the piece of timber whose length is 27·36 feet,—at the greater end the breadth is 1·78, and thickness 1·23; and at the less end the breadth is 1·04, and thickness 0·91? Ans. 41·278 feet.

PROBLEM III.

To find the solidity of round or unsquared timber.

Multiply the square of the quarter girt, or of $\frac{1}{4}$ of the mean circumference, by the length, for the content.

BY THE SLIDING RULE.

As the length upon C : 12 or 10 upon D ::
quarter girt, in 12ths or 10ths on D : content on C.

Note 1.—When the tree is tapering, take the mean dimensions as in the former problems, either by girting it in the middle, for the mean girt, or at the two ends, and take half the sum of the two. But when the tree is very irregular, divide it into several lengths, and find the content of each part separately.

Note 2.—This rule, which is commonly used, gives the answer about $\frac{1}{4}$ less than the true quantity in the tree, or nearly what the quantity would be after the tree is hewed square in the usual way; so that it seems intended to make an allowance for the squaring of the tree.

EXAMPLES.

Ex. I.—A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches; what is the content? Ans. $116\frac{1}{3}$ feet.

Ex. II.—The length of a tree is 24 feet, its girt at the thicker end 14 feet, and at the smaller end 2 feet required the content? Ans. 96 feet.

Ex. III.—What is the content of a tree, whose mean girt is 3·15 feet, and length 14 feet 6 inches? Ans. 8·9922 feet.

Ex. IV.—Required the content of a tree, whose length is $17\frac{1}{4}$ feet, which girts in five different places as follows; namely, in the first place 9·43 feet, in the second 7·92, in the third 6·15, in the fourth 4·74, and in the fifth 3·16?

Ans. 42·519525

PRACTICAL QUESTIONS IN MENSURATION

1. A plank is 14 feet 3 inches long, and I would have just a square yard slit off it; at what distance from the edge must the line be struck?

Ans. $7\frac{1}{19}$ inches.

2. A wooden trough cost 3*s.* 2*d.* painting within, at 6*d.* per yard; the length of it is 102 inches, and the depth 21 inches; what is the width?

Ans. $27\frac{1}{4}$ inches.

3. The paving of a triangular court, at 18*d.* per foot, came to £100; the longest of the three sides is 88 feet; required the sum of the other two equal sides? Ans. 106·85 feet.

4. What is the side of that equilateral triangle whose area cost as much paving at 8*d.* a foot, as the palisading the three sides did at a guinea a yard?

Ans. 72·746 feet.

5. Let a, b, c be the sides of a triangle respectively opposite to the angles A, B, C; then will the area of the triangle ABC be

$$= \tfrac{1}{2}\, a^2 \text{ sin. B sin. C cosec. A.}$$

6. Let a, b, c be the three sides of a triangle; put $h=b+c$ and $k=b-c$; then will the area of the triangle be $=\frac{1}{4}\sqrt{(h^2-a^2)(a^2-k^2)}$.

7. Let the three sides be $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$; then prove that the area of the triangle is $=\frac{1}{4}\sqrt{2(ab+bc+ca)-(a^2+b^2+c^2)}$.

8. A beam is $8\frac{1}{2}$ inches deep and $3\frac{1}{2}$ inches broad; what is the depth of another twice as large, which is $4\frac{3}{4}$ inches broad? Ans. 12·5263 inches.

9. Supposing the expense of paving a semicircular plot, at 2*s.* 4*d.* per foot, come to £10; what is its diameter? Ans. 14·7737 feet.

10. Two sides of an obtuse angled triangle are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land?

Ans. 58·876, or 23·099.

11. A circular fish-pond is to be made in a garden that shall enclose just half an acre; what must be the length of the chord that strikes the circle?

Ans. $27\frac{3}{4}$ yards.

12. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder, parallel to the longer side; and this alternately

repeated till there shall not be the quantity of a foot left; what will be the dimensions of the remaining piece? * Ans. 20·702 inches by 6·086.

13. If a round pillar, 7 inches across, have 4 feet of stone in it, what is the diameter of a column of equal length, that contains 10 times as much?

Ans. 22·136 inches.

14. Find the thickness of the lead in a pipe of an inch and a quarter bore, which weighs 14lb. per yard in length, the cubic foot of lead weighing 11325 ounces. Ans. ·20737 inches.

15. Let $C=$ outer circumference of a circular ring, $b=$ its breadth, and $\pi=3·1416$; then the area of the ring will be $=b(C-\pi b)$.

16. What will be the expense of a curb to a round well, at 8*d*. per foot square, the breadth of the curb being 8 inches, and the interior diameter $3\frac{1}{2}$ feet. Ans. 5*s*. $9\frac{3}{4}$*d*.

17. A garden is 100 feet long and 80 feet broad, and a gravel walk is to be made of an equal width half round it, so as to occupy half the garden; find both by *construction* and *calculation*, the breadth of the walk.

Ans. 25·968.

18. If the sides of a triangle are 28, 25, and 17, what is the area of its greatest inscribed square? Ans. 101·67124.

19. Let a, b, c be the distances between a tree and three corners of a square field in a successive order; then will the area of the field be

$$=a^2+b^2-ab\sqrt{2}\,(\cos.\,\phi-\sin\,\phi),\text{ where }\cos.\,\phi=\frac{a^2+2b^2-c^2}{2ab\sqrt{2}}.$$

20. The four sides of a field, whose diagonals are equal to each other, are known to be 25, 35, 31, and 19 poles, in a successive order; required the content of the field.

A. R. P.

Ans. $\frac{1}{2}(793+7\sqrt{8449})$ sq. poles, or 4 1 38.

21. The length and breadth of a vessel in the form of a rectangular parallelopepid are respectively 6 and 3 feet; what must be its depth, to contain exactly 200 imperial gallons? Ans. 1 foot 9·4 inches.

22. Seven men bought a grinding stone of 60 inches diameter, each paying $\frac{1}{7}$ part of the expense; what part of the diameter must each grind down for his share?

Ans. The 1st, 4·4508; 2d, 4·8400; 3d, 5·3535; 4th, 6·0765; 5th, 7·2079; 6th, 9·3935; 7th, 22·6778 inches.

23. Divide a cone into three equal parts by sections parallel to the base, and find the altitudes of the three parts, the height of the cone being 20 inches.

Ans. The upper part, 13·8672; the middle, 3·6044; the lower, 2·5284.

24. What quantity of canvass is necessary for a conical tent whose perpendicular height is 8 feet, and the radius at bottom $6\frac{1}{2}$ feet?

Ans. $210\frac{1}{2}$ sq. feet.

* This question admits of an elegant general solution, as may be seen in the "Ladies' Diary" for 1823. In this example, the student will find that the slab contains more than 10, but less than 11 feet; hence the operation must be performed 10 times, and the dimensions of the remaining part of a foot will be the answer.

25. A cable 3 feet long, and 9 inches in compass, weighs 22lb.; what will be the weight of a fathom of that cable whose circumference is a foot?

Ans. $78\frac{2}{9}$lb.

26. If R and r be the radii of two spheres inscribed in a cone, so that the greater may touch the less, and also the base of the cone; then will the capacity of the cone be $= \dfrac{2\pi R^5}{3r(R-r)}$.

27. If a heavy sphere whose diameter is 4 inches be let fall into a conical glass full of water, whose diameter is 5, and altitude 6 inches; it is required to determine how much water will run over. Ans. 26·272 cubic inches.

28. The dimensions of a sphere and cone being as in last question, and the cone only $\frac{1}{8}$ full of water; what part of the axis of the sphere is immersed in the water?* Ans. ·5459 inches.

29. Let A° represent the number of degrees in the arc of a segment of a circle whose radius is r; it is required to prove that

$$\text{area of segment} = \tfrac{1}{2} r^2 \{\text{arc A}^\circ \text{ (radius 1)} - \sin \text{A}^\circ\}.$$

30. What is the length of a chord which cuts off $\frac{1}{3}$ of the area from a circle of which the diameter is 289? Ans. 278·6716.

31. How high above the surface of the earth must a person be raised to see $\frac{1}{3}$ of its surface? Ans. The height of its diameter.

32. If a cubic foot of brass be drawn into wire of $\frac{1}{40}$ inch diameter, what will be the length of the wire, supposing no loss of metal in working?

Ans. 97784·5684 yards, or 55 miles 984·5684 yards.

33. Supposing the diameter of an iron 9lb. ball to be 4 inches, as it is very nearly; it is required to find the diameters of the several balls weighing 12, 18, 24, 32, and 36lb., and the calibre of their guns, allowing $\frac{1}{20}$ of the calibre, or $\frac{1}{19}$ of the ball's diameter for windage.

Answer.

Weight.	12	18	24	32	36
Diameter.	4·4026	5.0397	5·5469	6·1051	6·3496
Calibre.	4·4924	5·1425	5·6601	6·2297	6·4792

* This question admits of a beautiful algebraical solution, by putting x = the part of the axis immersed. The resulting cubic equation is easily rendered a complete power, and then the cube root being taken, gives finally a simple equation for the determination of x the part immersed.

ANALYTICAL

PLANE TRIGONOMETRY.

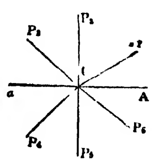
CHAPTER I.

INTRODUCTION AND DEFINITIONS.

Plane Trigonometry, as the name imports, was originally employed solely in determining, from certain data, the sides and angles of plane triangles. In modern analysis, however, its objects have been much extended, and the formulæ of this branch of Mathematics are extensively employed as instruments of calculation in almost every department of scientific investigation. From this circumstance, some writers wishing to change its designation to one which might more fully express its nature and applications, have proposed to term it the *Arithmetic of Sines*, others, the *Calculus of Angular Functions*, but the original appellation is still retained by the great majority of authors upon these subjects.

In treating of angular magnitude, we have hitherto confined ourselves to the consideration of angles less than two right angles, but in trigonometry it is frequently necessary to introduce angles which are greater than two, than three, or even than four right angles. We may take the following method of illustrating the generation of angular magnitude.

Let Aa be a fixed straight line, and let a line CP be supposed to revolve round the point C in Aa, and to assume in succession the different positions CP_1, CP_2, CP_3, ...



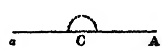
When CP coincides with CA, there is no angle contained between CP and CA, or the angle CAP is 0.

When CP begins to revolve round C, and comes into the position CP_1, it forms with CA an angle P_1CA less than a right angle.

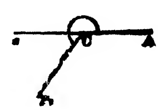
When CP has performed one-fourth part of an entire revolution, and has thus reached the position CP_2, where CP_2 is perpendicular to CA, it forms with CA the angle P_2CA, which is a right angle.

As CP continues its revolutions, it will assume the position CP_3, forming with CA the angle P_3CA, greater than one, and less than two right angles.

When CP coincides with Ca, it has performed one-half of an entire revolution, and forms with CA the angle aCA, equal to two right angles.

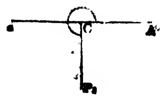


CP having passed Ca, will assume the position CP_4, forming with CA the angle P_4CA, greater than two, and less than three, right angles.

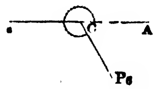


The *dotted space* indicates the angle which we are considering.

When CP has performed three-fourths of an entire revolution, it assumes the position CP_5, where CP_5 is perpendicular to Aa, forming with CA the angle P_5CA, equal to three right angles.



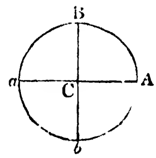
Passing beyond CP_5, the revolving line assumes the position CP_6, forming with CA the angle P_6CA, greater than three, and less than four, right angles.



Finally, when the line CP has completed an entire revolution, it will return to its original position CA, having formed with CA an angle equal to four right angles.

If we suppose the revolution to recommence, it is manifest that CP may be conceived to form with CA angles greater than four, than five, or than any given number of right angles.

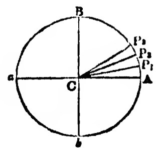
It is convenient in trigonometrical investigations, to draw two straight lines at right angles to each other, and from their point of intersection to describe a circle, with any radius cutting these lines in any points A, B, a, b.



The circumference of the circle will thus be divided into four equal arcs AB, Ba, ab, bA, each of which, being a fourth part of the whole circumference, is called *a quadrant*, and subtends a right angle at the centre of the circle.

AB is called the *first quadrant*, Ba the *second quadrant*, ab the *third quadrant*, and bA the *fourth quadrant*.

If each of these right angles be divided by straight lines CP_1, CP_2, ... into 90 equal angles, the whole circumference will be divided into a corresponding number of equal parts, each of which is called a *degree*. The whole circumference will thus contain 360 degrees, and each quadrant will contain 90 degrees.



The angles themselves, and the arcs which subtend them, are called *degrees* indifferently.

Angles are usually designated by the number of degrees which they contain; thus, a right angle is called an angle of 90 degrees; two right angles, an angle of 180 degrees, &c.

If each degree be divided into 60 equal parts, each of these smaller angles is called a *minute*.

If each minute be divided into 60 equal parts, each of these smaller angles is called a *second*.

Thus, four right angles, or the entire circumference of a circle, contains 360 degrees; 360×60, or 21,600 minutes; $360 \times 60 \times 60$, or 1,296,000 seconds.

Degrees are expressed in writing by placing a small cypher immediately above the number to the right; thus $90°$, $45°$, $63°$, signify 90 degrees, 45 degrees, 63 degrees, &c.

Minutes are expressed by placing one accent in the same manner above the number, and seconds by placing two accents: thus, $35'$, $40'$, &c. signify 35 minutes, 40 minutes, &c.; and $35''$. $40''$, signify 35 seconds, 40 seconds, &c.

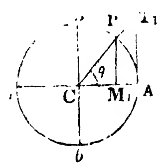
Any lower subdivision of a degree is usually expressed in decimal parts of a second.

Another division of the circle has been introduced by some modern authors, especially the French. They divide the whole circumference of the circle into four hundred equal parts or degrees, each degree into one hundred minutes, and each minute into one hundred seconds. This method possesses many practical advantages over the former; but the number of valuable tables calculated according to the former system, will in all probability prevent it from being generally adopted. It is called the *decimal division* of the circle, in contradistinction to the former which is called the *sexagesimal division.*

This being premised, we shall proceed to define the more important trigonometrical terms.

1. The *complement* of an angle is the defect of an angle from ninety degrees. Thus, if θ be any angle, the complement of θ is $(90^\circ - \theta)$.

2. The *supplement* of an angle is the defect of an angle from one hundred and eighty degrees. Thus, if θ be any angle, the supplement of θ is $(180^\circ - \theta)$.



Draw two straight lines Aa, Bb, at right angles to each other, intersecting in the point C. With centre C and any distance as radius, describe a circle, cutting the straight lines in the points A, B, a, b.

Draw the radius CP_1, forming with CA any angle $ACP_1 = \theta$.

From P_1 draw P_1M_1 perpendicular on Aa.

From A draw AT_1 a tangent to the circle at A.

Produce CP_1 to meet AT_1 in T_1.

3. Then the ratio of P_1M_1 to the radius of the circle, is called the *sine* of the angle P_1CA.

Or, $$\frac{P_1M_1}{CA} = \text{sin. } \theta$$

4. The ratio of AT_1 to the radius of the circle is called the *tangent* of the angle P_1CA.

Or, $$\frac{AT_1}{CA} = \text{tan. } \theta$$

5. The ratio of CT_1 to the radius of the circle is called the *secant* of the angle P_1CA.

Or, $$\frac{CT_1}{CA} = \text{sec. } \theta$$

6. The ratio of AM_1 to the radius of the circle, is called the *versed sine* of the angle P_1CA.

Or, $$\frac{AM_1}{CA} = \text{v. sin. } \theta$$

7. The sine of the complement of any given angle is called the *cosine* of that angle.

Or, $\text{sin. } (90^\circ - \theta) = \text{cos. } \theta$ and $\therefore \text{cos. } (90^\circ - \theta) = \text{sin. } \theta$.

8. The tangent of the complement of any given angle is called the *cotangent* of that angle.

Or, $\text{tan. } (90^\circ - \theta) = \text{cot. } \theta$. and $\therefore \text{cot. } (90^\circ - \theta) = \text{tan. } \theta$.

9. The secant of the complement of any given angle is called the *cosecant* of that angle.

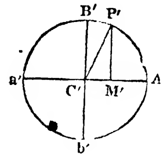
Or, $\text{sec. } (90^\circ - \theta) = \text{cosec. } \theta$ and $\therefore \text{cosec. } (90^\circ - \theta) = \text{sec. } \theta$

10. The versed sine of the complement of any angle is called the *co-versed sine* of that angle.

Or, v. sin. $(90^\circ - \theta)$ = co-v. sin. θ and $\therefore$ co-v. sin. $(90^\circ - \theta) =$ v sin. θ

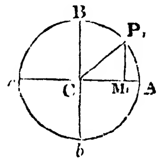
We shall now prove that the ratio of CM_1 to the radius of the circle, in the last figure, is the cosine of the angle P_1CA, that is, the sine of its complement.

Or that, $\dfrac{CM_1}{CA} = \cos.\ \theta.$



Draw a circle $A'B'a'b'$, equal to the circle $ABab$, and from C' the centre draw $C'P'$ making with $C'A'$ the angle $P'C'A'$ equal to the angle P_1CB, *i. e.* to the complement of P_1CA, or to $(90^\circ - \theta)$.

Then since CP_1 is equal to $C'P'$, and the angles at M_1 and M' right angles, and the angle CP_1M, equal to the angle $P'C'M'$, the two triangles P_1CM_1, $P'C'M'$, are equal in every respect, $P_1M_1 = C'M'$, $CM_1 = P'M'$.

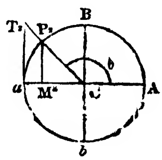


$$\therefore \frac{CM_1}{CA} = \frac{P'M'}{CA}$$
$= \sin.\ P'C'A'$ by Def.
$= \sin.\ (90^\circ - \theta)$ by construct.
$= \cos.\ \theta$ by Def.

We have hitherto considered an angle P_1CA less than a right angle, but the same definitions are applied whatever may be the magnitude of the angle

Thus, for example, let us take an angle P_2CA situated in the second quadrant, that is, an angle greater than one right angle and less than two.

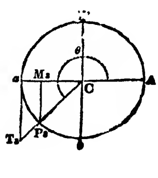
From P_2 let fall P_2M_2 perpendicular on Aa, from a draw aT_2 a tangent to the circle at a meeting CP produced in T_2; then as before,



$$\frac{P_2M_2}{CA} = \sin.\ P_2CA$$
$$\frac{CM_2}{CA} = \cos.\ P_2CA$$
$$\frac{aT_2}{CA} = \tan.\ P_2CA$$
$$\frac{CT_2}{CA} = \sec.\ P_2CA$$
$$\frac{AM_2}{CA} = \text{v sin. } P_2CA.$$

Again, let the angle in question be situated in the third quadrant, that is, let it be an angle greater than two and less than three right angles.

Making a construction analogous to that in the two former cases, we shall have



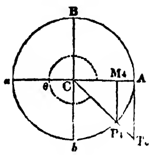
$$\frac{P_3M_3}{CA} = \sin.\ P_3CA$$
$$\frac{CM_3}{CA} = \cos.\ P_3CA$$
$$\frac{aT_3}{CA} = \tan.\ P_3CA$$

$$\frac{CT_3}{CA} = \text{sec. } P_3CA.$$

$$\frac{AM_3}{CA} = \text{v. sin. } P_3CA$$

Lastly, let the angle be situated in the fourth quadrant; that is, let it be an angle greater than three and less than four right angles, then as before

$$\frac{P_4M_4}{CA} = \text{sin. } P_4CA$$



$$\frac{CM_4}{CA} = \text{cos. } P_4CA$$

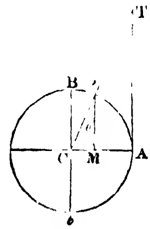
$$\frac{AT_4}{CA} = \text{tan. } P_4CA$$

$$\frac{CT_4}{CA} = \text{sec. } P_4CA$$

$$\frac{AM_4}{CA} = \text{v. sin. } P_4CA$$

We shall now proceed to establish some important general relations between the trigonometrical quantities which are immediately deducible from the above definitions, and from the principles of Geometry.

Resuming the figure of Def. (2):



Since CMP is a right-angled triangle and CP the hypotenuse,

$$PM^2 + CM^2 = CP^2$$

Dividing by CP^2,

$$\frac{PM^2}{CP^2} + \frac{CM^2}{CP^2} = 1$$

$$\therefore \text{sin.}^2\theta + \text{cos.}^2\theta = 1 \text{ .... } (1)^*$$

The triangles PMC, TAC, are equiangular and similar; hence

$$\frac{PM}{CM} = \frac{AT}{CA}$$

$$\therefore \frac{\frac{PM}{CA}}{\frac{CM}{CA}} = \frac{AT}{CA}$$

$$i.e. \ \frac{\text{sin. }\theta}{\text{cos. }\theta} = \text{tan. }\theta \text{ .... } (2)$$

In last case, for θ substitute $(90^\circ - \theta)$; then

$$\frac{\text{sin. }(90^\circ - \theta)}{\text{cos. }(90^\circ - \theta)} = \text{tan. }(90^\circ - \theta)$$

$$\text{Or, } \frac{\text{cos. }\theta}{\text{sin. }\theta} = \text{cot. }\theta \text{ .... } (3)$$

From (2) and (3) we have

$$\frac{\text{sin. }\theta}{\text{cos. }\theta} = \text{tan. }\theta, \text{ and, } \frac{\text{cos. }\theta}{\text{sin. }\theta} = \text{cot }\theta.$$

$$\textbf{Hence,} \quad \text{tan. }\theta = \frac{1}{\text{cot. }\theta}; \text{ or, tan. }\theta \text{ cot. }\theta = 1 \text{ .... } (4)$$

* The symbols $\text{sin.}^2\theta$, $\text{cos.}^2\theta$, $\text{tan.}^2\theta$, &c., signify *the square of sin. θ, the square of cos. θ*, &c., this is the common notation;—another, more strictly in accordance with analogy, is sometimes employed to express the same thing, viz. $(\text{sin. }\theta)^2$, $(\text{cos. }\theta)^2$, $(\text{tan. }\theta)^2$, &c.

By similar triangles CTA, CPM.

$$\frac{CT}{CA} = \frac{CP}{CM}$$

$$= \frac{1}{\frac{CM}{CP}}$$

Or, $$\sec.\ \theta = \frac{1}{\cos.\ \theta}, \text{ or, } \sec.\ \theta \cos.\ \theta = 1 \ \ldots\ldots\ldots(5)$$

By Definition,

$$\text{cosec.}\ \theta = \sec.\ (90^\circ - \theta)$$

$$= \frac{1}{\cos.\ (90^\circ - \theta)} \text{ by last case}$$

$$= \frac{1}{\sin.\ \theta} \text{ or, cosec. } \theta \sin.\ \theta = 1 \ldots(6)$$

Since CAT is a right-angled triangle and CT the hypotenuse,

$$CA^2 + AT^2 = CT^2$$

Dividing by CA^2,

$$1 + \frac{AT^2}{CA^2} = \frac{CT^2}{CA^2}$$

i. e. $$1 + \tan.^2\ \theta = \sec.^2\ \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

By (3) we have

$$\cot.\ \theta = \frac{\cos.\ \theta}{\sin.\ \theta}$$

$$\therefore \cot.^2\ \theta = \frac{\cos.^2\ \theta}{\sin.^2\ \theta}$$

Adding 1 to each side of the equation,

$$1 + \cot.^2\ \theta = 1 + \frac{\cos.^2\ \theta}{\sin.^2\ \theta}$$

$$= \frac{\sin.^2\ \theta + \cos.^2\ \theta}{\sin.^2\ \theta}$$

$$= \frac{1}{\sin.^2\ \theta} \text{ by (1)}$$

$$= \text{cosec.}^2\ \theta \text{ by (6)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

By Definition,

$$\text{versin.}\ \theta = \frac{MA}{CA}$$

$$= \frac{CA - CM}{CA}$$

$$= 1 - \frac{CM}{CA}$$

$$= 1 - \cos.\ \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

By Definition,

$$\text{coversin.}\ \theta = \text{versin.}\ (90^\circ - \theta)$$

$$= 1 - \cos.\ (90^\circ - \theta), \text{ by last case.}$$

$$= 1 - \sin.\ \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

The above results, which are of the highest importance in all trigonometrical investigations, may be collected and arranged in the following Table, which ought to be committed to memory:—

TABLE I.

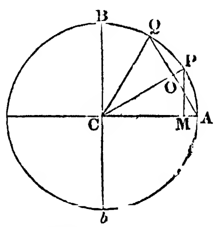
1.	$\sin.^2 \theta + \cos.^2 \theta$	$=$	1
2.	$\dfrac{\sin. \theta}{\cos. \theta}$	$=$	$\tan. \theta$
3.	$\dfrac{\cos. \theta}{\sin. \theta}$	$=$	$\cot. \theta$
4.	$\tan. \theta \cot. \theta$	$=$	1
5.	$\sec. \theta \cos. \theta$	$=$	1
6.	$\text{cosec.} \theta \sin. \theta$	$=$	1
7.	$1 + \tan.^2 \theta$	$=$	$\sec.^2 \theta$
8.	$1 + \cot.^2 \theta$	$=$	$\text{cosec.}^2 \theta$
9.	$\text{v. sin.} \theta$	$=$	$1 - \cos. \theta$
10.	$\text{coversin.} \theta$	$=$	$1 - \sin. \theta.$

11. The *chord* of an arc is the ratio of the straight line joining the two extremities of the arc to the radius of the circle.

PROPOSITION.

The chord of any arc is equal to twice the sine of half the arc.

Take any arc AQ, subtending at the centre of the circle the angle ACQ $= \theta$.



Draw the straight line CP bisecting the angle ACQ.

Join A, Q; from P let fall PM perpendicular on CA.

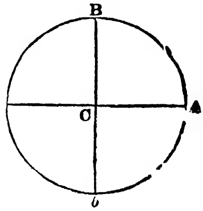
Since CP bisects ACQ, the vertical angle of the isosceles triangle ACQ, it bisects the base AQ at right angles.

$$\therefore \quad AO = OQ, \text{ and the angles at O are right angles.}$$

Again, since the triangles AOC, PMC, have the angles CMP, COA, right angles, and the angle PCM common to the two triangles, and also the side CP of the one equal to the side CA of the other, these triangles are in every respect equal.

$$\therefore \quad PM = AO = OQ$$
$$\therefore \quad AQ = 2PM$$
$$\therefore \quad \frac{AQ}{CA} = 2\frac{PM}{CA}$$
$$\text{or, chord } \theta = 2 \sin. PCA$$
$$= 2 \sin. \frac{\theta}{2}$$

We shall now proceed to explain the principle by which the *signs* of the trigonometrical quantities are regulated.



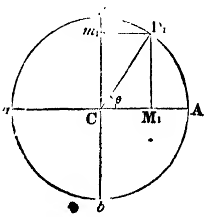
All lines measured from the point C along CA, that is, *to the right*, are considered positive, or have the sign $+$.

All lines measured from the point C along Ca, that is, in the opposite direction, *to the left*, are considered negative, or have the sign $-$.

All lines measured from the point C along CB, that is, *upwards*, are considered positive, or have the sign $+$.

All lines measured from the point C along Cb, that is, in the opposite direction *downwards*, are considered negative, or have the sign $-$.

Let us determine according to this principle, the signs of the sines and cosines of angles in the different quadrants.

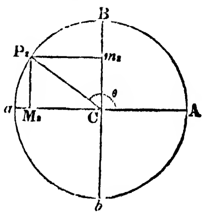


In the first quadrant, sin. $\theta = \frac{P_1M_1}{CA}$, cos. $\theta = \frac{CM_1}{CA}$

Here $P_1M_1 = Cm_1$ is reckoned from C along CB upwards, and is $\therefore$ positive.

CM_1 is reckoned from C along CA, to the right, and is $\therefore$ positive.

In the first quadrant, therefore, *the sine and cosine are both positive.*

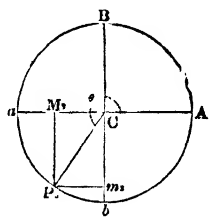


In the second quadrant, sin. $\theta = \frac{P_2M_2}{CA}$, cos. $\theta = \frac{CM_2}{CA}$

Here $P_2M_2 = Cm_2$ is reckoned from C along CB upwards, and is $\therefore$ positive.

CM_2 is reckoned from C along Ca, to the left, and is $\therefore$ negative.

In the second quadrant, therefore, *the sine is positive, and the cosine is negative.*

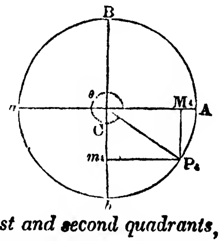


In the third quadrant, sin. $\theta = \frac{P_3M_3}{CA}$, cos. $\theta = \frac{CM_3}{CA}$

Here $P_3M_3 = Cm_3$ is reckoned from C along Cb, downwards, and is $\therefore$ negative.

CM_3 is reckoned from C along Ca, to the left, and is $\therefore$ negative.

In the third quadrant, therefore, *the sine and cosine are both negative.*



In the fourth quadrant, sin. $\theta = \frac{P_4M_4}{CA}$, cos. $\theta = \frac{CM_4}{CA}$

Here $P_4M_4 = Cm_4$ is reckoned from C along Cb, downwards, and is $\therefore$ negative.

CM_4 is reckoned from C along CA, to the right, and is $\therefore$ positive.

In the fourth quadrant, therefore, *the sine is negative and the cosine positive.*

Hence we conclude, that, *the sine is positive in the first and second quadrants, and negative in the third and fourth; and the cosine is positive in the first and fourth, and negative in the second and third;* or, in other words,

The sine of an angle less than 180° *is positive, and the sine of an angle greater than* 180° *and less than* 360° *is negative.*

The cosine of an angle less than 90° *is positive, the cosine of an angle greater than* 90° *and less than* 270° *is negative, and the cosine of an angle greater than* 270° *and less than* 360° *is positive.*

The signs of the sine and cosine being determined, the signs of all the other trigonometrical quantities may be at once established by referring to the relations in Table I.

Thus, for the tangent,

$$\tan.\ \theta = \frac{\sin.\ \theta}{\cos.\ \theta}$$

Hence, it appears that when the sine and cosine have the same sign, the tangent will be positive, and when they have different signs it will be negative.

Therefore, *the tangent is positive in the first and third quadrants, and negative in the second and fourth.*

The same holds good for the cotangent; for

$$\cot. \theta = \frac{\cos. \theta}{\sin. \theta}$$

Again, since

$$\sec. \theta = \frac{1}{\cos. \theta}$$

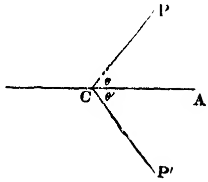
the *sign of the secant is always the same with that of the cosine;* and, since

$$\operatorname{cosec.} \theta = \frac{1}{\sin. \theta}$$

in like manner, *the sign of the cosecant is always the same with that of the sine.*

The versed sine is always positive, being reckoned from A always in the same direction.

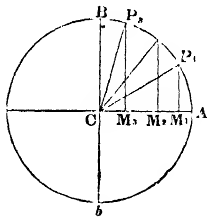
It is sometimes convenient to give different signs to angles themselves. We have hitherto supposed angles of different magnitudes to be generated by the revolution of the moveable radius CP round C in a direction from right to left; and the angles so formed have been considered positive, or affected with the sign $+$. If we now suppose an angle $\theta' = \theta$ to be generated by the revolution of the radius CP′ in the opposite direction, we may, upon a principle analogous to the former, consider the angle θ' as negative, and affect it with the sign $-$.



We shall now determine the variations in the magnitude of the sine and cosine for angles of different magnitudes.

In the first quadrant:

Let CP_1, CP_2, CP_3, ... be different positions of the revolving radius in the first quadrant; and from P_1, P_2, P_3, ... draw P_1M_1, P_2M_2, P_3M_3, ... perpendiculars on CA.



It is manifest, that as the angle increases the sine increases; for

$$\frac{P_2M_2}{CA} > \frac{P_1M_1}{CA} \text{ and } \frac{P_3M_3}{CA} > \frac{P_2M_2}{CA}$$

When the angle becomes very small, PM becomes very small also; and when the revolving radius coincides with CA, *i.e.* when the angle becomes 0, then PM disappears altogether, and is $= 0$.

Hence since, generally, $\sin. \theta = \frac{PM}{CA}$ and since, when $\theta = 0$, $PM = 0$;

$$\therefore \sin. 0 = \frac{0}{CA}$$
$$= 0$$

On the other hand, when the angle becomes equal to 90°, PM coincides with CB, and is equal to it.

Hence since, generally $\sin. \theta = \frac{PM}{CA}$, and since, when $\theta = 90°$, $PM = CB$,

$$\therefore \quad \sin. 90^\circ = \frac{CB}{CA}$$
$$= 1; \quad \because \quad CB = CA.$$

Again, it is manifest, that as the angle increases the cosine diminishes; for

$$\frac{CM_1}{CA} > \frac{CM_2}{CA} \text{ and } \frac{CM_2}{CA} > \frac{CM_3}{CA}$$

When the angle is very small, CM is very nearly equal to CA; and when the revolving radius coincides with CA, *i. e.* when the angle is 0, then CM coincides with CA and is equal to it.

Hence since, generally, $\cos. \theta = \frac{CM}{CA}$, and since, when $\theta = 0$, $CM = CA$;

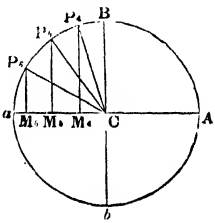
$$\therefore \quad \cos. 0 = \frac{CA}{CA}$$
$$= 1$$

On the other hand, as the angle increases, CM diminishes, and when the angle becomes equal to 90°, CM disappears altogether, and is $= 0$.

Hence since, generally, $\cos. \theta = \frac{CM}{CA}$, and since, when $\theta = 90^\circ$, $CM = 0$;

$$\therefore \quad \cos. 90^\circ = \frac{0}{CA}$$
$$= 0$$

Let us now take different positions of the revolving radius *in the second quadrant.*



It is manifest, that as the angle increases the sine diminishes; for

$$\frac{P_4M_4}{CA} > \frac{P_5M_5}{CA} \text{ and } \frac{P_5M_5}{CA} > \frac{P_6M_6}{CA}$$

As the angle goes on increasing, PM goes on diminishing; and when CP coincides with C*a*, *i. e.* when the angle becomes equal to 180°, PM disappears altogether, and is equal 0.

Hence since generally, $\sin. \theta = \frac{PM}{CA}$, and since, when $\theta = 180^\circ$ $PM = 0$;

$$\therefore \sin. 180^\circ = 0.$$

On the other hand, as the angle increases the cosine increases; for

$$\frac{CM_4}{CA} < \frac{CM_5}{CA} \text{ and } \frac{CM_5}{CA} < \frac{CM_6}{CA}$$

and when the revolving radius coincides with C*a* and the angle becomes 180°, CP coincides with C*a* and is equal to it.

Hence since, generally, $\cos. \theta = \frac{CM}{CA}$ and since, when $\theta = 180^\circ$, $CM = Ca$;

$$\therefore \quad \cos. 180^\circ = \frac{Ca}{CA}$$
$$= -1; \quad \because \quad Ca = CA.$$

The negative sign is here employed, because the cosine is reckoned *to the left* along C*a*.

Reasoning in the same manner for the third and fourth quadrants, we shall

$$\begin{aligned} \sin. 270° &= -1 \\ \cos. 270° &= 0 \\ \sin. 360° &= 0 \\ \cos. 360° &= 1. \end{aligned}$$

Thus, it appears,

That, as the angle increases in the first quadrant, from 0 up to 90°.
The sine, being positive, increases from 0 up to 1,
The cosine, being positive, decreases from 1 down to 0.

That, as the angle increases in the second quadrant, from 90° up to 180°,
The sine, being positive, decreases from 1 down to 0,
The cosine, being negative, increases * from 0 up to -1.

That, as the angle increases in the third quadrant, from 180° up to 270°
The sine, being negative, increases * from 0 up to -1,
The cosine, being negative, decreases * from -1 down to 0.

That, as the angle increases in the fourth quadrant, from 270° up to 360°,
The sine, being negative, decreases * from -1 down to 0,
The cosine, being positive, increases from 0 up to 1.

The variations in the magnitude of the sine and cosine being known, those of the other trigonometrical quantities may be determined by means of the relations in Table I.

Thus, since $\tan. \theta = \dfrac{\sin. \theta}{\cos. \theta}$

$$\tan. 0 = \frac{\sin. 0}{\cos. 0} = \frac{0}{1} = 0$$

$$\tan. 90° = \frac{\sin. 90°}{\cos. 90°} = \frac{1}{0} = \infty$$

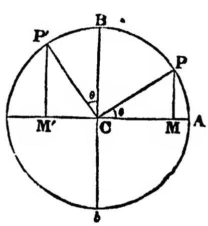
The truth of this last relation may be readily illustrated, by referring to the geometrical construction; when it will be seen, that for an angle of 90°, AT becomes parallel to CP; and therefore, the point T, in which the two lines meet, is at an infinite distance.

So, also, $\cot. 0 = \infty$

$\cot. 90° = 0$

and so for all the rest.

We shall next proceed to point out some important general relations, which exist between the trigonometrical functions of angles less than 90° and those of angles greater than 90°.



Draw CP, making with CA any angle PCA, which we may call θ; let fall PM perpendicular from P on CA. Draw CP', making with BC the angle BCP' $=$ PCA $= \theta$; and from P' let fall P'M' perpendicular on Ca.

Then the angle P'CA $= 90° + \theta$.

The two triangles PCM, P'CM', have the side PC of the one equal to the side P'C of the other, also the angles at M and M' right angles, and the angle CPM of the one equal to the angle P'CM' of the other; ∴ the two triangles are in every respect equal; and "

* That is, considered *absolutely* or independently of its sign.

$$PM = CM', \quad CM = P'M'.$$

$$\therefore \frac{P'M'}{CA} = \frac{CM}{CA}$$

$$\text{Or,} \quad \sin. P'CA = \cos. PCA,$$

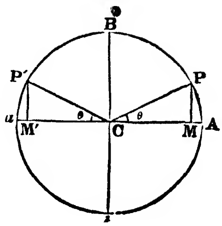
$$i.\,e. \quad \sin. (90^\circ + \theta) = \cos. \theta.$$

Again,

$$\frac{CM'}{CA} = \frac{PM}{CA}$$

$$\text{Or,} \quad -\cos. P'CA = \sin. PCA$$

$$i.\,e. \quad \cos. (90^\circ + \theta) = -\sin. \theta$$



As before, draw CP, making any angle θ with CA, and draw CP′, making with Ca the angle P′Ca, equal to θ.

Then the angle $P'CA = 180^\circ - \theta$.

The two triangles PCM, P′CM′, are manifestly in all respects equal; and

$$PM = P'M', \quad CM = CM'$$

$$\therefore \frac{PM}{CA} = \frac{P'M'}{CA}$$

$$i.\,e. \quad \sin. \theta = \sin. (180^\circ - \theta),$$

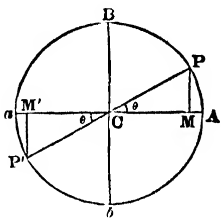
an important proposition, which enunciated in words is, *The sine of an angle is equal to the sine of its supplement.*

Again,

$$\frac{CM}{CA} = \frac{CM'}{CA}$$

$$\cos. \theta = -\cos. (180^\circ - \theta)$$

that is, *The cosine of an angle and the cosine of its supplement are equal in absolute magnitude, but have opposite signs.*



If, as in the annexed figure, we draw CP′, making with Ca an angle aCP′ equal to the angle θ, we shall find, in like manner,

$$\sin. (180^\circ + \theta) = -\sin. \theta$$

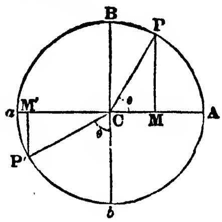
$$\cos. (180^\circ + \theta) = \cos. (180^\circ - \theta)$$

$$= -\cos. \theta.$$

If we draw CP′, making with Cb an angle $b\text{CP}' = \theta$, then

$$\sin. (270^\circ - \theta) = -\cos. \theta$$

$$\cos. (270^\circ - \theta) = -\sin. \theta$$



as is evident from the Def. 7, and the rule for signs; and, in like manner, we may proceed for angles in the fourth quadrant.

These relations being established between the sines and cosines, the corresponding relations between the other trigonometrical functions, may be deduced immediately from Table I.

Thus,

$$\tan. (90^\circ + \theta) = \frac{\sin. (90^\circ + \theta)}{\cos. (90^\circ + \theta)}$$

$$= \frac{\cos. \theta}{-\sin. \theta}$$

$$= -\cot. \theta$$

$$\tan.(180^\circ - \theta) = \frac{\sin.(180^\circ - \theta)}{\cos.(180^\circ - \theta)}$$
$$= \frac{\sin.\theta}{-\cos.\theta}$$
$$= -\tan.\theta$$

and so for all the rest.

The student may exercise himself, by verifying such of the results in the following table as have not been formally demonstrated.

TABLE II.

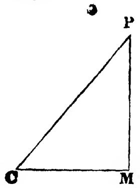
* $\sin. 0$	$= 0$	* $\sin.(180^\circ + \theta)$	$= -\sin.\theta$
* $\cos. 0$	$= 1$	* $\cos.(180^\circ + \theta)$	$= -\cos.\theta$
* $\tan. 0$	$= 0$	$\tan.(180^\circ + \theta)$	$= \tan.\theta$
* $\cot. 0$	$= \infty$	$\cot.(180^\circ + \theta)$	$= \cot.\theta$
$\sec. 0$	$= 1$	$\sec.(180^\circ + \theta)$	$= -\sec.\theta$
$\text{cosec}. 0$	$= \infty$	$\text{cosec}.(180^\circ + \theta)$	$= -\text{cosec}.\theta$
* $\sin.(90^\circ - \theta)$	$= \cos.\theta$	$\sin.(270^\circ - \theta)$	$= -\cos.\theta$
* $\cos.(90^\circ - \theta)$	$= \sin.\theta$	$\cos.(270^\circ - \theta)$	$= -\sin.\theta$
* $\tan.(90^\circ - \theta)$	$= \cot.\theta$	$\tan.(270^\circ - \theta)$	$= \cot.\theta$
* $\cot.(90^\circ - \theta)$	$= \tan.\theta$	$\cot.(270^\circ - \theta)$	$= \tan.\theta$
$\sec.(90^\circ - \theta)$	$= \text{cosec}.\theta$	$\sec.(270^\circ - \theta)$	$= -\text{cosec}.\theta$
$\text{cosec}.(90^\circ - \theta)$	$= \sec.\theta$	$\text{cosec}.(270^\circ - \theta)$	$= -\sec.\theta$
* $\sin. 90^\circ$	$= 1$	$\sin. 270^\circ$	$= -1$
* $\cos. 90^\circ$	$= 0$	$\cos. 270^\circ$	$= 0$
* $\tan. 90^\circ$	$= \infty$	$\tan. 270^\circ$	$= -\infty$
* $\cot. 90^\circ$	$= 0$	$\cot. 270^\circ$	$= 0$
$\sec. 90^\circ$	$= \infty$	$\sec. 270^\circ$	$= \infty$
$\text{cosec}. 90^\circ$	$= 1$	$\text{cosec}. 270$	$= -1$
* $\sin.(90^\circ + \theta)$	$= \cos.\theta$	$\sin.(270^\circ + \theta)$	$= -\cos.\theta$
* $\cos.(90^\circ + \theta)$	$= -\sin.\theta$	$\cos.(270^\circ + \theta)$	$= \sin.\theta$
* $\tan.(90^\circ + \theta)$	$= -\cot.\theta$	$\tan.(270^\circ + \theta)$	$= -\cot.\theta$
* $\cot.(90^\circ + \theta)$	$= -\tan.\theta$	$\cot.(270^\circ + \theta)$	$= -\tan.\theta$
$\sec.(90^\circ + \theta)$	$= -\text{cosec}.\theta$	$\sec.(270^\circ + \theta)$	$= \text{cosec}.\theta$
$\text{cosec}.(90^\circ + \theta)$	$= \sec.\theta$	$\text{cosec}.(270^\circ + \theta)$	$= -\sec.\theta$
* $\sin.(180^\circ - \theta)$	$= \sin.\theta$	$\sin.(360^\circ - \theta)$	$= -\sin.\theta$
* $\cos.(180^\circ - \theta)$	$= -\cos.\theta$	$\cos.(360^\circ - \theta)$	$= \cos.\theta$
* $\tan.(180^\circ - \theta)$	$= -\tan.\theta$	$\tan.(360^\circ - \theta)$	$= -\tan.\theta$
* $\cot.(180^\circ - \theta)$	$= -\cot.\theta$	$\cot.(360^\circ - \theta)$	$= -\cot.\theta$
$\sec.(180^\circ - \theta)$	$= -\sec.\theta$	$\sec.(360^\circ - \theta)$	$= \sec.\theta$
$\text{cosec}.(180^\circ - \theta)$	$= \text{cosec}.\theta$	$\text{cosec}.(360^\circ - \theta)$	$= -\text{cosec}.\theta$
* $\sin. 180^\circ$	$= 0$	$\sin. 360^\circ$	$= 0$
* $\cos. 180^\circ$	$= -1$	$\cos. 360^\circ$	$= 1$
* $\tan. 180^\circ$	$= 0$	$\tan. 360^\circ$	$= 0$
* $\cot. 180^\circ$	$= -\infty$	$\cot. 360^\circ$	$= \infty$
$\sec. 180^\circ$	$= -1$	$\sec. 360^\circ$	$= 1$
$\text{cosec}. 180^\circ$	$= \infty$	$\text{cosec}. 360$	$= \infty$

The results in the above table which are most frequently used, are marked with an asterisk, and ought to be committed to memory.

We have in the preceding pages confined ourselves to the consideration of angles not greater than 360°, but the student can find no difficulty in applying the above principles to angles of any magnitude whatsoever.

We shall conclude this introductory chapter, by demonstrating two propositions which are of the highest importance in our subsequent investigations. The first is,

In any right-angled triangle, the ratio which the side opposite to one of the acute angles bears to the hypotenuse, is the sine of that angle; the ratio which the side adjacent to one of the acute angles bears to the hypotenuse, is the cosine of that angle; and the ratio which the side opposite to one of the acute angles bears to the side adjacent to that angle, is the tangent of that angle.



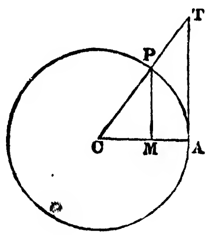
Let CMP be any plane triangle, right-angled at M.

Then,

$$\frac{PM}{CP} = \sin. C, \quad \frac{CM}{CP} = \cos. C, \quad \frac{PM}{CM} = \tan. C.$$

or,

$$\frac{CM}{CP} = \sin. P, \quad \frac{MP}{CP} = \cos. P, \quad \frac{MC}{MP} = \tan. P.$$



From C, as centre, with radius CP, describe a circle.

Produce CM to meet the circumference in A.

From A draw AT a tangent to the circle at A.

Produce CP to meet AT in T.

Then, from Definitions (1), (2), (3),

$$\frac{PM}{CP} = \sin. C, \quad \frac{CM}{CP} = \cos. C, \quad \frac{AT}{CP} = \tan. C,$$

for $CP = CA$.

But the triangles TAC, PMC, are similar;

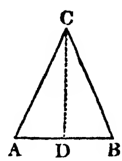
$$\therefore \quad \frac{AT}{CP} = \frac{PM}{CM} = \tan. C.$$

Corol.

$$PM = CP \sin. C = CP \cos. P$$
$$CM = CP \cos. C = CP \sin. P$$
$$PM = CM \tan. C = CM \cot. P$$

The second proposition is,

In any plane triangle, the ratio of any two of the sides, is equal to the ratio of the sines of the angles opposite to them.



Let ABC be a plane triangle; it is required to prove, that

$$\frac{CB}{CA} = \frac{\sin. A}{\sin. B}, \quad \frac{CB}{BA} = \frac{\sin. A}{\sin. C}, \quad \frac{CA}{BA} = \frac{\sin. B}{\sin. C}.$$

From C let fall CD perpendicular on AB.

Then, since CDB is a plane triangle right-angled at D, by last proposition,

$$CD = CB \sin. B \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Again, since CDA is a plane triangle right-angled at A,

$$CD = CA \sin. A \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Equating these two equal values of CD,

$$\text{CB sin. B} = \text{CA sin. A};$$

$$\therefore \quad \frac{\text{CB}}{\text{CA}} = \frac{\text{sin. A}}{\text{sin. B}}$$

In like manner, by dropping perpendiculars from B and A upon the sides AC, CB, we can prove,

$$\frac{\text{CB}}{\text{BA}} = \frac{\text{sin. A}}{\text{sin. C}}, \qquad \frac{\text{CA}}{\text{BA}} = \frac{\text{sin. B}}{\text{sin. C}}$$

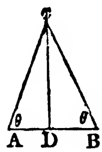
In treating of plane triangles, it is convenient to designate the three angles by the capital letters A, B, C, and the sides opposite to these angles by the corresponding small letters, a, b, c. According to this notation, the last proposition will be,

$$\frac{a}{b} = \frac{\text{sin. A}}{\text{sin. B}}, \qquad \frac{a}{c} = \frac{\text{sin. A}}{\text{sin. C}}, \qquad \frac{b}{c} = \frac{\text{sin. B}}{\text{sin. C}}.$$

CHAPTER II.

GENERAL FORMULÆ.

Given the sines and cosines of two angles, to find the sine of their sum.



Let ABC be a plane triangle; from C let fall CD perpendicular on AB,

Let angle CAB $= \theta$,
and angle CBA $= \theta'$,

$$\begin{aligned} \text{Then, AB} &= \text{BD} + \text{DA} \\ &= \text{BC cos. } \theta' + \text{AC cos. } \theta, \end{aligned}$$

because BDC and ADC are right-angled triangles.

Dividing each member of the equation by AB,

$$\begin{aligned} 1 &= \frac{\text{BC}}{\text{AB}} \text{cos. } \theta' + \frac{\text{AC}}{\text{AB}} \text{cos. } \theta \\ &= \frac{\text{sin. } \theta}{\text{sin. C.}} \text{cos. } \theta' + \frac{\text{sin. } \theta'}{\text{sin. C}} \text{cos. } \theta, \text{ by last Prop. in Chap. I.} \\ \therefore \text{ sin. C} &= \text{sin. } \theta \text{ cos. } \theta' + \text{sin. } \theta' \text{ cos. } \theta. \end{aligned}$$

But, since ABC is a plane triangle, $\theta + \theta' + \text{C} = 180°$

$$\begin{aligned} \therefore \text{C} &= 180° - (\theta + \theta') \\ \text{sin. C} &= \text{sin. } \{180° - (\theta + \theta')\} \\ &= \text{sin } (\theta + \theta'), \text{ because } 180° - (\theta + \theta') \text{ is the supplement of} \end{aligned}$$

$(\theta + \theta')$.

Hence, $\text{sin.} (\theta + \theta') = \text{sin. } \theta \text{ cos. } \theta' + \text{sin. } \theta' \text{ cos. } \theta$ .... (a)

This expression, from its great importance, is called the *fundamental formula* of Plane Trigonometry, and nearly the whole science may be derived from it.

Given the sines and cosines of two angles, to find the sine of their difference.

By formula (a).

$$\sin.(\theta + \theta') = \sin.\theta\cos.\theta' + \sin.\theta'\cos.\theta.$$

For θ substitute $180° - \theta$, the above will become

$$\sin.\{180° - (\theta - \theta')\} = \sin.(180° - \theta)\cos.\theta' + \sin.\theta'\cos.(180° - \theta)$$

But, $\sin.\{180° - (\theta - \theta')\} = \sin.(\theta - \theta')$ $\because$ $180° - (\theta - \theta')$ is the supplement of $(\theta - \theta')$.

And, $\sin.(180° - \theta) = \sin.\theta,$

And, $\cos.(180° - \theta) = -\cos.\theta$

Substitute, therefore, these values in the above expression, it becomes

$$\sin.(\theta - \theta') = \sin.\theta\cos.\theta' - \sin.\theta'\cos.\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (b)$$

Given the sines and cosines of two angles, to find the cosine of their sum.

By formula (a)

$$\sin.(\theta + \theta') = \sin.\theta\cos.\theta' + \sin.\theta'\cos.\theta.$$

For θ substitute $90° + \theta$, the above will become

$$\sin.\{90° + (\theta + \theta')\} = \sin.(90° + \theta)\cos.\theta' + \sin.\theta'\cos.(90° + \theta),$$

But, $\sin.\{90° + (\theta + \theta')\} = \cos.(\theta + \theta')$, by Table II.

And, $\sin.(90° + \theta) = \cos.\theta.$

And, $\cos.(90° + \theta) = -\sin.\theta.$

Substituting, therefore, these values in the above expression, it becomes

$$\cos.(\theta + \theta') = \cos.\theta\cos.\theta' - \sin.\theta\sin.\theta' \quad \ldots\ldots\ldots\ldots\ldots\ldots (c)$$

Given the sines and cosines of two angles, to find the cosine of their difference.

By formula (a):

$$\sin.(\theta + \theta') = \sin.\theta\cos.\theta' + \sin.\theta'\cos.\theta,$$

For θ substitute $90° - \theta$, the above will become

$$\sin.\{90° - (\theta - \theta')\} = \sin.(90° - \theta)\cos.\theta' + \sin.\theta'\cos.(90° - \theta)$$

But, $\sin.\{90° - (\theta - \theta')\} = \cos.(\theta - \theta')$, by Table II.

$\sin.(90° - \theta) = \cos.\theta$ ......

$\cos.(90° - \theta) = \sin.\theta$ ......

Substituting, therefore, these values in the above expression, it becomes

$$\cos.(\theta - \theta') = \cos.\theta\cos.\theta' + \sin.\theta\sin.\theta' \quad \ldots\ldots\ldots\ldots\ldots\ldots (d)$$

Given the tangents of two angles, to find the tangent of their sum.

By Table I.:

$$\tan.(\theta + \theta') = \frac{\sin.(\theta + \theta')}{\cos.(\theta + \theta')}$$

$$= \frac{\sin.\theta\cos.\theta' + \sin.\theta'\cos.\theta}{\cos.\theta\cos.\theta' - \sin.\theta\sin.\theta'} \text{ by } (a) \text{ and } (c)$$

Dividing both numerator and denominator of fraction by cos. θ cos. θ':

$$= \frac{\dfrac{\sin.\theta\cos.\theta'}{\cos.\theta\cos.\theta'} + \dfrac{\sin.\theta'\cos.\theta}{\cos.\theta'\cos.\theta}}{1 - \dfrac{\sin.\theta\sin.\theta'}{\cos.\theta\cos.\theta'}}$$

Simplifying, $$= \frac{\tan.\theta + \tan.\theta'}{1 - \tan.\theta\tan.\theta'} \quad \ldots\ldots\ldots\ldots \quad (e)$$

Given the tangents of two angles, to find the tangent of their difference.

By Table I.:

$$\tan.\ (\theta - \theta') = \frac{\sin.\ (\theta - \theta')}{\cos.\ (\theta - \theta')}$$

$$= \frac{\sin.\theta\ \cos.\theta' - \sin.\theta'\ \cos.\theta}{\cos.\theta\ \cos.\theta' + \sin.\theta\ \sin.\theta'} \text{ by } (b) \text{ and } (d)$$

Dividing both numerator and denominator by cos. θ cos. θ':

$$= \frac{\dfrac{\sin.\theta\ \cos.\theta'}{\cos.\theta\ \cos.\theta'} - \dfrac{\sin.\theta'\ \cos.\theta}{\cos.\theta\ \cos.\theta'}}{1 + \dfrac{\sin.\theta\ \sin.\theta'}{\cos.\theta\ \cos.\theta'}}$$

Simplifying, $$= \frac{\tan.\theta - \tan.\theta'}{1 + \tan.\theta\ \tan.\theta'} \quad \ldots\ldots\ldots\ldots \quad (f)$$

The student will have no difficulty in deducing the following:

$$\cot.\ (\theta + \theta') = \frac{\cot.\theta\ \cot.\theta' - 1}{\cot.\theta' + \cot.\theta}$$

$$\cot.\ (\theta - \theta') = \frac{\cot.\theta\ \cot.\theta' + 1}{\cot.\theta' - \cot.\theta}$$

$$\sec.\ (\theta + \theta') = \frac{\sec.\theta\ \sec.\theta'\ \text{cosec.}\theta\ \text{cosec.}\theta'}{\text{cosec.}\theta\ \text{cosec.}\theta' - \sec.\theta\ \sec.\theta'}$$

$$\sec.\ (\theta - \theta') = \frac{\sec.\theta\ \sec.\theta'\ \text{cosec.}\theta\ \text{cosec.}\theta'}{\text{cosec.}\theta\ \text{cosec.}\theta' + \sec.\theta\ \sec.\theta'}$$

$$\text{cosec.}\ (\theta + \theta') = \frac{\sec.\theta\ \sec.\theta'\ \text{cosec.}\theta\ \text{cosec.}\theta'}{\sec.\theta\ \text{cosec.}\theta' + \sec.\theta'\ \text{cosec.}\theta}$$

$$\text{cosec.}\ (\theta - \theta') = \frac{\sec.\theta\ \sec.\theta'\ \text{cosec.}\theta\ \text{cosec.}\theta'}{\sec.\theta\ \text{cosec.}\theta' - \sec.\theta'\ \text{cosec.}\theta}$$

To determine the sine of twice a given angle.

By formula (a):

$$\sin.\ (\theta + \theta') = \sin.\theta\ \cos.\theta' + \sin.\theta'\ \cos.\theta$$

Let $\theta = \theta'$, then the above becomes

$$\sin.\ 2\theta = \sin.\theta\ \cos.\theta + \sin.\theta\ \cos.\theta$$

$$= 2\sin.\theta\ \cos.\theta \quad \ldots\ldots\ldots\ldots \quad (g\ 1)$$

In the last formula, for θ substitute $\frac{\theta}{2}$; then,

$$\sin. 2 \times \frac{\theta}{2} = 2 \sin. \frac{\theta}{2} \cos. \frac{\theta}{2}$$

$$\text{Or, } \sin. \theta = 2 \sin. \frac{\theta}{2} \cos. \frac{\theta}{2} \quad \ldots\ldots\ldots\ldots \quad (g\,2)$$

To determine the cosine of twice a given angle.

By formula (c):

$$\cos. (\theta + \theta') = \cos. \theta \cos. \theta' - \sin. \theta \sin. \theta'$$

Let $\theta = \theta'$ then the above becomes

$$\cos. 2\theta = \cos.^2 \theta - \sin.^2 \theta \quad \ldots\ldots\ldots\ldots \quad (h\,1)$$

By table I. $\sin.^2 \theta = 1 - \cos.^2 \theta$; substituting this for $\sin.^2 \theta$:

$$\cos. 2\theta = 2 \cos.^2 \theta - 1 \quad \ldots\ldots\ldots\ldots \quad (h\,2)$$

Again, since $\cos.^2 \theta = 1 - \sin.^2 \theta$, substitute this for $\cos.^2 \theta$:

$$\cos. 2\theta = 1 - 2 \sin.^2 \theta \quad \ldots\ldots\ldots\ldots \quad (h\,3)$$

To determine the tangent of twice a given angle.

By formula (e):

$$\tan. (\theta + \theta') = \frac{\tan. \theta + \tan. \theta'}{1 - \tan. \theta \tan. \theta'}$$

Let $\theta = \theta'$, the above becomes

$$\tan. 2\theta = \frac{2 \tan. \theta}{1 - \tan.^2 \theta} \quad \ldots\ldots\ldots\ldots \quad (i)$$

The student will easily deduce the following:

$$\cot. 2\theta = \frac{\cot.^2 \theta - 1}{2 \cot. \theta} = \frac{\cot. \theta - \tan. \theta}{2}$$

$$\sec. 2\theta = \frac{\sec.^2 \theta \operatorname{cosec.}^2 \theta}{\operatorname{cosec.}^2 \theta - \sec.^2 \theta}$$

$$\operatorname{cosec.} 2\theta = \frac{\sec.^2 \theta \operatorname{cosec.}^2 \theta}{2 \sec. \theta \operatorname{cosec.} \theta} = \frac{\sec. \theta \operatorname{cosec.} \theta}{2}$$

To determine the sine of half a given angle.

By formula ($h\,3$):

$$\cos. 2\theta = 1 - 2 \sin.^2 \theta$$

For θ substitute $\frac{\theta}{2}$; the above becomes,

$$\cos. 2 \frac{\theta}{2} = 1 - 2 \sin.^2 \frac{\theta}{2}$$

$$\text{Or, } \cos. \theta = 1 - 2 \sin.^2 \frac{\theta}{2}$$

$$\therefore 2 \sin.^2 \frac{\theta}{2} = 1 - \cos. \theta$$

$$\sin. \frac{\theta}{2} = \sqrt{\frac{1 - \cos. \theta}{2}} \quad \ldots\ldots\ldots\ldots \quad ([illegible])$$

To determine the cosine of half a given angle.

By formula (h 2):

$$\cos.\ 2\,\theta = 2\cos.^2\theta - 1$$

For θ substitute $\frac{\theta}{2}$; the above becomes,

$$\cos.\ 2\,\frac{\theta}{2} = 2\cos.^2\frac{\theta}{2} - 1$$

Or, $$\cos.\ \theta = 2\cos.^2\frac{\theta}{2} - 1$$

$$\therefore\ 2\cos.^2\frac{\theta}{2} = 1 + \cos.\ \theta$$

$$\cos.\ \frac{\theta}{2} = \sqrt{\frac{1+\cos.\ \theta}{2}} \quad\ldots\ldots\ldots\ldots\ldots (k)$$

To determine the tangent of half a given angle.

Divide formula (j) by (k):

$$\frac{\sin.\ \frac{\theta}{2}}{\cos.\ \frac{\theta}{2}} = \sqrt{\frac{1-\cos.\ \theta}{1+\cos.\ \theta}}$$

Or, $$\tan.\ \frac{\theta}{2} = \sqrt{\frac{1-\cos.\ \theta}{1+\cos.\ \theta}} \quad\ldots\ldots\ldots\ldots\ldots (l\,1)$$

Multiply both numerator and denominator by $\sqrt{1-\cos.\ \theta}$; the above becomes,

$$\tan.\ \frac{\theta}{2} = \frac{1-\cos.\ \theta}{\sin.\ \theta} \quad\ldots\ldots\ldots\ldots\ldots (l\,2)$$

Multiply both numerator and denominator of (l 1) by $\sqrt{1+\cos.\ \theta}$; we have,

$$\tan.\ \frac{\theta}{2} = \frac{\sin.\ \theta}{1+\cos.\ \theta} \quad\ldots\ldots\ldots\ldots\ldots (l\,3)$$

The student will easily deduce the following:

$$\cot.\ \frac{\theta}{2} = \sqrt{\frac{1+\cos.\ \theta}{1-\cos.\ \theta}}$$

$$= \frac{1+\cos.\ \theta}{\sin.\ \theta}$$

$$= \frac{\sin.\ \theta}{1-\cos.\ \theta}$$

$$\sec.\ \frac{\theta}{2} = \sqrt{\frac{2\sec.\ \theta}{\sec.\ \theta + 1}}$$

$$\operatorname{cosec.}\ \frac{\theta}{2} = \sqrt{\frac{2\sec.\ \theta}{\sec.\ \theta - 1}}$$

To determine the sine of $(n+1)\,\theta$, in terms of $n\,\theta$, $(n-1)\,\theta$ and θ.

By formula (a) and (b):

$$\sin.\ (\theta' + \theta) = \sin.\ \theta' \cos.\ \theta + \sin.\ \theta \cos.\ \theta'$$
$$\sin.\ (\theta' - \theta) = \sin.\ \theta' \cos.\ \theta - \sin.\ \theta \cos.\ \theta'$$

Add these two equations,

$$\sin.\ (\theta' + \theta) + \sin.\ (\theta' - \theta) = 2\sin.\ \theta' \cos.\ \theta$$

Subtract sin $(\theta' - \theta)$ from each member,

$$\sin. (\theta' + \theta) = 2 \sin. \theta' \cos. \theta - \sin. (\theta' - \theta)$$

Let $\theta' = n\,\theta$, the above becomes

$$\sin. (n + 1)\,\theta = 2 \sin. n\,\theta \cos. \theta - \sin. (n - 1)\,\theta \ldots\ldots\ldots (m)$$

In the above formula, let $n = 1$; $\therefore n + 1 = 2, \quad n - 1 = 0$

$$\therefore \quad \sin. 2\,\theta = 2 \sin. \theta \cos. \theta - \sin. 0$$
$$= 2 \sin. \theta \cos. \theta, \text{ the same result as in } (g).$$

Let $n = 2$; $\therefore n + 1 = 3, \quad n - 1 = 1$;

$$\begin{aligned} \therefore \quad \sin. 3\,\theta &= 2 \sin. 2\,\theta \cos. \theta - \sin. \theta \\ &= 2 \times 2 \sin. \theta \cos. \theta \times \cos. \theta - \sin. \theta \\ &= 4 \sin. \theta \cos.^2 \theta - \sin. \theta \\ &= 4 \sin. \theta\,(1 - \sin.^2 \theta) - \sin. \theta \\ &= 3 \sin. \theta - 4 \sin.^3 \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (n) \end{aligned}$$

Let $n = 3$; $\therefore n + 1 = 4, \quad n - 1 = 2$;

$\therefore$ By formula (m):

$$\begin{aligned} \sin. 4\,\theta &= 2 \times \sin. 3\,\theta \times \cos. \theta - \sin. 2\,\theta \\ &= 2\,(3 \sin. \theta - 4 \sin.^3 \theta) \cos. \theta - 2 \sin. \theta \cos. \theta \\ &= (8 \cos.^3 \theta - 4 \cos. \theta) \sin. \theta \end{aligned}$$

It is manifest that, by continuing the same process, we may find in succession, $\sin. 5\,\theta$, $\sin. 6\,\theta$, ... &c.

To determine the cosine of $(n + 1)\,\theta$, in terms of $n\,\theta$, $(n - 1)\,\theta$, and θ.

By formula (c) and (d):

$$\begin{aligned} \cos. (\theta' + \theta) &= \cos. \theta' \cos. \theta - \sin. \theta' \sin. \theta \\ \cos. (\theta' - \theta) &= \cos. \theta' \cos. \theta + \sin. \theta' \sin. \theta \end{aligned}$$

Add these two equations,

$$\cos. (\theta' + \theta) + \cos. (\theta' - \theta) = 2 \cos. \theta' \cos. \theta$$

Subtract $\cos. (\theta' - \theta)$ from each member,

$$\cos. (\theta' + \theta) = 2 \cos. \theta' \cos. \theta - \cos. (\theta' - \theta)$$

Let $\theta' = n\,\theta$, the above becomes

$$\cos. (n + 1)\,\theta = 2 \cos. n\,\theta \cos. \theta - \cos. (n - 1)\,\theta \ldots\ldots\ldots (o)$$

In the above formula, let $n = 1$; $\therefore n + 1 = 2, \quad n - 1 = 0$;

$$\begin{aligned} \text{Then,} \quad \cos. 2\,\theta &= 2 \cos. \theta \cos. \theta - \cos. 0 \\ &= 2 \cos.^2 \theta - 1, \text{ the same result as in } (h\,2). \end{aligned}$$

Let $n = 2$, $\therefore n + 1 = 3, \quad n - 1 = 1$;

$$\begin{aligned} \therefore \quad \cos. 3\,\theta &= 2 \cos. 2\,\theta \cos. \theta - \cos. \theta \\ &= 2\,(2 \cos.^2 \theta - 1) \cos. \theta - \cos. \theta \\ &= 4 \cos.^3 \theta - 3 \cos. \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (p) \end{aligned}$$

Let $n = 3$; $\therefore n + 1 = 4, \quad n - 1 = 2$;

$$\begin{aligned} \therefore \quad \cos. 4\,\theta &= 2 \cos. 3\,\theta \cos. \theta - \cos. 2\,\theta \\ &= 2\,(4 \cos.^3 \theta - 3 \cos. \theta) \cos. \theta - (2 \cos.^2 \theta - 1) \\ &= 8 \cos.^4 \theta - 8 \cos.^2 \theta + 1 \end{aligned}$$

It is manifest that, by continuing the same process, we may find, in succession, $\cos. 5\,\theta$, $\cos. 6\,\theta$. .. .. &c.

By adding and subtracting (*a*) and (*b*), and by adding and subtracting (*c*) and (*d*), we obtain the following formulæ, which are of considerable utility.

$$\left.\begin{array}{lll}\sin.(\theta+\theta')+\sin.(\theta-\theta') & = & 2\sin.\theta\cos.\theta' \\ \sin.(\theta+\theta')-\sin.(\theta-\theta') & = & 2\sin.\theta'\cos.\theta \\ \cos.(\theta+\theta')+\cos.(\theta-\theta') & = & 2\cos.\theta\cos.\theta' \\ \cos.(\theta+\theta')-\cos.(\theta-\theta') & = & -2\sin.\theta\sin.\theta'\end{array}\right\}\ldots\ldots(q)$$

Any angle θ may, by a simple artifice, be put under the form,

$$\theta = \frac{\theta+\theta'}{2} + \frac{\theta-\theta'}{2}$$

And, in like manner,

$$\theta' = \frac{\theta+\theta'}{2} - \frac{\theta-\theta'}{2}$$

$$\therefore \sin.\theta = \sin.\left\{\frac{\theta+\theta'}{2} + \frac{\theta-\theta'}{2}\right\}$$

$$= \sin.\frac{\theta+\theta'}{2}\cos.\frac{\theta-\theta'}{2} + \sin.\frac{\theta-\theta'}{2}\cos.\frac{\theta+\theta'}{2}\ldots\ldots(1)$$

$$\sin.\theta' = \sin.\left\{\frac{\theta+\theta'}{2} - \frac{\theta-\theta'}{2}\right\}$$

$$= \sin.\frac{\theta+\theta'}{2}\cos.\frac{\theta-\theta'}{2} - \sin.\frac{\theta-\theta'}{2}\cos.\frac{\theta+\theta'}{2}\ldots\ldots(2)$$

$$\cos.\theta = \cos.\left\{\frac{\theta+\theta'}{2} + \frac{\theta-\theta'}{2}\right\}$$

$$= \cos.\frac{\theta+\theta'}{2}\cos.\frac{\theta-\theta'}{2} - \sin.\frac{\theta+\theta'}{2}\sin.\frac{\theta-\theta'}{2}\ldots\ldots(3)$$

$$\cos.\theta' = \cos.\left\{\frac{\theta+\theta'}{2} - \frac{\theta-\theta'}{2}\right\}$$

$$= \cos.\frac{\theta+\theta'}{2}\cos.\frac{\theta-\theta'}{2} + \sin.\frac{\theta+\theta'}{2}\sin.\frac{\theta-\theta'}{2}\ldots\ldots(4)$$

Add together (1) and (2):

$$\sin.\theta + \sin.\theta' = 2\sin.\frac{\theta+\theta'}{2}\cos.\frac{\theta-\theta'}{2}\ldots\ldots(r)$$

Subtract (2) from (1),

$$\sin.\theta - \sin.\theta' = 2\sin.\frac{\theta-\theta'}{2}\cos.\frac{\theta+\theta'}{2}\ldots\ldots(s)$$

Add together (3) and (4),

$$\cos.\theta + \cos.\theta' = 2\cos.\frac{\theta+\theta'}{2}\cos.\frac{\theta-\theta'}{2}\ldots\ldots(t)$$

Subtract (4) from (3),

$$\cos.\theta - \cos.\theta' = -2\sin.\frac{\theta+\theta'}{2}\sin.\frac{\theta-\theta'}{2}\ldots\ldots(v)$$

These formulæ, which are of the greatest importance, might have been immediately deduced from the group (q), by changing $\theta+\theta'$ into θ, $\theta-\theta'$ into θ', θ into $\frac{\theta+\theta'}{2}$, θ' into $\frac{\theta-\theta'}{2}$.

Divide (r) by (s):

$$\frac{\sin.\ \theta + \sin.\ \theta'}{\sin.\ \theta - \sin.\ \theta'} = \frac{2\sin.\ \frac{\theta+\theta'}{2}\cos.\ \frac{\theta-\theta'}{2}}{2\sin.\ \frac{\theta-\theta'}{2}\cos.\ \frac{\theta+\theta'}{2}}$$

$$= \frac{\tan.\ \frac{\theta+\theta'}{2}}{\tan.\ \frac{\theta-\theta'}{2}} \quad\ldots\ldots\ldots\ldots\ldots(w)$$

Multiply (a) by (b); then,

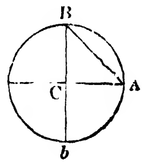
$$\sin.\ (\theta+\theta')\sin.\ (\theta-\theta') = \sin.^2\theta\cos.^2\theta' - \sin.^2\theta'\cos.^2\theta$$
$$= \sin.^2\theta - \sin.^2\theta' \quad\ldots\ldots\ldots\ldots(x)$$

Multiply (c) by (d); then,

$$\cos.\ (\theta+\theta')\cos.\ (\theta-\theta') = \cos.^2\theta\cos.^2\theta' - \sin.^2\theta\sin.^2\theta'$$
$$= \cos.^2\theta - \sin.^2\theta' \quad\ldots\ldots\ldots\ldots(y)$$

To find the numerical value of the sine, cosine, &c. of 45°.

In the circle ABa, draw CA, CB, radii at right angles; join AB.



Then by Definition (11),

$$\text{Chord ACB } (90^\circ) = \frac{AB}{AC}$$

$$\text{Chord}^2\ 90^\circ = \frac{AB^2}{AC^2}$$
$$= \frac{AC^2 + BC^2}{AC^2}$$
$$= \frac{2\ AC^2}{AC^2} \quad \because BC = AC$$
$$= 2 \quad\ldots\ldots\ldots\ldots(1)$$

Now, the chord of an arc is equal to twice the sine of half the arc therefore,

$$2\sin.\ 45^\circ = \text{chord } 90^\circ$$
$$4\sin.^2 45^\circ = \text{chord}^2\ 90^\circ$$
$$= 2, \text{ by Equation (1)};$$
$$\therefore \quad \sin.\ 45^\circ = \frac{1}{\sqrt{2}}$$

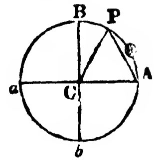
Again, by table I.:

$$\sin.^2\theta + \cos.^2\theta = 1$$
$$\therefore \quad \cos.^2 45^\circ = 1 - \sin.^2 45^\circ$$
$$= 1 - \frac{1}{2}$$
$$= \frac{1}{2}$$
$$\therefore \quad \cos.\ 45^\circ = \frac{1}{\sqrt{2}} = \sin.\ 45^\circ$$

Also,

$$\tan.\ 45^\circ = \frac{\sin.\ 45^\circ}{\cos.\ 45^\circ}$$
$$= 1 = \cot.\ 45^\circ.$$

To find the numerical value of the sine, cosine, &c. of 30°.



In the circle ABa, draw CP, making with CA the angle ACP = 60°; join A, P.

Now,

$$2 \sin. 30^\circ = \text{chord } 60^\circ$$
$$= \frac{AP}{AC}$$
$$= \frac{AC}{AC} \because AP = AC, \because \text{the trian. APC is equiangular, and therefore equilateral.}$$
$$= 1$$
$$\therefore \sin. 30^\circ = \frac{1}{2}$$

Again,

$$\cos. 30^\circ = \sqrt{1 - \sin.^2 30^\circ}$$
$$= \sqrt{1 - \tfrac{1}{4}}$$
$$= \frac{\sqrt{3}}{2}$$

Also,

$$\tan. 30^\circ = \frac{\sin. 30^\circ}{\cos. 30^\circ}$$
$$= \frac{1}{\sqrt{3}}$$
$$\cot. 30^\circ = \frac{1}{\tan. 30^\circ}$$
$$= \sqrt{3}$$

To find the numerical value of the sine, cosine, &c. of 60°.

$$\sin. 60^\circ = \cos. (90^\circ - 60^\circ)$$
$$= \cos. 30^\circ$$
$$= \frac{\sqrt{3}}{2}, \text{ by last art.}$$

Again,

$$\cos. 60^\circ = \sin. (90^\circ - 60^\circ)$$
$$= \sin. 30^\circ$$
$$= \frac{1}{2}$$

Also

$$\tan. 60^\circ = \sqrt{3}$$
$$\cot. 60^\circ = \frac{1}{\sqrt{3}}$$

It may be useful to exhibit the most useful results in this chapter, in the following

TABLE III.

(1.) $\sin. (\theta \pm \theta') = \sin. \theta \cos. \theta' \pm \sin. \theta' \cos. \theta$

(2.) $\cos. (\theta \pm \theta') = \cos. \theta \cos. \theta' \mp \sin. \theta \sin. \theta'$

(3.) $\tan. (\theta \pm \theta') = \dfrac{\tan. \theta \pm \tan. \theta'}{1 \mp \tan. \theta \tan. \theta'}$

(4.) $\sin. 2\theta = 2 \sin. \theta \cos. \theta$

(5.) $\cos. 2\theta = \cos.^2 \theta - \sin.^2 \theta = 2 \cos.^2 \theta - 1 = 1 - 2 \sin.^2 \theta$

(6.) $\tan. 2\theta = \dfrac{2 \tan. \theta}{1 - \tan.^2 \theta}$

(7.) $\sin. \dfrac{\theta}{2} = \sqrt{\dfrac{1 - \cos. \theta}{2}}$

(8.) $\cos. \dfrac{\theta}{2} = \sqrt{\dfrac{1 + \cos. \theta}{2}}$

(9.) $\tan. \dfrac{\theta}{2} = \sqrt{\dfrac{1 - \cos. \theta}{1 + \cos. \theta}} = \dfrac{1 - \cos. \theta}{\sin. \theta} = \dfrac{\sin. \theta}{1 + \cos. \theta}$

(10.) $\sin. \theta = 2 \sin. \dfrac{\theta}{2} \cos. \dfrac{\theta}{2}$

(11.) $\sin. 3\theta = 3 \sin. \theta - 4 \sin.^3 \theta$

(12.) $\cos. 3\theta = 4 \cos.^3 \theta - 3 \cos. \theta$

(13.) $\sin. (n + 1)\theta = 2 \sin. n\theta \cos. \theta - \sin. (n - 1)\theta$

(14.) $\cos. (n + 1)\theta = 2 \cos. n\theta \cos. \theta - \cos. (n - 1)\theta$

(15.) $\sin. \theta + \sin. \theta' = 2 \sin. \dfrac{\theta + \theta'}{2} \cos. \dfrac{\theta - \theta'}{2}$

(16.) $\sin. \theta - \sin. \theta' = 2 \sin. \dfrac{\theta - \theta'}{2} \cos. \dfrac{\theta + \theta'}{2}$

(17.) $\cos. \theta + \cos. \theta' = 2 \cos. \dfrac{\theta + \theta'}{2} \cos. \dfrac{\theta - \theta'}{2}$

(18.) $\cos. \theta - \cos. \theta' = -2 \sin. \dfrac{\theta + \theta'}{2} \sin. \dfrac{\theta - \theta'}{2}$

(19.) $\dfrac{\sin. \theta + \sin. \theta'}{\sin. \theta - \sin. \theta'} = \dfrac{\tan. \dfrac{\theta + \theta'}{2}}{\tan. \dfrac{\theta - \theta'}{2}}$

(20.) $\sin. (\theta + \theta') + \sin. (\theta - \theta') = 2 \sin. \theta \cos. \theta'$

(21.) $\sin. (\theta + \theta') - \sin. (\theta - \theta') = 2 \sin. \theta' \cos. \theta$

(22.) $\cos. (\theta + \theta') + \cos. (\theta - \theta') = 2 \cos. \theta \cos. \theta'$

(23.) $\cos. (\theta + \theta') - \cos. (\theta - \theta') = -2 \sin. \theta \sin. \theta'$

(24.) $\sin. (\theta + \theta') \cos. (\theta - \theta') = \sin.^2 \theta - \sin.^2 \theta' = \cos.^2 \theta' - \cos.^2 \theta$

(25.) $\cos. (\theta + \theta') \cos. (\theta - \theta') = \cos.^2 \theta - \sin.^2 \theta' = \cos.^2 \theta + \cos.^2 \theta' - 1$

(26.) $\sin. 45^\circ = \cos. 45^\circ = \dfrac{1}{\sqrt{2}}$

(27.) $\tan. 45^\circ = \cot. 45^\circ = 1$

(28.) $\sin. 30^\circ = \cos. 60^\circ = \dfrac{1}{2}$

(29.) $\cos. 30^\circ = \sin. 60^\circ = \dfrac{\sqrt{3}}{2}$

(30.) $\tan. 30^\circ = \cot. 60^\circ = \dfrac{1}{\sqrt{3}}$

(31.) $\cot. 30^\circ = \tan. 60^\circ = \sqrt{3}$

The formulæ of Trigonometry may be multiplied to almost any extent, and the same quantity may be expressed in a vast number of different ways. An intimate acquaintance with those given in the above table is *essential* to the progress of the student.

The following, although of less frequent occurrence, may occasionally be found useful, and can be readily deduced from the above.

$$(32.)\quad \left.\begin{matrix}\sin.\,(45^\circ \pm \theta)\\ \cos.\,(45^\circ \mp \theta)\end{matrix}\right\} = \frac{\cos.\,\theta \pm \sin.\,\theta}{\sqrt{2}}$$

$$(33.)\quad \tan.\,(45^\circ \pm \theta) = \frac{1 \pm \tan.\,\theta}{1 \mp \tan.\,\theta}$$

$$(34.)\quad \tan.^2\left(45^\circ \pm \frac{\theta}{2}\right) = \frac{1 \pm \sin.\,\theta}{1 \mp \sin.\,\theta}$$

$$(35.)\quad \tan.\left(45^\circ \pm \frac{\theta}{2}\right) = \frac{1 \pm \sin.\,\theta}{\cos.\,\theta} = \frac{\cos.\,\theta}{1 \mp \sin.\,\theta}$$

$$(36.)\quad \frac{\sin.\,(\theta + \theta')}{\sin.\,(\theta - \theta')} = \frac{\tan.\,\theta + \tan.\,\theta'}{\tan.\,\theta - \tan.\,\theta'} = \frac{\cot.\,\theta' + \cot.\,\theta}{\cot.\,\theta' - \cot.\,\theta}$$

$$(37.)\quad \frac{\cos.\,(\theta + \theta')}{\cos.\,(\theta - \theta')} = \frac{\cot.\,\theta' - \tan.\,\theta}{\cot.\,\theta' + \tan.\,\theta} = \frac{\cot.\,\theta - \tan.\,\theta'}{\cot.\,\theta + \tan.\,\theta}$$

$$(38.)\quad \frac{\sin.\,\theta + \sin.\,\theta'}{\cos.\,\theta + \cos.\,\theta'} = \tan.\,\frac{\theta + \theta'}{2}$$

$$(39.)\quad \frac{\sin.\,\theta + \sin.\,\theta'}{\cos.\,\theta - \cos.\,\theta'} = -\cot.\,\frac{\theta - \theta'}{2}$$

$$(40.)\quad \frac{\sin.\,\theta - \sin.\,\theta'}{\cos.\,\theta + \cos.\,\theta'} = \tan.\,\frac{\theta - \theta'}{2}$$

$$(41.)\quad \frac{\sin.\,\theta - \sin.\,\theta'}{\cos.\,\theta - \cos.\,\theta'} = -\cot.\,\frac{\theta + \theta'}{2}$$

$$(42.)\quad \frac{\cos.\,\theta + \cos.\,\theta'}{\cos.\,\theta - \cos.\,\theta'} = -\cot.\,\frac{\theta + \theta'}{2}\,\cot.\,\frac{\theta - \theta'}{2}$$

$$(43.)\quad \tan.\,\theta + \tan\,\theta' = \frac{\sin.\,(\theta + \theta')}{\cos.\,\theta\,\cos.\,\theta'}$$

$$(44.)\quad \cot.\,\theta + \cot.\,\theta' = \frac{\sin.\,(\theta + \theta')}{\sin.\,\theta\,\sin.\,\theta'}$$

$$(45.)\quad \tan.\,\theta - \tan.\,\theta' = \frac{\sin.\,(\theta - \theta')}{\cos.\,\theta\,\cos.\,\theta'}$$

$$(46.)\quad \cot.\,\theta - \cot.\,\theta' = -\frac{\sin.\,(\theta - \theta')}{\sin.\,\theta\,\sin.\,\theta'}$$

$$(47.)\quad \tan.^2\,\theta - \tan.^2\,\theta' = \frac{\sin.\,(\theta + \theta')\,\sin.\,(\theta - \theta')}{\cos.^2\,\theta\,\cos.^2\,\theta'}$$

$$(48.)\quad \cot.^2\,\theta - \cot.^2\,\theta' = -\frac{\sin.\,(\theta + \theta')\,\sin.\,(\theta - \theta)}{\sin.^2\,\theta\,\sin.^2\,\theta'}$$

In order to become familiar with the various combinations, and dexterous in the application of these expressions, the student will do well to exercise himself

by verifying the following values of Sin. θ, Cos. θ, Tan. θ, which are extracted from the large work of Cagnoli.

TABLE OF THE MOST USEFUL ANALYTICAL VALUES OF SIN. θ, COS. θ, TAN. θ.

VALUES OF SIN. θ.	VALUES OF COS. θ.	VALUES OF TAN. θ.
1. $\cos.\theta \tan.\theta$	16. $\dfrac{\sin.\theta}{\tan.\theta}$	31. $\dfrac{\sin.\theta}{\cos.\theta}$
2. $\dfrac{\cos.\theta}{\cot.\theta}$	17. $\sin.\theta \cot.\theta$	32. $\dfrac{1}{\cot.\theta}$
3. $\sqrt{1-\cos.^2\theta}$	18. $\sqrt{1-\sin.^2\theta}$	33. $\sqrt{\dfrac{1}{\cos.^2\theta}-1}$
4. $\dfrac{1}{\sqrt{1+\cot.^2\theta}}$	19. $\dfrac{1}{\sqrt{1+\tan.^2\theta}}$	34. $\dfrac{\sin.\theta}{\sqrt{1-\sin.^2\theta}}$
5. $\dfrac{\tan.\theta}{\sqrt{1+\tan.^2\theta}}$	20. $\dfrac{\cot.\theta}{\sqrt{1+\cot.^2\theta}}$	35. $\dfrac{\sqrt{1-\cos.^2\theta}}{\cos.\theta}$
6. $2\sin.\dfrac{\theta}{2}\cos.\dfrac{\theta}{2}$	21. $\cos.^2\dfrac{\theta}{2}-\sin.^2\dfrac{\theta}{2}$	36. $\dfrac{2\tan.\frac{\theta}{2}}{1-\tan.^2\frac{\theta}{2}}$
7. $\dfrac{\sqrt{1-\cos.2\theta}}{2}$	22. $1-2\sin.^2\dfrac{\theta}{2}$	
8. $\dfrac{2\tan.\frac{\theta}{2}}{1+\tan.^2\frac{\theta}{2}}$	23. $2\cos.^2\dfrac{\theta}{2}-1$	37. $\dfrac{2\cot.\frac{\theta}{2}}{\cot.^2\frac{\theta}{2}-1}$
	24. $\sqrt{\dfrac{1+\cos.2\theta}{2}}$	
9. $\dfrac{2}{\cot.\frac{\theta}{2}+\tan.\frac{\theta}{2}}$	25. $\dfrac{1-\tan.^2\frac{\theta}{2}}{1+\tan.^2\frac{\theta}{2}}$	38. $\dfrac{2}{\cot.\frac{\theta}{2}-\tan.\frac{\theta}{2}}$
10. $\dfrac{\sin.(30^\circ+\theta)-\sin.(30^\circ-\theta)}{\sqrt{3}}$		39. $\cot.\theta-2\cot.2\theta$
11. $2\sin.^2(45^\circ+\dfrac{\theta}{2})-1$	26. $\dfrac{\cot.\frac{\theta}{2}-\tan.\frac{\theta}{2}}{\cot.\frac{\theta}{2}+\tan.\frac{\theta}{2}}$	40. $\dfrac{1-\cos.2\theta}{\sin.2\theta}$
		41. $\dfrac{\sin.2\theta}{1+\cos.2\theta}$
12. $1-2\sin.^2(45^\circ-\dfrac{\theta}{2})$	27. $\dfrac{1}{1+\tan.\theta\tan.\frac{\theta}{2}}$	42. $\sqrt{\dfrac{1-\cos.2\theta}{1+\cos.2\theta}}$
13. $\dfrac{1-\tan.^2(45^\circ-\frac{\theta}{2})}{1+\tan.^2(45^\circ-\frac{\theta}{2})}$	28. $\dfrac{2}{\tan.(45^\circ+\frac{\theta}{2})+\cot.(45^\circ+\frac{\theta}{2})}$	43. $\dfrac{\tan.(45^\circ+\frac{\theta}{2})-\tan.(45^\circ-\frac{\theta}{2})}{2}$
14. $\dfrac{\tan(45^\circ+\frac{\theta}{2})-\tan.(45^\circ-\frac{\theta}{2})}{\tan.(45^\circ+\frac{\theta}{2})+\tan.(45^\circ-\frac{\theta}{2})}$	29. $2\cos.(45^\circ+\dfrac{\theta}{2})\cos.(45^\circ-\dfrac{\theta}{2})$	
15. $\sin.(60^\circ+\theta)-\sin.(60^\circ-\theta)$	30. $\cos.(60^\circ+\theta)+\cos.(60^\circ-\theta)$	

To develop sin. x *and cos.* x *in a series ascending by the powers of* x.

The series for sin. x must vanish when $x=0$, and therefore no term in the series can be independent of x, nor can the even powers of x occur in the series; for if we suppose

$$
\begin{aligned}
\sin. x &= a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots \\
\text{then } \sin. (-x) &= -a_1x + a_2x^2 - a_3x^3 + a_4x^4 - a_5x^5 + \ldots \\
\text{but } \sin. (-x) &= -\sin. x \\
&= -a_1x - a_2x^2 - a_3x^3 - a_4x^4 - a_5x^5 - \ldots \\
\therefore \quad a_2 &= -a_2,\ a_4 = -a_4, \ldots\ldots;\ \text{hence } a_2 = 0,\ a_4 = 0, \ldots\ldots \\
\therefore \sin. x &= a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \ldots\ldots\ldots\ldots \quad (1)
\end{aligned}
$$

Again, the series for cos. x must $= 1$ when $x = 0$, and therefore the series must contain a term independent of x, and it must be 1; also, the series can contain no odd powers of x, for if we suppose

$$
\begin{aligned}
\cos. x &= 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots \\
\text{then } \cos. (-x) &= 1 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 - \ldots\ldots \\
\text{but } \cos. (-x) &= \cos. x \\
&= 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots \\
\therefore \quad a_1 &= -a_1,\ a_3 = -a_3, \ldots\ldots \quad \therefore a_1 = 0,\ a_3 = 0 \ldots\ldots \\
\therefore \cos. x &= 1 + a_2x^2 + a_4x^4 + a_6x^6 + \ldots\ldots\ldots \quad (2)
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence } \cos. x + \sin. x &= 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots \quad (3) \\
\cos. x - \sin. x &= 1 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 - a_5x^5 + \ldots\ldots \quad (4)
\end{aligned}
$$

Now in equation (3) write $x + h$ for x, and we have

$$\cos.(x+h)+\sin.(x+h)=1+a_1(x+h)+a_2(x+h)^2+a_3(x+h)^3+\ldots\ldots \quad (5)$$

$$
\begin{aligned}
\text{but } \cos.(x+h)+\sin.(x+h) &= \cos. x \cos. h - \sin. x \sin. h + \sin. x \cos. h + \cos. x \sin. h \\
&= \cos. h\,(\cos. x + \sin. x) + \sin. h\,(\cos. x - \sin. x) \\
&= (1+a_2h^2+a_4h^4+\ldots).(1+a_1x+a_2x^2+a_3x^3+\ldots) \\
&\quad + (a_1h+a_3h^3+a_5h^5+\ldots)(1-a_1x+a_2x^2-a_3x^3+\ldots) \\
&= \left.\begin{array}{l}
1+a_1x+a_2x^2+a_3x^3 + \ldots \\
+a_1h-a_1^2xh+a_1a_2x^2h + \ldots \\
\quad + a_2h^2 + a_1a_2xh^2 + \ldots \\
\quad\quad + a_3h^3 + \ldots \\
\quad\quad\quad + \ldots
\end{array}\right\} \ldots\ldots\ldots \quad (6)
\end{aligned}
$$

Comparing equations (5) and (6) we have

$$
\left.\begin{array}{l}
1+a_1x+a_2x^2+a_3x^3+\ldots \\
+a_1h+2a_2xh+3a_3x^2h+\ldots \\
\quad + a_2h^2+3a_3xh^2+\ldots \\
\quad\quad + a_3h^3+\ldots \\
\quad\quad\quad +\ldots
\end{array}\right\} = \left\{\begin{array}{l}
1 + a_1x + a_2x^2 + a_3x^3 + \ldots \\
+ a_1h - a_1a_1xh + a_1a_2x^2h - \ldots \\
\quad + a_2h^2 + a_1a_2xh^2 + \ldots \\
\quad\quad + a_3h^3 - \ldots \\
\quad\quad\quad + \ldots
\end{array}\right\}
$$

and equating the coefficients of the terms involving the same powers of x and h, we have

$$
\begin{aligned}
2a_2 &= -a_1a_1;\ \text{therefore } a_2 = -\frac{a_1a_1}{2} = -\frac{a_1^2}{1.2} \\
3a_3 &= a_1a_2 \ldots\ldots\ldots a_3 = \frac{a_1a_2}{3} = -\frac{a_1^3}{1.2.3} \\
4a_4 &= -a_1a_3 \ldots\ldots\ldots a_4 = -\frac{a_1a_3}{4} = +\frac{a_1^4}{1.2.3.4} \\
5a_5 &= a_1a_4 \ldots\ldots\ldots a_5 = \frac{a_1a_4}{5} = +\frac{a_1^5}{1.2.3.4.5}
\end{aligned}
$$

$$\text{hence sin. } x = a_1 x - \frac{a_1^3}{1.2.3}x^3 + \frac{a_1^5}{1.2.3.4.5}x^5 - \frac{a_1^7}{1.2.3.4.5.6.7}x^7 + \ldots$$

$$\cos. x = 1 - \frac{a_1^2}{1.2}x^2 + \frac{a_1^4}{1.2.3.4}x^4 - \frac{a_1^6}{1.2.3.4.5.6}x^6 + \ldots$$

and we have only to determine the value of a_1. To effect this we have

$$\sin. x = a_1 x - \frac{a_1^3}{1.2.3}x^3 + \frac{a_1^5}{1.2.3.4.5}x^5 - \ldots.$$

$$= a_1 x\left(1 - \frac{a_1^3}{1.2.3}x^2 + \frac{a_1^4}{1.2.3.4.5}x^4 - \ldots.\right)$$

Now the value of x may be assumed so small that the series in the parenthesis, and sin. x, shall differ from 1 and x respectively, by less than any assignable quantities; hence ultimately

$$x = a_1 x, \text{ and therefore } a_1 = 1; \text{ whence}$$

$$\sin. x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \ldots$$

$$\cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \ldots.$$

To develop tan. x *and cot.* x *in a series ascending by the powers of* x.

The development may be obtained from those of sin. x and cos. x, already found.

$$\tan. x = \frac{\sin. x}{\cos. x} = \frac{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.}{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.}$$

and the series will therefore be of the form $x + a_3x^3 + a_5x^5 + a_7x^7 + \ldots.$

$$\text{Hence, let } x + a_3x^3 + a_5x^5 + \ldots = \frac{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots.}{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots.}$$

$$\therefore x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots = \left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots\right)\left(x + a_3x^3 + a_5x^5 + \ldots\right)$$

$$= x + \begin{array}{l|l|l} a_3 & a_5 & \\ -\frac{1}{1\cdot 2} & -\frac{a_3}{1\cdot 2} & \\ & +\frac{1}{1.2.3.4} & \end{array} \begin{array}{l} x^3 + \\ \\ \\ \end{array} \begin{array}{l} x^5 + \\ - \\ + \\ - \end{array}$$

Hence, equating the coefficients of the like terms, we have

$$a_3 - \frac{1}{1.2} = -\frac{1}{1.2.3} \qquad \therefore a_3 = \frac{2}{1.2.3}$$

$$a_5 - \frac{a_3}{1.2} + \frac{1}{1.2.3.4} = \frac{1}{1.2.3.4.5} \qquad \therefore a_5 = \frac{2^4}{1.2.3.4.5} \ \&c.$$

$$\therefore \tan. x = x + \frac{2x^3}{1.2.3} + \frac{2^4x^5}{1.2.3.4.5} + \ldots\ldots\ldots\ldots$$

$$\text{Sim. cot. } x = \frac{1}{x} - \frac{2x}{1.2.3} - \frac{2^4x^3}{1.2.3.4.5} - \ldots\ldots\ldots\ldots$$

CHAPTER III.

FORMULÆ FOR THE SOLUTION OF TRIANGLES.

We shall here repeat the enunciations of the two propositions established in Chapter I.

PROP. I.

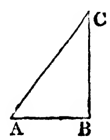
In any right-angled plane triangle,

1°. *The ratio which the side opposite to one of the acute angles has to the hypothenuse, is the sine of that angle.*

2°. *The ratio which the side adjacent to one of the acute angles has to the hypothenuse, is the cosine of that angle.*

3°. *The ratio which the side opposite to one of the acute angles has to the side adjacent to that angle, is the tangent of that angle.*

Thus, in any right-angled triangle ABC,



$$\frac{CB}{CA} = \sin. A, \quad \frac{BA}{AC} = \cos. A, \quad \frac{CB}{BA} = \tan. A$$
$$= \cos. C, \quad = \sin. C, \quad = \cot. C$$

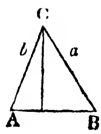
Or,
$$\left.\begin{aligned} CB &= AC \sin. A \\ &= AC \cos. C \end{aligned}\right\}$$
$$\left.\begin{aligned} BA &= AC \cos. A \\ &= AC \sin. C \end{aligned}\right\}$$
$$\left.\begin{aligned} CB &= BA \tan. A \\ &= BA \cot. C \end{aligned}\right\} \quad \ldots\ldots (\alpha)$$

PROP. II.

In any plane triangle, the sides are to each other as the sines of the angles opposite to them.

We shall, henceforth, in treating of triangles, make use of the following notation. We shall denote the angles of the triangle by the large letters at the angular points, and the sides of the triangle opposite to these angles, by the corresponding small letters.

Thus, in the triangle ABC, we shall denote the angles BAC, CBA, BCA, by the letters A, B, C, respectively, and the sides BC, AC, AB, by the letters a, b, c, respectively.



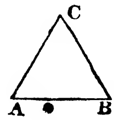
According to this, we shall have, by the proposition,

$$\left.\begin{aligned} \frac{a}{b} &= \frac{\sin. A}{\sin. B} \\ \frac{a}{c} &= \frac{\sin. A}{\sin. C} \\ \frac{b}{c} &= \frac{\sin. B}{\sin. C} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

PROP. III.

In any plane triangle, the sum of any two sides, is to their difference, as the tangent of half the sum of the angles opposite to them, is to the tangent of half their difference.

Let ABC be any plane triangle, then, by proposition, II



$$\frac{a}{b} = \frac{\sin. A}{\sin. B}$$

$$\frac{a+b}{a-b} = \frac{\sin. A + \sin. B}{\sin. A - \sin. B}$$

But, by Trigonometry, Chap. II. (r),

$$\sin. A + \sin. B = 2 \sin. \frac{A+B}{2} \cos. \frac{A-B}{2}$$

$$\sin. A - \sin. B = 2 \cos. \frac{A+B}{2} \sin. \frac{A-B}{2}$$

$$\therefore \frac{a+b}{a-b} = \frac{2 \sin. \frac{A+B}{2} \cos. \frac{A-B}{2}}{2 \cos. \frac{A+B}{2} \sin. \frac{A-B}{2}}$$

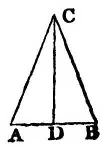
$$= \tan. \frac{A+B}{2} \cot. \frac{A-B}{2}$$

$$\left.\begin{aligned} &= \frac{\tan. \frac{A+B}{2}}{\tan. \frac{A-B}{2}} \\ &\text{And, in like manner,} \\ \frac{a+c}{a-c} &= \frac{\tan. \frac{A+C}{2}}{\tan. \frac{A-C}{2}} \\ \frac{b+c}{b-c} &= \frac{\tan. \frac{B+C}{2}}{\tan. \frac{B-C}{2}} \end{aligned}\right\} \ldots\ldots\ldots\ldots (\gamma)$$

PROP. IV.

To express the cosine of an angle of a plane triangle in terms of the sides of the triangle.

Let ABC be a triangle; A, B, C, the three angles; a, b, c, the corresponding sides.



1. Let the proposed angle (A) be acute.

From C draw CD perpendicular to AB, the base of the triangle.

Then,

$$BC^2 = AC^2 + AB^2 - 2AB \,.\, AD \text{ (Geom.)}$$

Or,

$$a^2 = b^2 + c^2 - 2c \,.\, AD$$

But, since CDA is a right-angled triangle,

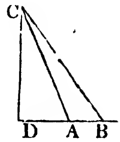
$$AD = AC \cos. CAD = b \cos. A.$$

$$\therefore\ a^2 = b^2 + c^2 - 2bc \cos. A$$

$$\therefore\ \cos. A = \frac{b^2 + c^2 - a^2}{2bc}$$ which is the expression required.

2. Let the proposed angle (A) be obtuse.

From C, draw CD perpendicular to AB produced.



Then,

$$BC^2 = AC^2 + AB^2 + 2AB.AD$$

Or,

$$a^2 = b^2 + c^2 + 2c\ .AD$$

But, since CDA is a right-angled triangle,

$$AD = AC \cos. CAD$$
$$= AC \times - \cos. CAB \quad \because \text{CAB is the supplement of CAD.}$$
$$= - b \cos. A$$

$$\therefore\ a^2 = b^2 + c^2 - 2bc \cos. A$$

$$\therefore\ \cos. A = \frac{b^2 + c^2 - a^2}{2bc}$$

It will be seen that this result is identical with that which we deduced in the last case, so that, whether A be acute or obtuse, we shall have,

Proceeding in the same manner for the other angles, we shall find, (0)

$$\cos. B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos. C = \frac{a^2 + b^2 - c^2}{2ab}$$

PROP. V.

To express the sine of an angle of a plane triangle in terms of the sides of the triangle.

Let A be the proposed angle; then by last prop.,

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}$$

Adding unity to each member of the equation,

$$1 + \cos. A = 1 + \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{b^2 + 2bc + c^2 - a^2}{2bc}$$
$$= \frac{(b+c)^2 - a^2}{2bc}$$
$$= \frac{(b+c+a)(b+c-a)}{2bc} \quad \text{.......................(1)}$$

Again, $$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}$$

Subtracting each member of the equation from unity,

$$1 - \cos. A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{(a + b - c)(a + c - b)}{2bc} \quad (2)$$

Multiplying together equations (1) and (2),

$$(1 + \cos. A) \times (1 - \cos. A) = \frac{(a + b + c)(b + c - a)(a + c - b)(a + b - c)}{4 b^2 c^2}$$

$$\text{But } (1 + \cos. A)(1 - \cos. A) = 1 - \cos.^2 A$$

$$= \sin.^2 A \quad \text{(Table I.)}$$

$$\therefore \sin.^2 A = \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4 b^2 c^2}$$

Extracting the root on both sides,

$$\sin. A = \frac{1}{2bc} \cdot \sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)} \ldots (3)$$

The above expression, for the sine of an angle of a triangle in terms of the sides, is sometimes exhibited under a form somewhat different.

Let s denote the semiperimeter, that is to say, half the sum of the sides of the triangle; then

$$s = \frac{a + b + c}{2}, \text{ and,} \qquad 2s = a + b + c$$

$$s - a = \frac{b + c - a}{2} \ \ldots \ 2(s - a) = b + c - a$$

$$s - b = \frac{a + c - b}{2} \ \ldots \ 2(s - b) = a + c - b$$

$$s - c = \frac{a + b - c}{2} \ \ldots \ 2(s - c) = a + b - c$$

Substituting $2s$, $2(s - a)$,...... for $a + b + c$, $b + c - a$,...... in the expression for $\sin.^2 A$, it becomes

$$\sin.^2 A = \frac{16 s (s - a)(s - b)(s - c)}{4 b^2 c^2}$$

And extracting the root on both sides,

$$\sin. A = \frac{2}{bc} \cdot \sqrt{s (s - a)(s - b)(s - c)}$$

Proceeding in the same manner for the other angles, we shall find ......(s)

$$\sin. B = \frac{2}{ac} \cdot \sqrt{s (s - a)(s - b)(s - c)}$$

$$\sin. C = \frac{2}{ab} \cdot \sqrt{s (s - a)(s - b)(s - c)}$$

By equation (1) we have .

$$1 + \cos. A = \frac{(a+b+c)(b+c-a)}{2bc}$$

$$= \frac{4s(s-a)}{2bc}.$$

But, by Chap. II,

$$1 + \cos. A = 2\cos.^2 \frac{A}{2}$$

$$\therefore \ 2\cos^2 \frac{A}{2} = \frac{4s(s-a)}{2bc}$$

Extracting the root on both sides,

$$\left.\begin{array}{l} \cos. \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}} \\ \text{And in like manner,} \\ \cos. \dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ac}} \\ \cos. \dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (\sigma)$$

By equation (2) we have

$$1 - \cos. A = \frac{(a+c-b)(a+b-c)}{2bc}$$

$$= \frac{4(s-b)(s-c)}{2bc}$$

But, by Chap. II,

$$1 - \cos. A = 2\sin.^2 \frac{A}{2}$$

$$\therefore \ 2\sin.^2 \frac{A}{2} = \frac{4(s-b)(s-c)}{2bc}$$

Extracting the root on both sides,

$$\left.\begin{array}{l} \sin. \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}} \\ \text{And in like manner,} \\ \sin. \dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{ac}} \\ \sin. \dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (\zeta)$$

Dividing the formulæ marked (ζ) by those marked (σ), we have

$$\left.\begin{array}{l} \tan. \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} \\ \tan. \dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}} \\ \tan. \dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (\eta)$$

CHAPTER IV.

ON THE CONSTRUCTION OF TRIGONOMETRICAL TABLES.

Before proceeding to apply the formulæ deduced in the last chapter to the solution of triangles, we shall make a few remarks upon the construction of those tables, by means of which we are enabled to reduce our trigonometrical calculations to numerical results.

It is manifest, from definitions 1°, 2°, 3°, &c. that the various trigonometrical quantities, the sine, the cosine, the tangent, &c. are abstract numbers representing the comparative length of certain lines. We have already obtained the numerical value of these quantities in a few particular cases, and we shall now show how the numbers, corresponding to angles of every degree of magnitude, may be obtained by the application of the most simple principles.

The numbers corresponding to the sine, cosine, &c. of all angles from 1′ up to 90°, when arranged in a table, form what is called the *Trigonometrical Canon.*

The first operation to be performed is

To compute the numerical value of the sine and cosine of 1′.

We have seen, Chap. II. formula (*j*) that

$$\sin.\frac{\theta}{2} = \sqrt{\frac{1-\cos.\theta}{2}}$$
$$= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\sin.^2\theta}}$$

By which formula the sine of any angle is given in terms of the sine of twice that angle.

Now substitute $\frac{\theta}{2}$ for θ and it becomes

$$\text{Sin.}\frac{\theta}{4} \text{ or } \sin.\frac{\theta}{2^2} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\sin.^2\frac{\theta}{2}}}$$

$$\text{In like manner, } \sin.\frac{\theta}{2^3} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\sin.}}$$

$$\sin.\frac{\theta}{2^4} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\sin.^2\frac{\theta}{2^3}}}$$

$$\&c. = \&c.$$

$$\text{And generally, } \sin.\frac{\theta}{2^n} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\sin.^2\frac{\theta}{2^{n-1}}}}$$

$$\text{Now let } \theta = 30° \quad \therefore \quad \frac{\theta}{2} = 15°$$

and applying the above formula, we have

$$\sin.15° = \sqrt{\tfrac{1}{2} - \sqrt{1-\sin.^2 30°}}$$

But by Chap. II. $\sin. 30^\circ = \frac{1}{2}$ $\therefore \sin.^2 30^\circ = \frac{1}{4}$

$$\therefore \sin. 15^\circ = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \tfrac{1}{4}}}$$
$$= \tfrac{1}{2}\sqrt{2 - \sqrt{3}}$$
$$= \cdot 2588190$$

Similarly, $\sin. 7^\circ\ 30' = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \sin.^2 15^\circ}}$
$$= \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - (\cdot 2588190)^2}}$$
$$= \cdot 1305268 \ \ldots\ldots$$
$$\&c. = \&c.$$

It is manifest, that, by continuing the process, we shall obtain in succession the sines of $3^\circ\ 45'$, of $1^\circ\ 52'\ 30''$, &c.

In this way we find

$$\text{Sin. } \frac{30^\circ}{2^{11}} \text{ or sin. } 1'\ 45''\ 28'''\ 7^{iv}\ 30^{v} = \cdot 0005113269, \&c.$$

$$\text{Sin. } \frac{30^\circ}{2^{12}} \text{ or sin. } 52''\ 44'''\ 3^{iv}\ 45^{v} = \cdot 0002556634, \&c.$$

From which it appears, that, when the operation above-mentioned has been repeated so many times, the sine of the arc is halved at the same time that the arc itself is bisected: that is,

*The sines of very small arcs are proportional to the arcs themselves.**

Hence we shall have

$$\text{Sin. } 52''\ 44'''\ 3^{iv}\ 45^{v} : \text{sin. } 1' :: 52''\ 44'''\ 3^{iv}\ 45^{v} : 1'$$
$$:: \frac{60}{2^{12}} \qquad 60$$
$$\qquad\qquad 60 \times 60$$
$$:: 3600 \quad : 4096$$

$$\therefore \sin. 1' = \frac{\sin. 52''\ 44'''\ 3^{iv}\ 45^{v} \times 4096}{3600}$$
$$= \frac{\cdot 0002556634 \times 4096}{3600}$$
$$= \cdot 000290888204 \ldots\ldots = \cos. 89^\circ\ 59' \quad \because \sin. \theta = \cos. (90^\circ - \theta)$$

Again, $\because \cos. \theta = \sqrt{1 - \sin.^2 \theta}$
$$\cos. 1' = \sqrt{1 - (.000290888204 \ldots)^2}$$
$$= \cdot 999999915384 \ldots\ldots\ldots$$

The sine and cosine of $1'$ being thus determined, we shall proceed to show in what manner we shall now be enabled to compute the sines and cosines of all superior angles.

By formula (m) Chap. II.

$$\text{Sin. } (n+1)\,\theta = 2 \cos. \theta \sin. n\,\theta - \sin. (n-1)\,\theta$$

If we suppose $\theta = 1'$ and n to be taken = to the numbers 1, 2, 3, ... in succession, we find

$$\text{Sin. } 2' = 2 \cos. 1' \sin. 1' - \sin. 0 = \cdot 0005817764 \ldots = \cos. 89^\circ\ 58'$$
$$\text{Sin. } 3' = 2 \cos. 1' \sin. 2' - \sin. 1' = \cdot 0008726645 \ldots = \cos. 89^\circ\ 57'$$
$$\text{Sin. } 4' = 2 \cos. 1' \sin. 3' - \sin. 2' = \cdot 0011635526 \ldots = \cos. 89^\circ\ 56$$
$$\&c. = \&c.$$

* This proposition is not *accurately* true, but no appreciable error will be introduced into our calculations by employing it, for it holds good as far as ten places of decimals.

Again, by employing formula (o), Chap. II.

$$\text{Cos.}\ (n+1)\,\theta = 2\cos.\theta\cos.n\,\theta - \cos.(n-1)\,\theta$$

If, as before, we suppose $\theta = 1'$ and $n = 1, 2, 3, \ldots\ldots$ in succession,

$$\begin{aligned}
\text{Cos.}\ 2' &= \quad\ 2\cos.^2 1' - \cos.\ 0 = \cdot 999999830 \ldots\ldots = \sin.\ 89^\circ\ 58' \\
\text{Cos.}\ 3' &= 2\cos.\ 1'\cos.\ 2' - \cos.\ 1' = \cdot 999999619 \ldots\ldots = \sin.\ 89^\circ\ 57' \\
\text{Cos.}\ 4' &= 2\cos.\ 1'\cos.\ 3' - \cos.\ 2' = \cdot 999999323 \ldots\ldots = \sin.\ 89^\circ\ 56' \\
\&c. &= \&c.
\end{aligned}$$

It is manifest, that, by continuing the above processes, we shall obtain the numerical values of the sines and cosines of all angles from 1′ up to 90°. These being determined, the tangents, cotangents, &c. may be calculated by means of the relations established in table I.

The above operations are exceedingly laborious, but require a knowledge of the fundamental rules of arithmetic alone. It is manifest that, in employing this method, an error committed in the sine or cosine of an inferior arc, will entail errors on the sines or cosines of all succeeding arcs. Hence is created the necessity of some check on the computist, and of some independent mode of examining the accuracy of the computation. For this purpose, formulæ, derived immediately from established properties, are employed; if the numerical results from these formulæ agree with the results obtained by a regular process of computation, then it is almost a certain conclusion that the latter process has been rightly conducted.

Formulæ employed for this purpose are called *formulæ of verification*, and of these any number may be obtained; it will be sufficient for our present purpose to give one.

$$\begin{aligned}
&\text{Sin.}^2\,\theta + \cos.^2\,\theta = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \text{tab. I.} \\
\text{And}\quad &2\sin.\theta\cos\theta = \sin.\ 2\,\theta \\
\text{Hence}\quad &\sin.\theta = \tfrac{1}{2}\sqrt{1+\sin.\ 2\,\theta} \pm \tfrac{1}{2}\sqrt{1-\sin.\ 2\,\theta} \\
&\cos.\theta = \tfrac{1}{2}\sqrt{1+\sin.\ 2\,\theta} \mp \tfrac{1}{2}\sqrt{1-\sin.\ 2\,\theta} \\
\text{Now we if suppose}\quad &\theta = 12^\circ\ 30' \\
&\sin.\ 12^\circ\ 30' = \tfrac{1}{2}\sqrt{1+\sin.\ 25^\circ} \pm \tfrac{1}{2}\sqrt{1-\sin.\ 25^\circ} \\
&\cos.\ 12^\circ\ 30' = \tfrac{1}{2}\sqrt{1+\sin.\ 25^\circ} \mp \tfrac{1}{2}\sqrt{1-\sin.\ 25^\circ}
\end{aligned}$$

Hence, if the values of the sine and cosine of 12° 30′, and of the sine of 25 obtained by the method already explained, when substituted in these equations, render the two members identical, we conclude that our operations are correct.

The values of the sine and cosine of 30°, 45°, 60°, &c. which were obtained in Chap. II., may be employed as formulæ of verification.*

Such is the formation of the trigonometrical canon.

* We can obtain finite expressions, although under an incommensurable form, for the sines of arcs of 3°, and all the multiples of 3°, i. e. for

3°, 6°, 9°, 12°, 15°, 18°, 21°, 24°, 27°, 30°, 33°, 36°, 39°, 42°, 45°, 48°, 51°, 54°, 57°, 60°, 63°, 66°, 69°, 72°, 75°, 78°, 81°, 84°, 87°, 90°.

We first obtain the values of the sines 30°, 45°, 60°, 18°, and from these we obtain all the others, by means of the formulæ, for

$$\text{Sin.}\ (\theta+\theta'),\quad \sin.\ (\theta-\theta'),\ \&c.$$

The numerical value of the trigonometrical functions have been calculated by some to ten places of figures, by others as far as twelve. We must have tables calculated to ten places to have the seconds and tenths of a second with precision, when we make use of the sines of angles which differ but little from 90°, or of the cosines of angles of a few seconds only. Tables in general, however, are calculated as far as seven places only, and these give results sufficiently accurate for all ordinary purposes.

Since the properties of Logarithms afford great facilities in performing complicated arithmetical operations upon large numbers, it becomes desirable to have the Logarithms of sines, cosines, tangents, &c. computed and arranged in tables; but most of these numbers being less than unity, their Logarithms would, of course, be negative. To avoid this inconvenience, all the trigonometrical functions calculated in the manner above explained are multiplied by a large number, and, the operation being performed upon all, their relative value is not altered. This number may, of course, be any whatever, provided it be so large, that, when the numerical values of trigonometrical quantities are multiplied by it, their logarithms may be positive numbers.

The number employed for this purpose in the common tables is 10000000000 or 10^{10}, which is usually represented by the symbol R.

The sine of 1′, as computed above, is

$$\text{Sin. } 1' = \cdot 0002908882 \ldots\ldots\ldots$$

a number much smaller than unity, and whose logarithm would consequently be negative.

When multiplied by 10^{10} it becomes

$$= 2908882 \ldots\ldots\ldots$$

a number whose logarithm is 6·4637261, and consequently we find in our tables

$$\log. \sin. 1' = 6 \cdot 4637261$$

A table constituted upon this principle is called a *Table of Logarithmic Sines, Cosines, Tangents,* &c. and by this nearly all the practical operations of trigonometry are usually performed.

It is manifest, from these remarks, that, before we can apply formulæ deduced in the preceding chapters to practical purposes, we must transform them in such a manner as to render the several trigonometrical quantities identical with those registered in our tables, The sines, cosines, &c. we have hitherto employed, are called *Trigonometrical quantities calculated to a radius unity;* those registered in the tables, *Trigonometrical quantities calculated to radius* R.

The problem to be solved therefore is

To transform an expression calculated to a radius unity, to another calculated to a radius R.

Let us represent sin. θ to radius unity by m.

.. .. .. R by n.

Then the relation between them is

$$n = \text{R}\, m$$

$$m = \frac{n}{\text{R}}$$

and so for all the other trigonometrical quantities.

Hence, *in order to transform an expression calculated to radius unity, to another calculated to radius* R, *we must divide each of the trigonometrical quantities by* R.

If any of the trigonometrical quantities enter in the square, cube, &c. these must of course be divided by R^2, R^3, &c. ....

As observed above, R may be any given number whatever, the number usually employed in the ordinary tables being 10^{10}, and therefore

$$\log. \text{R} = 10$$

Take as an example such an expression as

$$a \sin. \theta = b \tan.^2 \varphi$$

in order to reduce this to an expression which we can compute by our tables we must, according to the above rule, divide each of the trigonometrical quantities by the proper power of R: the expression then becomes

$$a \frac{\sin. \theta}{R} = b \frac{\tan.^2 \phi}{R^2}$$

Or, clearing of fractions,

$$a \, R \sin. \theta = b \tan.^2 \phi$$

Or,

$$\log. a + \log. R + \log. \sin. \theta = \log. b + 2 \log. \tan. \phi$$

an expression which may be calculated by the tables.

If the expression calculated to radius unity be of the form

$$m = \frac{\sin. \theta}{\sin. \phi}$$

it requires no modification, for if we divide both terms of the fraction $\frac{\sin. \theta}{\sin. \phi}$ by R, we shall not alter its value.

We need not prosecute this subject farther, as numerous examples of these transformations will occur at every step in the succeeding chapters.

CHAPTER V.

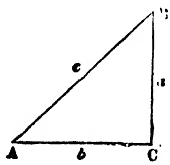
ON THE SOLUTION OF RIGHT ANGLED TRIANGLES.

Every plane triangle being considered to consist of 6 parts, the three sides, and the three angles, if any three of these parts be given, we can, in general, determine the remaining parts by trigonometry.

In right angled triangles, the right angle is always known, and therefore any two other parts being given, we can, in general, determine the rest. We shall thus have five different cases.

1. When one of the acute angles and the hypotenuse is given.
2. When one of the acute angles and a side is given.
3. When the hypotenuse and one side is given.
4. When the two sides are given.
5. When the two acute angles are given.

Let A, B, C, be a right angled triangle, C the right angle.



Let the sides opposite to the angles A and B be denoted respectively by a, b, and let the hypotenuse be called c.

Case 1, Given A, c, required B, a, b.

Since C is a right angle

$$A + B = 90°$$

$$\therefore \quad B = 90° - A \text{ whence B is known } \ldots\ldots\ldots \quad (1)$$

By Chap. III. prop. 1,

$$a = c \sin. A$$

Adapting this expression to computation by the tables.

$$\text{log. } a = c \, \frac{\text{sin. A}}{\text{R}}$$

$\therefore$ log. a = log. c + log. sin. A — log. R, whence a is known .... (2)

In like manner,

$$b = c \, \frac{\text{cos. A}}{\text{R}}$$

$\therefore$ log. b = log. c + log. cos. A — log. R, whence b is known......... (3)

If B, c, are given, and A, a, b required, we shall have precisely in the same manner.

A = 90° — B .... (4)

log. a = log. c + log. cos. B — log. R .... (5)

log. b = log. c + log. sin. B — log. R .... (6)

Case 2. Given A, a, required B, b, c.

B = 90° — A, whence B is known .... (7)

$$b = a \text{ cot. A}$$

Adapting the expression to computation by the tables.

$$b = a \, \frac{\text{cot. A}}{\text{R}}$$

log. b = log. a + log. cot. A — log. R, whence b is known,......... (8)

Again,

$$a = c \text{ sin. A}$$

$$\therefore \quad c = \frac{a}{\text{sin. A}}$$

Adapting it to computation,

$$c = \text{R} \, . \, \frac{a}{\text{sin. A}}$$

$\therefore$ log. c = log. R + log. a — log. sin. A, whence c is known, ... (9)

If A, b, be given, and B, a, c, we shall have in like manner,

B = 90° — A .... (10)

log. a = log. b + log. tan. A — log. R .... (11)

log. c = log. R + log. b — log. cos. A .... (12)

If B, b be given, and A, a, c required.

A = 90° — B .... (13)

log. a = log. b + log. cot. B — log. R .... (14)

log. c = log. R + log. b — log. sin. B .... (15)

If B, a be given, and A, b, c, required.

A = 90° — B .... (16)

log. b = log. a + log. tan. B — log. R .... (17)

log. c = log. R + log. a — log. cos. B .... (18)

Case 3. Let a, c be given, required b, A, B.

$$b^2 = c^2 - a^2$$

$$= (c + a)\,(c - a)$$

$\therefore$ 2 log. b = log. $(c + a)$ + log. $(c - a)$ whence b is known .... (19)

$$\text{sin. A} = \frac{a}{c}$$

Adapting the expression to computation.

$$\frac{\sin. A}{R} = \frac{a}{c}$$

$\therefore$ log. sin. A $=$ log. R $+$ log. a - log. c, whence A is known ...... (20)

So also,

$$\frac{\cos. B}{R} = \frac{a}{c}$$

$\therefore$ log. cos. B $=$ log. R $+$ log. a — log. c, whence B is known, ... (21)

If b, c be given, and a, A, B required, we shall have

2 log. a $=$ log. $(c+b)$ $+$ log. $(c-b)$ .... (22)

log. cos. A $=$ log. R $+$ log. b — log. c .... (23)

log. sin. B $=$ log. R $+$ log. b — log. c .... (24)

Case 4. Let a, b be given, required A, B, c.

$$\tan. A = \frac{a}{b}$$

Adapting the expression to computation.

$$\frac{\tan. A}{R} = \frac{a}{b}$$

$\therefore$ log. tan. A $=$ log. R $+$ log. a — log. b, whence A is known, ... (25)

So also,

$$\frac{\tan. B}{R} = \frac{b}{a}$$

$\therefore$ log. tan. B $=$ log. R $+$ log. b — log. a, whence B is known, ... (26)

$c = \sqrt{a^2 + b^2}$, whence c is known, ... (27)

Case 5. Given A, B, required a, b, c.

It is manifest that this case does not admit of solution, for *any number* of unequal similar triangles may be constructed, having their angles equal to the angles A, B, C.

We shall conclude this chapter by giving one or two numerical examples.

Example 1. Given A $= 26^\circ\ 41'\ 6''$, $c = 6539{\cdot}76$ yards, required a

Then by (2).

log. a $=$ log. c. $+$ log. sin. A — log. R

By the tables, log. c $=$ 3·8155618

log. sin. A $=$ 9·6523286

13·4678904

log. R $=$ 10·

$\therefore$ log. a $=$ 3·4678904

The number in the tables corresponding to the logarithm 3·4678904 is found to be 2936·91

$a = 2936{\cdot}91$ yards.

In like manner, the side b may be determined, if required.

Example 2. Given $c = 6539{\cdot}76$ yards, $a = 2936{\cdot}91$ yards, required b, A, B.

By (19).

log. b $=$ log. $(c+a)$ $+$ log. $(c-a)$ $c + a = 9476{\cdot}67$

$c - a = 3602{\cdot}85$

By the tables,

$$\log.(c+a) = 3{\cdot}9766557$$
$$\log.(c-a) = 3{\cdot}5566462$$

$$\therefore \quad 2\log. b = 7{\cdot}5333019$$
$$\log. b = 3{\cdot}7666509$$

The number in the tables corresponding to the logarithm 3·7666509 is 5843·2

$\therefore \qquad b = 5843{\cdot}2$ yards.

To determine A we have (20).

$$\log. \sin. A = \log. R + \log. a - \log. c$$

By the tables, $\log. a = 3{\cdot}4678904$

$$\log. R = 10.$$

$$13{\cdot}4678904$$
$$\log. c = 3{\cdot}8155618$$

$$\therefore \quad \log. \sin. A = 9{\cdot}6523286$$

On referring to our tables, we shall find that the angle whose logarithmic sine is 9.6523286 is 26° 41′ 6″, which is consequently the value of A.

A being known, B is determined at once by subtracting the value of A from 90°, or B may be determined independently of A by (21).

$$\log. \cos. B = \log. R + \log. a - \log. c$$

Example 3. Let a, b, in the last example, be given, and c required.

Then by (27).

$$c = \sqrt{a^2+b^2} \text{ or } c^2 = a^2 + b^2$$

The calculation in this case is not so simple, for the quantity under the radical cannot be easily adapted to logarithmic calculation.

We have, $\log. a^2 = 6{\cdot}9357808 \quad \therefore \quad a^2 = 8625400$

$\log. b^2 = 7{\cdot}5333019 \quad \therefore \quad b^2 = 34143000$

$\therefore \quad c^2 = 42768400$

$$\therefore \quad \log. c^2 = 7{\cdot}6311230$$
$$\log. c = 3{\cdot}8155615$$
$$c = 6539{\cdot}76$$

Example 4. Given $c = 6512.4$ yards, $b = 6510.6$, to find A.

By (23).

$$\log. \cos. A = \log. R + \log. b - \log. c$$

Now, $\log. R = 10.$

$$\log. b = 3{\cdot}8136210$$

$$13{\cdot}8136210$$
$$\log. c = 3.8137411$$

$$\therefore \quad \log. \cos. A = 9{\cdot}9998799$$
$$A = 1^\circ\ 20'\ 50''$$

Upon inspecting the tables that are calculated to seven places of decimals only, it will be seen that, when the angles become very small, the cosines differ very little from each other. The same remark applies, of course, to the sines of angles nearly 90°. In cases, therefore, where great accuracy is required, we may commit an important error by calculating a small angle from its cosine, or a large one from its sine. We must consequently endeavour to avoid this,

by transforming our expression by help of the relations established in chapter first and second.

In the example before us, A is a small angle which has been calculated from its cosine; we must therefore, if possible, calculate this angle by means of its sine, or some other trigonometrical function.

Now, by formula (j), chap. II. we have generally

$$\sin. \frac{A}{2} = \sqrt{\frac{1 - \cos. A}{2}}$$

In the present case, $\cos. A = \frac{a}{c}$, substituting this in the above equation,

$$\sin. \frac{A}{2} = \sqrt{\frac{c - a}{2 a}}$$

$$\therefore \quad \log. \sin. \frac{A}{2} = \tfrac{1}{2} \log. (c - a) - \tfrac{1}{2} \log. 2 a + \log. R.$$

From which we find,

$$\frac{A}{2} = 40' \, 24''$$

$$\text{And} \therefore \quad A = 1^\circ \, 20' \, 48''$$

Instead of $1^\circ \, 20' \, 50''$, as obtained by the former process.

No angle which is nearly 90° ought to be calculated from its tangent, for the tangents of large angles increase with so much rapidity, that the results derived from the column of proportional parts found in the tables cannot be depended on as accurate.

CHAPTER VI.

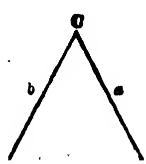
ON THE SOLUTION OF OBLIQUE ANGLED TRIANGLES

Six different cases present themselves.

1. When two angles and the side between them are given.
2. When two angles and the side opposite to one of them are given.
3. When two sides and the included angle are given.
4. When two sides and the angle opposite to one of them are given.
5. When the three sides are given.
6. When the three angles are given.

Let A, B, C be a plane triangle.

Let the angles be denoted by the large letters A, B, C, and the sides opposite to these angles by the corresponding small letters *a*, *b*, *c*.



Case 1. Given A, B, c, required C, a, b.

Since $A + B + C = 180^\circ$

$C = 180^\circ - (A + B)$, whence

C is known.

C being thus determined we have, by chap. III. prop. 2,

$$\frac{a}{c} = \frac{\sin. A}{\sin. C}$$

$$a = c \cdot \frac{\sin. A}{\sin. C}$$

An expression which is in a form adapted to computation by the tables.

$\therefore$ log. a = log. c + log. sin. A — log. sin. C, whence a is known.

Again,

$$\frac{b}{c} = \frac{\sin. B}{\sin. C}$$

$$b = c \cdot \frac{\sin. B}{\sin. C}$$

$\therefore$ log. b = log. c + log. sin. B — log. sin. C, whence b is known.

If any other two angles and the side between them be given, we may determine the remaining angle and sides in a manner precisely similar.

Case 2. Given A, B, a, required C, b, c.

Since $A + B + C = 180°$

$\therefore \quad C = 180° - (A + B)$, whence C is known.

Again,
$$\frac{b}{a} = \frac{\sin. B}{\sin. A}$$

$$\therefore \quad b = a \cdot \frac{\sin. B}{\sin. A}$$

$\therefore$ log. b = log. a + log. sin. B — log. sin. A, whence b is known.

Also, C being known,

$$\frac{c}{a} = \frac{\sin. C}{\sin. A}$$

$$\therefore \quad c = a \frac{\sin. C}{\sin. A}$$

$\therefore$ log. c = log. a + log. sin. C — log. sin. A, whence c is known.

If any two other angles and the side opposite to one of them are given, the remaining angle and sides may be determined in a manner precisely similar.

Case 3. Given a, b, C, required A, B, c.

By prop. 3, chap. III.

$$\frac{\tan. \frac{A + B}{2}}{\tan. \frac{A - B}{2}} = \frac{a + b}{a - b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now,
$$A + B + C = 180°$$

$$\therefore \quad \frac{A + B}{2} = 90° - \frac{C}{2}$$

$$\tan. \frac{A + B}{2} = \tan. \left(90 - \frac{C}{2}\right)$$

$$= \cot. \frac{C}{2}$$

Substituting this value of tan. $\frac{A+B}{2}$ in (1).

$$\frac{\cot.\frac{C}{2}}{\tan.\frac{A-B}{2}} = \frac{a+b}{a-b}$$

$$\tan.\frac{A-B}{2} = \frac{a-b}{a+b}\cot.\frac{C}{2}$$

$$\log.\tan.\frac{A-B}{2} = \log.(a-b) + \log.\cot.\frac{C}{2} - \log.(a+b)$$

And we can thus calculate the value of the angle $\frac{A-B}{2}$ from our tables; let the angle thus found be called ϕ $\therefore$ $A - B = 2\,\phi$.

Now $\quad A + B = 180° - C$

And $\quad A - B = 2\,\phi$

$\therefore$ adding and subtracting

$$A = 90° + \phi - \frac{C}{2}$$

$$B = 90° - (\phi + \frac{C}{2})$$

The angles A and B will thus become known, and, these being determined, we can find the side c from the relation,

$$\frac{c}{a} = \frac{\sin.C}{\sin.A}$$

$$c = a.\frac{\sin.C}{\sin.A}$$

$$\therefore \quad \log.c = \log.a + \log.\sin.C - \log.\sin.A$$

If a, c, B, or b, c, A be given, the remaining angle and sides may be determined in a similar manner by aid of the formula (j) in chap. III.

Case 4. Given a, b, A to determine B, C, c.

$$\frac{\sin.B}{\sin.A} = \frac{b}{a}$$

$$\therefore \quad \sin.B = \sin.A.\frac{b}{a}$$

$\therefore$ $\log.\sin.B = \log.\sin.A + \log.b - \log.a$, whence c is known.

B being known, $\quad C = 180° - (A + B)$, whence C is known.

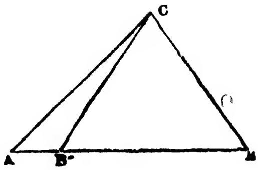
C being known, $\quad \frac{c}{a} = \frac{\sin.C}{\sin.A}$

$$c = a.\frac{\sin.C}{\sin.A}$$

$\log.c = \log.a + \log.\sin.C - \log.\sin.A$, whence b is known.

If any two other sides and the angle opposite to one of them be given, the remaining angles and side may be determined in a manner precisely similar.

It must be remarked, that, in the above case, we determine the angle B from the logarithm of its sine; but since the sine of any angle, and the sine of its supplement are equal to one another, and since it is not always possible for us to ascertain *a priori* whether the angle B is acute or obtuse, the solution will be sometimes ambiguous.



In fact, two different and unequal triangles may be constructed, having two sides and the angle opposite to one of these sides in one triangle, equal to the corresponding sides and angle of the other; one of these triangles will be obtuse-angled, and the other acute-angled, and the angles opposite the remaining given sides in each will be supplemental.

Thus let A, B, C, be a plane triangle.

With centre C and radius equal to CB describe a circle cutting AB in B'.

Join CB'.

Then it is manifest that the two unequal triangles CBA, CB'A, have the two sides CB, CA of the one, equal to the two sides CB', CA of the other, and the angle A, opposite the equal sides CB, CB', in each, common.

It is manifest from this, that it is impossible to determine generally, from the data of this case, which of the two triangles is the solution of the problem. There are certain considerations, however, by which the ambiguity may sometimes be removed.

1. If the given angle be obtuse, then both of the remaining angles must be acute, and the species of B will be determined.

2. If the given angle be acute, but the side opposite the given angle greater than the given side opposite the required angle, then the required angle is acute. For since in every triangle the greater side has the greater angle opposite to it, and since the side opposite to the given angle, which is acute, is greater than the side opposite to the required angle, it follows, a *fortiori*, that the required angle is acute.

But if the given angle be acute, and the side opposite to the given angle less than the side opposite to the required angle, then we have no means of ascertaining the species of the required angle, and the solution in this case is ambiguous.

Case 5. Given the three sides a, b, c, required the three angles A, B, C.

By formula (ε) chap. III.

$$\sin. A = \frac{2}{bc}\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

$$\sin. B = \frac{2}{ac}\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

$$\sin. C = \frac{2}{ab}\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

Adapting these expressions to computation by the tables, and taking the logs.

log. sin. A

$= \log.R + \log.2 + \frac{1}{2}\{\log.s + \log.(s-a) + \log.(s-b) + \log.(s-c)\} - \log.b - \log.c$

log. sin. B

$= \log.R + \log.2 + \frac{1}{2}\{\log.s + \log.(s-a) + \log.(s-b) + \log.(s-c)\} - \log.a - \log.c$

log. sin. C

$= \log.R + \log.2 + \frac{1}{2}\{\log.s + \log.(s-a) + \log.(s-b) + \log.(s-c)\} - \log.a - \log.b$

Whence the three angles are known.

The three angles may also be obtained from any of the groups of formulæ (σ), (ζ), (η), in chap. III.

It is manifest, from the remarks made at the conclusion of the last chapter, that, when one or more of the required angles is very small, the group (σ) may

be used with greatest advantage, and when one or more of the angles is nearly 90°, we ought to employ the group (ζ). The group (η) may be made use of in any case.

Case 6. Given the three angles A, B, C, required the three sides a, b, c.

It is manifest that this case does not admit of solution, for *any number* of unequal similar triangles may be constructed, having their angles equal to the angles A, B, C.

We shall conclude this chapter by giving one or two numerical examples.

Example 1. Given A $= 68° 2' 24''$, B $= 57° 53' 16''\cdot8$, $a = 3754$ feet, required C, b, c.

Then by case 2.

$$C = 180° - (A + B)$$
$$= 180° - 125° 55' 40''\cdot8$$
$$= 54° 4' 19''\cdot2$$
$$b = a \cdot \frac{\sin. B}{\sin. A}$$
$$\log. b = \log. a + \log. \sin. B - \log. \sin. A$$

Now

$$\begin{aligned} \log. a &= 3\cdot5744943 \\ \log. \sin. B &= 9\cdot9278888 \\ &\overline{13\cdot5023831} \\ \log. \sin. A &= 9\cdot9672882 \\ \therefore \quad \log. b &= \overline{3\cdot5350949} = \log. 3428\cdot43 \\ \therefore \quad b &= 3428\cdot43 \end{aligned}$$

Similarly,

$$\begin{aligned} \log. c &= \log. a + \log. \sin. C - \log. \sin. A \\ \log. a &= 3\cdot5744943 \\ \log. \sin. C &= 9\cdot9083536 \\ &\overline{13\cdot4828479} \\ \log. \sin. A &= 9\cdot9672882 \\ \therefore \quad \log. c &= \overline{3\cdot5155597} = \log. 3277\cdot628 \\ \therefore \quad c &= 3277\cdot628 \text{ feet.} \end{aligned}$$

Example 2. Given $a = 145$, $b = 178\cdot3$, A $= 41° 10'$, required B, C.

This example belongs to case 4, and since the given angle A is acute, and the side b opposite to the required angle B greater than the side a, the solution will be ambiguous.

We have

$$\begin{aligned} \log. \sin. B &= \log. \sin. A + \log. b - \log. a \\ \log. \sin. A &= 9\cdot8183919 \\ \log. b &= 2\cdot2511513 \\ &\overline{12\cdot0695432} \\ \log. a &= 2\cdot1613680 \\ \therefore \quad \log. \sin. B &= \overline{9\cdot9081752} \end{aligned}$$

The angle in the tables corresponding to this logarithm is $54° 2' 22''$, but we cannot determine *a priori* whether the angle sought be this angle, or its supplement $125° 57' 38''$.

$$\therefore \quad B = 54° 2' 22''$$

Or

$$B = 125° 57' 38''$$

If we take the 1st value,

$C = 84° 47' 38''$ and the triangle required is ABC
If we take the second value, — see last figure.
$C = 12° 52' 22''$ and the triangle required is AB′C

Example 3. Given $a = 178{\cdot}3$, $b = 145$, $A = 41° 10'$, required B.

This example also belongs to case 4, but since the given angle A is acute, and the side b opposite the required angle B less than the side a, it follows that the angle B must be an acute angle, and the solution will not be ambiguous.

We have $\quad \log. \sin. B = \log. \sin. A + \log. b - \log. a$

$$\begin{aligned}
\text{But} \quad \log. \sin. A &= 9{\cdot}8183919 \\
\log. b &= 2{\cdot}1613680 \\
&\overline{\;11{\cdot}9797599\;} \\
\log. a &= 2{\cdot}2511513 \\
\therefore \quad \log. \sin. B &= \overline{\;9{\cdot}7286086\;}
\end{aligned}$$

The angle in the tables corresponding to this logarithm is $32° 21' 54''$, and since, in the present instance, the supplement of $32° 21' 54''$ cannot belong to the case proposed, the solution is not ambiguous.

Example 4. Given $a=3754$, $b=3277{\cdot}628$, and the included angle $57° 53' 16''{\cdot}8$ required A, C, b.

By case 3 we have

$$\log. \tan. \frac{A-B}{2} = \log. (a-b) + \log. \cot. \frac{C}{2} - \log. (a+b)$$

$$\begin{aligned}
a-b = 476{\cdot}372, \therefore \log. (a-b) &= 2{\cdot}6779444 \\
\log. \cot. \frac{C}{2} &= 10{\cdot}2572497 \\
&\overline{\;12{\cdot}9351941\;} \\
a+b=7031{\cdot}628, \log. (a+b) &= 3{\cdot}8470543 \\
\therefore \quad \log. \tan. \frac{A-B}{2} &= \overline{\;9{\cdot}0881398\;}
\end{aligned}$$

$$\begin{aligned}
\text{Whence} \quad \frac{A-B}{2} &= 6° 59' 2''{\cdot}4 \\
\text{And since} \quad A+B &= 122° 6' 43''{\cdot}2 \\
\text{And} \quad A-B &= 13° 58' 4''{\cdot}8 \\
\therefore \quad 2A &= 136° 4' 48'' \\
2B &= 108° 8' 38''{\cdot}4 \\
\therefore \quad A &= 68° 2' 24'', \; B = 54° 4' 19''{\cdot}2
\end{aligned}$$

The angles A and B being determined, the side c may be readily found from the equation.

$$\frac{c}{a} = \frac{\sin. C}{\sin. A}$$

$$\log c = \log. a + \log. \sin. C - \log. \sin. A$$

Example 5. Given $a = 33$, $b = 42{\cdot}6$, $c = 53{\cdot}6$, required A, B, C.

Taking the formula marked (ϵ) in chap. III. we have

$$\log. \sin. A = \log. R + \log. 2 + \tfrac{1}{2} \{\log. s + \log. (s-a) + \log. (s-b) + \log. (s-c)\} - \{\log. b + \log. c\}$$

$$\log. \sin. B = \log. R + \log. 2 + \tfrac{1}{2} \{\log. s + \log. (s-a) + \log. (s-b) + \log. (s-c)\} - \{\log. a + \log. c\}$$

$$\log. \sin. C = \log. R + \log. 2 + \tfrac{1}{2} \{\log. s + \log. (s-a) + \log. (s-b) + \log. (s-c)\} - \{\log. a + \log. b\}$$

Now

$$\log.\mathrm{R} = 10.$$
$$\log.2 = 0{\cdot}3010300$$

$$\left.\begin{array}{lll} a = 33 & \therefore \log.a & = 1{\cdot}5185139 \\ b = 42{\cdot}6 & \therefore \log.b & = 1{\cdot}6294096 \\ c = 53{\cdot}6 & \therefore \log.c & = 1{\cdot}7291648 \end{array}\right\} \therefore \begin{array}{l} \log.b+\log.c=3{\cdot}3585714 \\ \log.a+\log.c=3{\cdot}2476787 \\ \log.a+\log.b=3{\cdot}1479235 \end{array}$$

$$\begin{array}{lll} s = 64{\cdot}6 & \therefore \log.s & = 1{\cdot}8102325 \\ s-a = 31{\cdot}6 & \therefore \log.s-a & = 1{\cdot}4996871 \\ s-b = 22 & \therefore \log.s-b & = 1{\cdot}3424227 \\ s-c = 11 & \therefore \log.s-c & = 1{\cdot}0413927 \end{array}$$

$$\therefore \log.s+\log.(s-a)+\log.(s-b)+\log.(s-c) = 5{\cdot}6937350$$

And

$$\tfrac{1}{2}\,\{\log.s+\log.(s-a)+\log.(s-b)+\log.(s-c)\} = 2{\cdot}8468675$$

$$\therefore \log.\mathrm{R}+\log.2+\tfrac{1}{2}\{\log.s+\log.(s-a)+\log.(s-b)+\log.(s-c)\} = 13{\cdot}1478975$$

Subtracting from this number the values $\log.\ b + \log.\ c$; $\log.\ a + \log.\ c$; $\log.\ a + \log.\ b$; in succession we find,

$$\begin{array}{lll} \log.\sin.\mathrm{A} = 9{\cdot}7893231 & \therefore & \mathrm{A} = 37^\circ\,59'\,53'' \\ \log.\sin.\mathrm{B} = 9{\cdot}9002188 & \therefore & \mathrm{B} = 52^\circ\,37'\,46''\tfrac{9}{10} \\ \log.\sin.\mathrm{C} = 9{\cdot}9999740 & \therefore & \mathrm{C} = 89^\circ\,22'\,20''\tfrac{7}{10} \end{array}$$

Having determined A and B by the above method, we find the above accurate value of C, by subtracting the sum of A and B from 180°. If, however, it had been required to determine C alone (being an angle nearly equal to 90°) we could not have found its value with sufficient accuracy from the common tables, for it will be seen, upon referring to them, that the number 9.9999740 may be the logarithm of the sine of any angle from 89° 22′ 20″ up to 89° 22′ 25″, consequently the above method cannot be applied with propriety to determine the exact value of C, unless we previously determine A and B.

The angle C may however be determined directly, and with great accuracy, from any of the three formulæ (σ), (ζ), (η), in Chap. III.

Let us take these in succession, by (σ).

$$\log.\cos.\frac{\mathrm{C}}{2} = \log.\mathrm{R} + \tfrac{1}{2}\ \{\log.s + \log.(s-c)\} - \tfrac{1}{2}\left(\log.a + \log.b\right)$$

$$\left.\begin{array}{rl} \log.s = & 1{\cdot}8102325 \\ \log.(s-c) = & 1{\cdot}0413927 \end{array}\right\} \therefore \tfrac{1}{2}\left(\log.s + \log.(s-c)\right) = 1{\cdot}4258126$$

$$\log.\mathrm{R} = 10$$

$$2{\cdot}8516252 \qquad\qquad 11{\cdot}4258126$$

$$\log.a + \log.b = 3{\cdot}1479235 \quad \therefore \quad \tfrac{1}{2}\left(\log.a + \log.b\right) = 1{\cdot}5739617$$

$$\therefore \quad \log.\cos.\frac{\mathrm{C}}{2} = 9{\cdot}8518509$$

$$\therefore \quad \frac{\mathrm{C}}{2} = 44^\circ\,41'\,10''\text{[illegible]}$$

$$\mathrm{C} = 89^\circ\,22'\,20''\text{[illegible]}$$

By (ζ).

$$\log.\sin.\frac{C}{2} = \log.R + \tfrac{1}{2}\{\log.(s-a) + \log.(s-b)\} \quad \tfrac{1}{2}\{\log.a + \log.b\}$$

$$\left.\begin{array}{l} \log.(s-a) = 1\cdot4996871 \\ \log.(s-b) = 1\cdot3424227 \end{array}\right\} \quad \therefore \tfrac{1}{2}\{\log.(s-a)+\log.(s-b)\} = 1\cdot4210549$$

$$\log.R = 10.$$

$$\begin{array}{lll} & 9\cdot8421098 & \qquad\qquad\qquad\qquad\qquad 11\cdot4210549 \\ \log.a + \log.b = & 3\cdot1479235 & \quad \therefore \tfrac{1}{2}\{\log.a + \log.b\} \quad = \underline{1\cdot5739617} \end{array}$$

$$\therefore \qquad \log.\sin.\frac{C}{2} = 9\cdot8470932$$

$$\therefore \qquad \frac{C}{2} = 44^\circ\, 41'\, 10''\tfrac{8}{33}$$

$$C = 89^\circ\, 22'\, 20''\tfrac{16}{33}$$

By (η).

$$\log.\tan.\frac{C}{2} = \log.R + \tfrac{1}{2}\{\log.(s-a)+\log.(s-b)\} - \tfrac{1}{2}\{\log.s + \log.(s-c)\}$$

$$\left.\begin{array}{l} \log.(s-a) = 1\cdot4996871 \\ \log.(s-b) = \underline{1\cdot3424227} \\ \qquad\qquad\quad 2\cdot8421098 \end{array}\right\} \therefore \tfrac{1}{2}\left\{\begin{array}{r} \log.(s-a)+\log.(s-b) \\ \log.R \end{array}\right\} \begin{array}{l} = 1\cdot4210549 \\ = 10. \end{array}$$

$$\left.\begin{array}{l} \log.s = 1\cdot8102325 \\ \log.s-c = 1\cdot0413927 \end{array}\right\} \therefore \tfrac{1}{2}\{\log.s + \log.(s-c)\} = \begin{array}{r} 11\cdot4210549 \\ \underline{1\cdot4258126} \end{array}$$

$$2\cdot8516252$$

$$\therefore \qquad \log.\tan.\frac{C}{2} = 9\cdot9952423$$

$$\therefore \qquad \frac{C}{2} = 44^\circ\, 41'\, 10''\tfrac{8}{33}$$

$$C = 89^\circ\, 22'\, 20\, \tfrac{16}{33}$$

CHAPTER VII.

ON THE USE OF SUBSIDIARY ANGLES.

Subsidiary Angles are angles which, although not immediately connected with a given problem, are introduced by the computist in order to simplify his calculations. Their use, and the method in which they are employed, will be understood from what follows.

When two sides of a triangle, and the included angle, are given, according to the method pursued in the last chapter, we must determine the two remaining angles before we can compute the third side. It frequently happens, however, in practice, that the side only is required, and it therefore becomes desirable to have some direct method of computing the side independently of the two angles.

Suppose that a, b, C are given, and c is required. By chap. III. prop. 4,

$$c^2 = a^2 + b^2 - 2\,ab\cos.C$$

the side c is determined *theoretically* at once by this expression, but the formula

is not adapted to logarithmic computation, and would, if employed practically lead to a very tedious and complicated calculation. We can, however, put this expression under a form adapted to logarithmic calculation, by having recourse to an algebraical artifice, and introducing a subsidiary angle.

$$c^2 = a^2 + b^2 - 2\,ab \cos. C$$

Adding and subtracting 2 ab on the right hand side.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2\,ab + 2\,ab - 2\,ab \cos. C \\ &= (a-b)^2 + 2\,ab\,(1 - \cos. C) \\ &= (a-b)^2 + 2\,ab \times 2 \sin.^2 \frac{C}{2} \\ &= (a-b)^2 \left\{ 1 + \frac{4\,ab \sin.^2 \frac{C}{2}}{(a-b)^2} \right\} \end{aligned}$$

Assume $$\frac{4\,ab \sin.^2 \frac{C}{2}}{(a-b)^2} = \tan.^2 \varphi$$

$$\begin{aligned} c^2 &= (a-b)^2\,(1 + \tan.^2 \varphi) \\ &= (a-b)^2 \sec.^2 \varphi \\ c &= (a-b) \sec. \varphi \end{aligned}$$

$$\therefore \log. c = \log. (a-b) + \log. \sec. \varphi - \log. R$$

The angle φ is known from the equation.

$$\tan. \varphi = \frac{2\sqrt{ab} \,.\, \sin. \frac{C}{2}}{(a-b)}$$

Whence,

$$\log. \tan. \varphi = \log. 2 + \tfrac{1}{2}(\log. a + \log. b) + \log. \sin. \frac{C}{2} - \log. (a-b)$$

φ being thus determined, log. sec. φ can be found from the tables, and the value of c becomes known.

The angle φ, which is introduced into the above calculation, in order to render the expression convenient for logarithmic computation, is called a *subsidiary angle.*

The above transformation may be effected in a manner somewhat different, as before.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2\,ab \cos. C \\ &= a^2 + b^2 + 2\,ab - 2\,ab - 2\,ab \cos. C \\ &= (a+b)^2 - 2\,ab\,(1 + \cos. C) \\ &= (a+b)^2 - 2\,ab \times 2 \cos.^2 \frac{C}{2} \\ &= (a+b)^2 \left(1 - \frac{4\,ab \cos^2 \frac{C}{2}}{(a+b)^2}\right) \end{aligned}$$

Assume $$\frac{4\,ab \cos.^2 \frac{C}{2}}{(a+b)^2} = \sin.^2 \varphi$$

$$\begin{aligned} \therefore\quad c^2 &= (a+b)^2\,(1 - \sin.^2 \varphi) \\ &= (a+b)^2 \cos.^2 \varphi \\ c &= (a+b) \cos. \varphi \end{aligned}$$

$$\therefore \log. c = \log. (a+b) + \log. \cos. \varphi - \log. R$$

As before the angle φ must be determined from the equation.

$$\sin. \varphi = \frac{2\sqrt{ab} \cdot \cos. \frac{C}{2}}{(a+b)}$$

In order to prove that we can always assume

$$\frac{2\sqrt{ab} \cdot \cos. \frac{C}{2}}{(a+b)} = \sin. \varphi$$

we must show that $\frac{2\sqrt{ab} \cos. \frac{C}{2}}{(a+b)}$ is always less than unity, or, in other words, that $2\sqrt{ab}$ is always less than $(a+b)$, this is easily done.

$$\begin{array}{lll} \text{If} & a+b & > 2\sqrt{ab} \\ \text{Then} & a^2+2ab+b^2 & > 4ab \\ & a^2+b^2 & > 2ab \\ & a^2+b^2-2ab & > 0 \\ \text{Or} & (a-b)^2 & > 0 \end{array}$$

But since $(a-b)^2$ is necessarily a positive quantity, it must always be greater than 0 (except in the particular case $a=b$, where it is $=0$), and therefore $\frac{2\sqrt{ab} \cos. \frac{C}{2}}{(a+b)}$ is always less than unity, and consequently an angle may always be found whose sine is equal to it.

In solving the same case of oblique-angled triangles, we determined the difference of the angles A, B from the equation.

$$\tan. \frac{A-B}{2} = \frac{a-b}{a+b} \cot. \frac{C}{2}$$

$$\text{Whence} \quad \log. \tan. \frac{A-B}{2} = \log. (a-b) + \log. \cot. \frac{C}{2} - \log. (a+b)$$

In the solution of certain astronomical problems, the logarithms of the sides a, b are given, but not the sides themselves, and these logarithms being given, we can very easily calculate $\frac{A-B}{2}$ without knowing the sides.

$$\tan. \frac{A-B}{2} = \frac{a-b}{a+b} \cot. \frac{C}{2}$$

$$= \frac{1-\frac{b}{a}}{1+\frac{b}{a}} \cot. \frac{C}{2}$$

Assume $\frac{b}{a} = \tan. \varphi$

$$\tan. \frac{A-B}{2} = \frac{1-\tan. \varphi}{1+\tan. \varphi} \cdot \cot. \frac{C}{2}$$

$$= \tan. (45^\circ - \varphi) \cot. \frac{C}{2}$$

$$\therefore \quad \log. \tan. \frac{A-B}{2} = \log. \tan. (45^\circ - \varphi) + \log. \cot. \frac{C}{2} - \log. R$$

The angle φ is known from the equation

$$\tan. \varphi = \frac{b}{a}$$

Whence $\qquad \log. \tan. \varphi = \log. R + \log. b - \log. a$

The angle $\frac{A-B}{2}$ thus becomes known from the logs. of a and b, withou calculating a and b. In the same way we may have,

$$\cot. \frac{A-B}{2} = \tan. (45^\circ + \varphi) \tan. \frac{C}{2}$$

And $\therefore \log. \cot. \frac{A-B}{2} = \log. \tan. (45^\circ + \varphi) + \log. \tan. \frac{C}{2} - \log. R.$

CHAPTER VIII.

ON THE

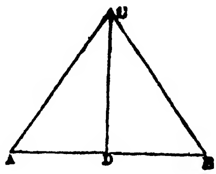
SOLUTION OF GEOMETRICAL PROBLEMS BY TRIGONOMETRY.

A GREAT variety of geometrical problems may be solved with much elegance by the introduction of trigonometrical formulæ. We shall give a few examples

PROB. I.

To express the area of a plane triangle in terms of the sides of the triangle.

Let CD be a perpendicular from C upon AB.



Area of triangle ABC $= \frac{AB \times CD}{2}$

$= \frac{c}{2} . AC \sin. A$

$= \frac{bc}{2} . \sin. A$

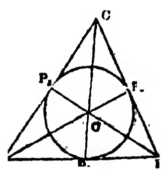
$= \frac{bc}{2} . \frac{2}{bc} . \sqrt{s(s-a)(s-b)(s-c)}$ ... Chap. III

$= \sqrt{s(s-a)(s-b)(s-c)}$

PROB. II.

To express the radius of a circle inscribed in a given triangle, in terms of the sides of the triangle.

Let the radius required be called r.



Area of AOC $= \frac{rb}{2}$

.. AOB $= \frac{rc}{2}$

.. COB $= \frac{ra}{2}$

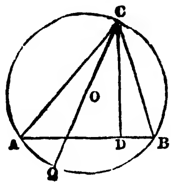
$\therefore$ Whole area of triangle ABC $= \frac{r}{2}(a+b+c) = r . s$

i. e. $\sqrt{s(s-a)(s-b)(s-c)} = r \cdot s$ by last prob.

$$\therefore \quad r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

PROB. III.

To express the radius of a circle circumscribed about a given triangle, in terms of the sides of the triangle.

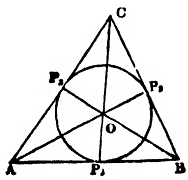


Let fall CD perpendicular on AB
Let the radius be called R.
By a well known geometrical prop.

$$CQ \cdot CD = AC \cdot CB$$
$$\therefore CQ \cdot CD \cdot AB = AC \cdot CB \cdot AB$$
$$2\,R \times 2 \text{ area} = abc$$
$$R = \frac{abc}{4 \text{ area}}$$
$$= \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

PROB. IV.

Given the three angles of a plane triangle, and the radius of the inscribed circle, to find the sides of the triangle.



Let A, B, C, be the three given angles, r the radius

$$AB \text{ or } c = AP_1 + P_1B$$
$$= r\left(\cot.\frac{A}{2} + \cot.\frac{B}{2}\right)$$
$$= r \cdot \frac{\sin.\left(\frac{A}{2} + \frac{B}{2}\right)}{\sin.\frac{A}{2}\,\sin.\frac{B}{2}}$$

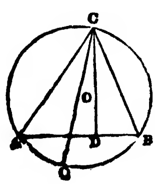
So,

$$AC \text{ or } b = r \cdot \frac{\sin.\left(\frac{A}{2} + \frac{C}{2}\right)}{\sin.\frac{A}{2}\,\sin.\frac{C}{2}}$$

$$BC \text{ or } a = r \cdot \frac{\sin.\left(\frac{B}{2} + \frac{C}{2}\right)}{\sin.\frac{B}{2}\,\sin.\frac{C}{2}}$$

PROB. V.

Given the three angles of a plane triangle, and the radius of the circumscribing circle, to find the sides of the triangle.



As in Prob. 3.

$$CQ \cdot CD = AC \cdot CB$$
$$CQ \cdot CB \sin. B = AC \cdot CB$$
$$\therefore \quad AC = 2\,R \sin. B$$

So,

$$BC = 2\,R \sin. A$$
$$AB = 2\,R \sin. C$$

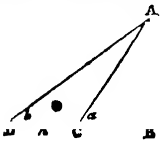
CHAPTER IX.

PROBLEMS IN TRIGONOMETRICAL SURVEYING.

Problem I.

To determine the height of an inaccessible object.

Let AB be the object, and in a straight line towards it measure any distance DC, and observe the angles of elevation ADB, ACB at the stations D, C. Put $CD = h$, $\angle ACB = a$, $\angle ADB = b$; then $\angle DAC = a - b$; hence we have



$$\frac{AB}{AC} = \sin a; \quad \frac{AC}{CD} = \frac{\sin b}{\sin (a-b)};$$

and multiplying these two equations, we have

$$\frac{AB}{CD} = \frac{\sin a \sin b}{\sin (a-b)}, \text{ or } AB = h \sin a \sin b \operatorname{cosec} (a-b)$$

$\therefore \log. AB = \log. h + \log. \sin a + \log \sin b + \log. \operatorname{cosec} (a-b) - 30 \ldots\ldots (1)$

Cor. Since $DB = AB \cot b$, and $CB = AB \cot a$; therefore, by subtraction, $CD = AB (\cot b - \cot a)$, or $AB = \dfrac{h}{\cot b - \cot a} \ldots\ldots\ldots (2)$.

Ex. Let $DC = h = 200$, $\angle BDA = b = 31°$, $\angle BCA = a = 46°$; to find AB and CB.

$$\begin{aligned}
\log. h \quad \ldots &= \log. 200 = 2.3010300 \\
\log. \sin a \quad \ldots &= \log. \sin 46° = 9.8569341 \\
\log. \sin b \quad \ldots &= \log. \sin 31° = 9.7118393 \\
\log. \operatorname{cosec} (a-b) &= \log. \operatorname{cosec} 15° = 10.5870038 \\
\hline
\log. AB \quad \ldots &= \log. 286.29 = 2.4568072 \quad \therefore AB = 286.29
\end{aligned}$$

Also $BC = AB \cot a$ $\therefore \log. BC = \log. AB + \log. \cot a - 10$.

Problem II.

To determine the height of an inaccessible object, which has no level ground before it.

Let AB be the object, and CD two stations in a vertical plane passing through AB; measure the distance CD, and at C take the angles of elevation or depression of the station D, and the top and bottom of the object. Also at D take the elevation or depression of the top of AB.

Put $CD = h$, $\angle DCH = a$, $\angle BCK = b$, $\angle ACK = c$, $\angle ADG = d$; then $\angle ACB = c - b$, $\angle ADC = a + d$, and $\angle CAD = c - d$.

Now, $\dfrac{AB}{AC} = \dfrac{\sin ACB}{\sin ABC} = \dfrac{\sin (c-b)}{\sin ABK} = \dfrac{\sin (c-b)}{\cos b}$;

$\dfrac{AC}{CD} = \dfrac{\sin ADC}{\sin CAD} = \dfrac{\sin (a+d)}{\sin (c-d)}$; hence, multiplying these equations,

we have $\dfrac{AB}{CD} = \dfrac{\sin (c-b) \sin (a+d)}{\cos b \sin (c-d)}$; and, therefore,

$$AB = h \sin (c-b) \sin (a+d) \sec b \operatorname{cosec} (c-d)$$

$\therefore \log. AB = \log. h + \log. \sin (c-b) + \log. \sin (a+d) + \log. \sec b + \log. \operatorname{cosec} (c-d) - 40 \ldots\ldots\ldots\ldots (1)$

Cor. When $a = 90°$, and $b = 0°$, then we have

$\log. AB = \log. h + \log. \sin c + \log. \cos d + \log. \operatorname{cosec} (c-d) - 30 \ldots\ldots (2)$

Ex. 1. Let $h=18$ feet; $c=40^\circ$; $d=37^\circ 30'$, $a=90^\circ$ and $b=0^\circ$; to find AB.

$$\begin{aligned}
&\log. h \ldots &&= \log. 18 \ldots &&= 1.2552725\\
&\log. \sin c \ldots &&= \log. \sin 40^\circ &&= 9.8080675\\
&\log. \cos d \ldots &&= \log. \cos 37^\circ 30' &&= 9.8994667\\
&\log. \operatorname{cosec} (c-d) &&= \log. \operatorname{cosec} 2^\circ 30' &&= 11.3603204\\
\hline
&\log. AB \ldots &&= \log. 210{\cdot}4394 &&= 2{\cdot}3231271 \quad \therefore AB=210{\cdot}4394.
\end{aligned}$$

Ex. 2. The angle of elevation of the top of a tower, standing on a hill, was 33° 45′, and, measuring on level ground 300 yards directly towards the tower, the angles of elevation of the top and bottom of the tower were 51° and 40° respectively. What is the height of the tower? Ans. 140 yds.

Remark.—When the station D is higher than A, the top of the tower, then the angle d must be considered negative, and therefore we should have

$$AB = h \sin (c-b) \sin (a-d) \sec b \operatorname{cosec} (c+d).$$

Lemma.

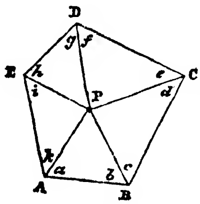
If straight lines be drawn from any point, either within, or out of, a polygon, to all the angular points, the continued products of the sines of the alternate angles, made by the sides of the polygon, and the lines so drawn, will be equal.

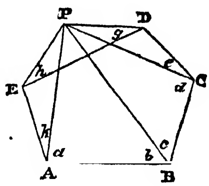
Let angle CDP$=f$, and PEA$=i$;

then $\dfrac{PA}{PB}=\dfrac{\sin b}{\sin a}$, $\dfrac{PB}{PC}=\dfrac{\sin d}{\sin c}$,

$\dfrac{PC}{PD}=\dfrac{\sin f}{\sin e}$, $\dfrac{PD}{PE}=\dfrac{\sin h}{\sin g}$,

$\dfrac{PE}{PA}=\dfrac{\sin k}{\sin i}$.

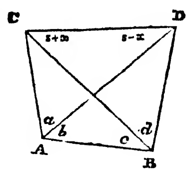




But PA . PB . PC . PD . PE = PB . PC . PD . PE . PA;
that is, the product of the numerators = product of the denominators, in the first members of these equations; hence, this being true in the second members also, $\sin b. \sin d. \sin f. \sin h. \sin k = \sin a. \sin c. \sin e. \sin g. \sin i$.

Problem III.

Given AB, *and the angles* a, b, c, d, *to find* x, *and thence* CD.



Put $BCD+ADC = b+c = 2s$

$BCD-ADC = \ldots \quad 2x$

Then $BCD = s+x$, $ADC = s-x$; also, $\sin ADB = \sin (b+c+d)$, and $\sin ACB = \sin (a+b+c)$; hence, by the lemma, we have

$\sin a. \sin c. \sin (b+c+d) \sin (s+x) = \sin b. \sin d. \sin (a+b+c) \sin (s-x)$;

or, $\sin a \sin c \sin (b+c+d) \{\sin s \cos x + \cos s \sin x\} = \sin b \sin d \sin (a+b+c) \{\sin s \cos x - \cos s \sin x\}$.

Then, dividing by $\cos s \cos x$, we have

$\sin a \sin c \sin (b+c+d) (\tan s + \tan x) = \sin b \sin d \sin (a+b+c) (\tan s - \tan x)$,

$$\tan x = \frac{\sin b \sin d \sin (a+b+c) - \sin a \sin c \sin (b+c+d)}{\sin b \sin d \sin (a+b+c) + \sin a \sin c \sin (b+c+d)} \tan s.$$

Dividing numerator and denominator by $\sin a \sin c \sin (b+c+d)$, and putting $\frac{\sin b \sin d \sin (a+b+c)}{\sin a \sin c \sin (b+c+d)} = \tan \beta$ and $1 = \tan 45°$; then

$$\tan x = \frac{\tan \beta - \tan 45°}{\tan \beta + \tan 45°} \tan s = \frac{\sin (\beta - 45°)}{\sin (\beta + 45°)} \tan s;$$

hence x, $s+x$, $s-x$ are all known, and thence CD is known.

For $\frac{CD}{BD} = \frac{\sin d}{\sin (s+x)}$, and $\frac{BD}{AB} = \frac{\sin b}{\sin (b+c+d)}$; therefore,

$$CD = AB \sin b \sin d \operatorname{cosec} (s+x) \operatorname{cosec} (b+c+d).$$

Cor. When CD is given, and the same angles, to find AB, we have

$$AB = CD \sin (b+c+d) \sin (s+x) \operatorname{cosec} b \operatorname{cosec}. d.$$

EXAMPLE.

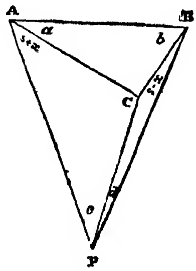
Given AB = 600 yards, $a = 37°$, $b = 58° 20'$, $c = 53° 30'$, $d = 45° 15'$, to find CD.

Here, $\tan \beta = \operatorname{cosec} a \sin b \operatorname{cosec} c \sin d \sin (a+b+c) \operatorname{cosec} (b+c+d)$,

log. cosec a .... =	log. cosec	37° ..... =	10.2205370	
$\sin b$ .... =	sin	58° 20′... =	9.9299891	9.9299891
cosec c......... =	cosec	53 30 ... =	10.0948213	
$\sin d$ .... =	sin	45 15 ... =	9.8513717	9.8513717
$\sin (a+b+c)$ ... =	sin	148 50 ... =	9.7139349	
cosec $(b+c+d)$. =	cosec	157 5 ... =	10.4096131	10.4096131
$\tan \beta$ .... =	tan.	58° 56′ 39″ =	10.2202671	
$\sin (\beta - 45°)$ .... =	sin	13 56 39 =	9.3819742	
cosec $(\beta + 45°)$ .. =	cosec	103 56 39 =	10.0129906	
$\tan s$ .... =	tan	55 55 ... =	10.1696508	
$\tan x$ .... =	tan	20 9 3 =	9.5646156	
cosec $(s+x)$ .... =	cosec	76 4 3 =	....	10.0129687
AB ...... =		600 ...... =	....	2.7781513
∴ CD .... =		959.608 ... =	....	2.9820939

PROBLEM IV.

Given AB, a, b, *and the angles* c, d, *taken at some point* P *in the same plane* ABC, *to find* x; *and thence* PA, PB, PC.



Put $PAC + PBC = 180° - (a+b+c+d) = 2s$

$PAC - PBC = \dots\dots\dots\dots\dots = 2x$;

Then, $PAC = s+x$, $PBC = s-x$, and, by the lemma,

$\sin a \sin c \sin (s-x) = \sin b \sin d \sin (s+x)$

$$\therefore \frac{\sin b \sin d}{\sin a \sin c} = \frac{\sin (s-x)}{\sin (s+x)} = \frac{\tan s - \tan x}{\tan s + \tan x}.$$

Put $\tan \beta = \frac{\sin b \sin d}{\sin a \sin c} = \operatorname{cosec} a \sin b \operatorname{cosec} c \sin d$; then we have

$$\frac{\tan s - \tan x}{\tan s + \tan x} = \tan \beta \ \therefore \frac{\tan x}{\tan s} = \frac{1 - \tan \beta}{1 + \tan \beta} = \frac{\tan 45° - \tan \beta}{\tan 45° + \tan \beta};$$

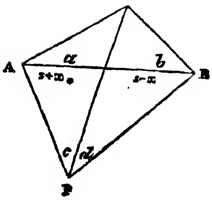
$$\therefore \tan x = \frac{\tan 45° - \tan \beta}{\tan 45° + \tan \beta} \tan. s = \frac{\sin (45° - \beta)}{\sin (45° + \beta)} \tan s.$$

Hence x is known, and thence $s+x$ and $s-x$ are known.

Then $\frac{PC}{AC} = \frac{\sin(s+x)}{\sin c}$, $\frac{AC}{AB} = \frac{\sin b}{\sin(a+b)}$; hence
$$PC = AB \operatorname{cosec}(a+b) \sin b \operatorname{cosec} c \sin(s+x).$$

Problem V.

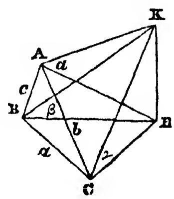
When the points P *and* C *are on opposite sides of* AB.



Put $PAB+PBA=180^\circ-(c+d)=2s$
$PAB-PBA= \dots\dots\dots =2x$;
then, $PAB=s+x$, $PBA=s-x$; and, by the lemma
$$\sin a \sin c \sin(s-x) = \sin b \sin d \sin(s+x);$$
hence, as in the last problem, we have
$$\tan x = \frac{\sin(45^\circ-\beta)}{\sin(45^\circ+\beta)} \tan s;$$
where $\tan \beta = \operatorname{cosec} a \sin b \operatorname{cosec} c \sin d$; and $2s = 180^\circ-(c+d)$.

Problem VI.

Given the angles of elevation of any distant object, taken at three places on a level plane, no two of which are in the same vertical plane with the object; to find the height of the object, and its distance from either station.



Let A, B, C, be the three stations, K the object, and KH perpendicular to the plane of the triangle ABC.

Put $BC=a$, $AC=b$, $AB=c$, $HAK=\alpha$, $HBK=\beta$, $HCK=\gamma$, and $HK=x$; then the angles AHK, BHK, CHK being right angles, we have $AH=x\cot\alpha$, $BH=x\cot\beta$, $CH=x\cot\gamma$; hence,

$$\cos HBA = \frac{AB^2+BH^2-AH^2}{2AB\cdot BH} = \frac{c^2+x^2(\cot^2\beta-\cot^2\alpha)}{2cx\cot\beta}$$

$$\cos HBC = \frac{BC^2+BH^2-CH^2}{2BC\cdot BH} = \frac{a^2+x^2(\cot^2\beta-\cot^2\gamma)}{2ax\cot\beta}$$

$$\cos ABC = \frac{AB^2+BC^2-AC^2}{2AB\cdot BC} = \frac{a^2+c^2-b^2}{2ac}$$

But $\cos ABC = \cos(ABH+CBH) = \cos ABH \cos CBH - \sin ABH \sin CBH$; and by transposing $\sin ABH \sin CBH$ to one side of the equation, then squaring both sides, substituting $1-\cos^2 ABH$, and $1-\cos^2 CBH$ for $\sin^2 ABH$ and $\sin^2 CBH$, we have

$$1-\cos^2 ABH-\cos^2 CBH-\cos^2 ABC+2\cos ABH \cos CBH \cos ABC=0.$$

Substituting the above values of $\cos ABH$, $\cos CBH$, $\cos ABC$, in this equation, and reducing to a common denominator, we get

$$\left.\begin{array}{r} a^2(\cot^2\beta-\cot.^2\alpha)^2 \\ c^2(\cot^2\beta-\cot.^2\gamma)^2 \\ (b^2-c^2-a^2)(\cot^2\beta-\cot^2\alpha)(\cot^2\beta-\cot.^2\gamma) \end{array}\right\} x^4+\left\{\begin{array}{l} a^2(a^2-b^2-c^2)\cot^2\alpha \\ +b^2(b^2-c^2-a^2)\cot^2\beta \\ +c^2(c^2-a^2-b^2)\cot^2\gamma \end{array}\right\} x^2+a^2b^2c^2=0\ldots(1.)$$

Assume $h=a\dfrac{\cot\alpha}{\cot\beta}$ and $k=c\dfrac{\cot\gamma}{\cot\beta}$; then the last equation becomes

$$\left.\begin{array}{r} a^2(c^2-k^2)^2 \\ c^2(a^2-h^2)^2 \\ (b^2-c^2-a^2)(c^2-k^2)(a^2-h^2) \end{array}\right\} x^4\cot^4\beta+\left\{\begin{array}{l} a^2b^2c^2(b^2-c^2-a^2) \\ +a^2c^2k^2(c^2-a^2-b^2) \\ +a^2c^2h^2(a^2-b^2-c^2) \end{array}\right\} x^2\cot^2\beta+b^2a^4c^4=0\ \ldots\ldots(2.)$$

Let us again assume $k^2=b^2+h^2-2\,bh\cos\phi$, where b, h, k, are the three sides of a triangle, and ϕ the angle opposite to k; and there fore b, h, k, being known, the angle ϕ may be computed by logarithms. Also, in the triangle ABC, we have $c^2=a^2+b^2-2\,ab\cos C$, and substituting these values of k^2 and c^2 in the preceding equation, we have the coefficient of $x^4\cot^4\beta=b^2\{(a^2-h^2)^2-4\,ah\,(a^2+h^2)\cos C$ $\cos\phi+4\,a^2h^2(\cos^2 C+\cos^2\phi)\}$; which, by adding and subtracting $b^2(4\,a^2h^2+4\,a^2h^2\cos^2 C\cos^2\phi)$, is transformed to

$$b^2\{(a^2+h^2)^2-4ah\,(a^2+h^2)\cos C\cos\phi+4\,a^2h^2\cos^2 C\cos^2\phi-4\,a^2h^2(1-\cos^2 C)(1-\cos^2\phi)\}=$$
$$b^2\{(a^2+h^2-2ah\cos C\cos\phi)^2-(2ah\sin C\sin\phi)^2\}.$$

In like manner the coefficient of $x^2\cot^2\beta$ becomes $-2a^2b^2c^2(a^2+h^2-2ah\cos C\cos\phi)$; and, therefore, the last equation is transformed to

$$\{(a^2+h^2-2\,ah\cos C\cos\phi)^2-(2\,ah\sin C\sin\phi)^2\}\,x^4\cot^4\beta-2a^2c^2(a^2+h^2-2ah\cos C\cos\phi)\,x^2\cot^2\beta+a^4c^4=0;$$

hence, by transposing $(2ah\sin C\sin\phi)^2x^4\cot^2\beta$, we have both sides of the equation complete squares; and, extracting the roots, we get

$$(a^2+h^2-2\,ah\cos C\cos\phi)\,x^2\cot^2\beta-a^2c^2=\pm 2\,ah\sin C\sin\phi\,x^2\cot^2\beta$$
$$\therefore\ \{a^2+h^2-2\,ah\,(\cos C\cos\phi\pm\sin C\sin\phi)\}\,x^2\cot^2\beta=a^2c^2;$$
$$\text{or, }\{a^2+h^2-2\,ah\cos(C\mp\phi)\}\,x^2\cot^2\beta=a^2c^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

Now, $a^2+h^2-2\,ah\cos(C\mp\phi)$ is an expression for the square of a side of a triangle opposite to the angle $(C\mp\phi)$, and whose containing sides are a and h; and we may therefore assume $a^2+h^2-2\,ah\cos(C\mp\phi)=m^2$, and compute m by the usual trigonometrical theorem; hence we have, finally, $m^2x^2\cot^2\beta=a^2c^2$

$$\therefore\ x\ =+\frac{ac\ \tan\beta}{m}\ \ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

Cor. 1. When the three places are in the same straight line, we have $b = a+c$, and angle $C = 0$; therefore $b^2-c^2-a^2 = 2ac$; $c^2-a^2-b^2 = -2ab$; $a^2-b^2-c^2 = -2bc$; and hence in this case equation (1) becomes a complete square, whose root, extracted, gives

$$\{a \cot^2 \alpha - (a+c) \cot^2 \beta + c \cot^2 \gamma\}\, x^2 - ac\,(a+c) = 0 \quad \ldots\ldots (5.)$$

$$\text{and} \ldots\ldots\ldots\ldots x = \pm \frac{ac \tan \beta}{m} \quad \ldots\ldots (6.)$$

where $\cos \phi = \dfrac{(a+c)^2+h^2-k^2}{2h\,(a+c)}$, and $m = \pm\sqrt{a^2+h^2-2\,ah \cos \phi}$.

Cor. 2. When $b = a+c$, and also $a = c$; then we have

$$(\cot^2 \alpha - 2 \cot^2 \beta + \cot^2 \gamma)\, x^2 - 2\,a^2 = 0 \quad \ldots\ldots (7.)$$

$$\text{and } x = \pm \frac{a^2 \tan \beta}{m} \quad \ldots\ldots (8.)$$

where $\cos \phi = \dfrac{4\,a^2+h^2-k^2}{4\,ah}$, and $m = \pm \sqrt{a^2+h^2-2\,ah \cos \phi}$

Scholium.

In all these cases the computation can be conducted entirely by logarithms; for ϕ is the value of the angle opposite to the side k of the triangle whose three sides are b, h, k; and m is the third side of the triangle whose two sides are a, h, and included angle $C \mp \phi$. The following table exhibits the several steps of the computation for each case, and though the expressions for cos. C, cos. ϕ, and m are put down as employed in the investigation, still they should be calculated by the usual trigonometrical rules adapted to logarithmic computation.

Table of Formulæ.

	(1.) For any three stations.	(2.) In the same straight line.
(1)	$\begin{cases} h = a \cot \alpha \tan \beta \\ k \doteq c \cot. \gamma \tan \beta \end{cases}$	$\begin{cases} h = a \cot \alpha \tan \beta \\ k = c \cot \gamma \tan \beta \end{cases}$
(2)	$\cos C = \dfrac{a^2+b^2-c^2}{2\,ab}$	$C = 0$
(3)	$\cos \phi = \dfrac{b^2+h^2-k^2}{2\,bh}$	$\cos \phi = \dfrac{(c+c)^2+h^2-k^2}{2\,h\,(a+c)}$
(4)	$m = \pm\sqrt{a^2+h^2-2ah\cos(C\mp\phi)}$	$m = \pm\sqrt{a^2+h^2-2\,ah\cos \phi}$
(5)	$x = \pm \dfrac{ac \tan. \beta}{m}$	$x = \pm \dfrac{ac \tan \beta}{m}$.

When the stations are at equal intervals in the same straight line, then c is equal to a, and the necessary modification of the formulæ in the second column of the table is obvious.

The two cases in which the three stations are in the same straight line may, however, be investigated in a different manner; for, resuming equation (5), we have

$$\{a\,(\cot^2\alpha - \cot^2\beta) + c\,(\cot^2\gamma - \cot^2\beta)\}\,x^2 = ac\,(a+c) \ldots\ldots\ldots\ldots (9.)$$

But, $\cot^2\alpha - \cot^2\beta = (\cot\alpha + \cot\beta)(\cot\alpha - \cot\beta) = \dfrac{\sin(\beta+\alpha)\sin(\beta-\alpha)}{\sin^2\alpha\sin^2\beta}$

$$\cot^2\gamma - \cot^2\beta = (\cot\gamma + \cot\beta)(\cot\gamma - \cot\beta) = \frac{\sin(\beta+\gamma)\sin(\beta-\gamma)}{\sin^2\beta\sin^2\gamma}.$$

Hence equation (5) becomes $\left\{1 + \dfrac{a\sin(\beta+\alpha)\sin(\beta-\alpha)\sin^2\gamma}{c\sin(\beta+\gamma)\sin(\beta-\gamma)\sin^2\alpha}\right\}.$

$$\frac{\sin(\beta+\gamma)\sin(\beta-\gamma)}{\sin^2\beta\sin^2\gamma}\,x^2 = a\,(a+c) \ldots\ldots\ldots\ldots (10.)$$

$$\therefore\ x^2 = \frac{a\,(a+c)\dfrac{\sin^2\beta\sin^2\gamma}{\sin(\beta+\gamma)\sin(\beta-\gamma)}}{1+\dfrac{a\sin(\beta+\alpha)\sin(\beta-\alpha)\sin^2\gamma}{c\sin(\beta+\gamma)\sin(\beta-\gamma)\sin^2\alpha}} \ldots\ldots\ldots\ldots (11.)$$

Now, when β exceeds α, as well as γ, we may assume

$$\tan^2\theta = \frac{a\sin(\beta+\gamma)\sin(\beta-\alpha)\sin^2\gamma}{c\sin(\beta+\gamma)\sin(\beta-\gamma)\sin^2\alpha};$$

and when β is less than α, and greater than γ, or less than γ and greater than α, then the fractions in the numerator and denominator of equation (11) are negative, and we must assume

$$\sec^2\theta = \frac{a\sin(\beta+\alpha)\sin(\beta-\alpha)\sin^2\gamma}{c\sin(\beta+\gamma)\sin(\beta-\gamma)\sin^2\alpha};$$

and therefore, in the former case, we have

$$\left\{\begin{array}{l}\tan\theta = \pm\operatorname{cosec}\alpha\sin\gamma\sqrt{\dfrac{a}{c}\sin(\beta+\alpha)\sin(\beta-\alpha)\operatorname{cosec}(\beta+\gamma)\operatorname{cosec}(\beta-\gamma)}\\ x = \pm\sin\beta\sin\gamma\cos\theta\sqrt{a\,(a+c)\operatorname{cosec}(\beta+\gamma)\operatorname{cosec}(\beta-\gamma)}\end{array}\right\} \ldots (12.)$$

And, in the latter,

$$\left\{\begin{array}{l}\sec\theta = \pm\operatorname{cosec}\alpha\sin\gamma\sqrt{\dfrac{a}{c}\sin(\beta+\alpha)\sin(\beta-\alpha)\operatorname{cosec}(\beta+\gamma)\operatorname{cosec}(\beta-\gamma)}\\ x = \pm\sin\beta\sin\gamma\cot\theta\sqrt{a\,(a+c)\operatorname{cosec}(\beta+\gamma)\operatorname{cosec}(\beta-\gamma)}\end{array}\right\} \ldots (13.)$$

When $a = c$, then equations (12) and (13) become

$$\left\{\begin{array}{l}\tan\theta = \pm\operatorname{cosec}\alpha\sin\gamma\sqrt{\sin(\beta+\alpha)\sin(\beta-\alpha)\operatorname{cosec}(\beta+\gamma)\operatorname{cosec}(\beta-\gamma)}\\ x = \pm a\sin\beta\sin\gamma\cos\theta\sqrt{2\operatorname{cosec}(\beta+\gamma)\operatorname{cosec}(\beta-\gamma)}\end{array}\right\} \ldots\ldots (14.)$$

$$\left\{\begin{array}{l}\sec\theta = \pm\operatorname{cosec}\alpha\sin\gamma\sqrt{\sin(\beta+\alpha)\sin(\beta-\alpha)\operatorname{cosec}(\beta+\gamma)\operatorname{cosec}(\beta-\gamma)}\\ x = \pm a\sin\beta\sin\gamma\cot\theta\sqrt{2\operatorname{cosec}(\beta+\gamma)\operatorname{cosec}(\beta-\gamma)}\end{array}\right\} \ldots\ldots (15).$$

Example.

Given $\alpha = 30^\circ 40'$, $\beta = 40^\circ 33'$, $\gamma = 50^\circ 23'$; find x, when the three stations are in the same straight line, AB being $= 50$, and BC $= 60$ yards.

This example corresponds to the second case in the preceding table of formulæ, and may be resolved by the formulæ there given. It may also be resolved by the equations in (13), for β is less than γ, and greater than α; hence, employing the latter of these methods, we have the subsequent computation.

$\beta+\alpha = 71^\circ\ 13'$ sin	$= 9.9762321$			
$\beta-\alpha = 9^\circ\ 53'$ sin	$= 9.2346249$			
$\beta+\gamma = 90^\circ\ 56'$ cosec	$= 10.0000576$			10.0000576
$\beta-\gamma = 9^\circ\ 50'$ cosec	$= 10.7675560$			10.7675560
$a = 60$ log.	$= 1.7781513$	log. 60	$=$	1.7781513
$c = 50$ ar. co. log.	$= 8.3010300$	50		
	2)20.0576519	log. 110	$=$	2.0413927
	10.0288259			2)4.5871576
$\alpha = 30^\circ\ 40'$ cosec	$= 10.2923936$			2.2935788
$\gamma = 50^\circ\ 23'$ sin	$= 9.8866756$..........			9.8866756
$\theta = 51^\circ\ 42'\ 49\frac{1}{2}''$ sec	$= 10.2078951$	cot θ	$=$	9.8972767
$\beta = 40^\circ\ 33'$ sin			$=$	9.8129878
$x = 77.7175$ yards		log.	$=$	1.8905189

Whence the height of the object is nearly 77.7175 yards, and its distances from the three stations are easily found. This example is from Bonnycastle's Trigonometry, 2nd Edition, p. 72, where the height is stated to be 79.029 yards, which differs from the true height by 3 feet $11\frac{1}{4}$ inches, or 4 feet nearly; a very considerable error, arising purely from the inconvenient mode of solution by means of natural cotangents.

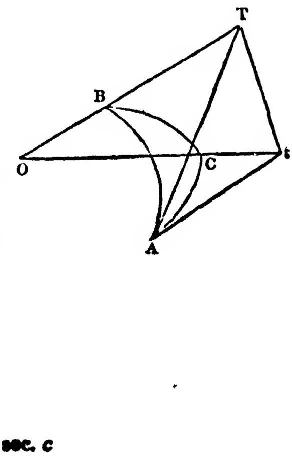
This method of solution, combining great accuracy with simplicity, is preferable to every other method by which the solution has been attempted. Meyer Hirsch, one of the ablest of the Continental mathematicians, and one of the most successful teachers of our time, has given an elegant solution of this problem in his "Geometry," p. 78, somewhat analogous to the preceding, but altogether different in the resulting equations, arising from a different mode of substitution in the investigation of equation (2), and embracing only one case of the problem. A beautiful and simple geometrical construction of this problem may be seen in *Ingram's Concise System of Mathematics*, page 269, fifth Edition.

SPHERICAL TRIGONOMETRY.

Having demonstrated in the treatise on Spherical Geometry, several important properties of the circle of the sphere, and of spherical triangles, we shall now proceed to deduce various relations which exist between the several parts of a spherical triangle. These constitute what is called *Spherical Trigonometry;* and enable us, when a certain number of the parts are given, to determine the rest. The first formula which we shall establish, serves as a key to all the rest, and is to spherical trigonometry what the expression for the sine of the sum of two angles is to plane trigonometry.

CHAPTER I.

1. *To express the cosine of an angle of a spherical triangle in terms of the sines and cosines of the sides.*



Let A B C be a spherical triangle, O the centre of the sphere.

Let the angles of the triangle be denoted by the large letters A, B, C, and the sides opposite to them by the corresponding small letters, a, b, c.

At the point A, draw A T a tangent to the arc A B, and A t a tangent to the arc A C.

Then the spherical angle A is equal to the angle T A t between the tangents (Spher. Geom. prop. IV.).

Join O B, and produce it to meet A T in T.

Join O C, and produce it to meet A t in t.

Join T, t;

Then,

$$\frac{OT}{OC} = \sec. AB = \sec. c$$

$$\frac{Ot}{OC} = \sec. AC = \sec. b$$

$$\frac{AT}{OC} = \tan. AB = \tan. c$$

$$\frac{At}{OC} = \tan. AC = \tan. b$$

Then in triangle T O t

$$Tt^2 = OT^2 + Ot^2 - 2\,OT \cdot Ot \cos. TOt$$

$$\therefore \frac{Tt^2}{OC^2} = \frac{OT^2}{OC^2} + \frac{Ot^2}{OC^2} - 2 \cdot \frac{OT}{OC} \cdot \frac{Ot}{OC} \cos. TOt$$

$$= \sec.^2 c + \sec.^2 b - 2 \sec. c \sec. b \cos. a, \quad - \; - \quad - \; (1)$$

Again, in triangle T A t

$$T t^2 = A T^2 + A t^2 - 2 A T \,.\, A t \cos. T A t$$

$$\therefore \frac{T t^2}{O C^2} = \frac{A T^2}{O C^2} + \frac{A t^2}{O C^2} - 2 \cdot \frac{A T}{O C} \cdot \frac{A t}{O C} \cos. T A$$

$$= \tan.^2 c + \tan.^2 b - 2 \tan. c \tan. b \cos. A \quad . \quad .$$

Equating (1) and (2)

$$\tan.^2 c + \tan.^2 b - 2 \tan. c \tan. b \cos. A = \sec.^2 c + \sec.^2 b - 2 \sec. c \sec. b \cos$$

$$= 1 + \tan.^2 c + 1 + \tan.^2 b - 2 \sec. c \sec. b \cos. a.$$

$$\therefore - 2 \tan. c \tan. b \cos A = 2 - 2 \sec. c \sec. b \cos. a.$$

$$\text{or,} - \frac{\sin. c}{\cos. c} \cdot \frac{\sin. b}{\cos. b} \cos. A = 1 - \frac{1}{\cos. c} \cdot \frac{1}{\cos. b} \cos. a.$$

$$\therefore \cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$$

Similarly we shall have,

$$\left.\begin{aligned} \cos. B &= \frac{\cos. b - \cos. a \cos. c}{\sin. a \sin. c} \\ \cos. C &= \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b} \end{aligned}\right\} (\alpha)$$

2\. *To express the cosine of a side of a spherical triangle, in terms of the sines and cosines of the angles.*

Let A, B, C, a, b, c, be the angles and sides of a spherical triangle; A′, B′, C′, a', b', c', the corresponding quantities in the Polar triangle,

Then, by (α),

$$\cos. A' = \frac{\cos. a' - \cos. b' \cos. c'}{\sin. b' \sin. c'.}$$

But (Spherical Geometry, prop. VI.), $A' = (180^\circ - a)$, $a' = (180^\circ - A)$, $b' = (180^\circ - B)$, $c' = (180^\circ - C)$,

$$\therefore \cos. (180^\circ - a) = \frac{\cos. (180^\circ - A) - \cos. (180^\circ - B) \cos. (180^\circ - C)}{\sin. (180^\circ - B) \sin. (180^\circ - C)}$$

$$\therefore \cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C.}$$

Similarly,

$$\left.\begin{aligned} \cos. b &= \frac{\cos. B + \cos. A \cos. C}{\sin. A \sin. C} \\ \cos. c &= \frac{\cos. C + \cos. A \cos. B}{\sin. A \sin. B.} \end{aligned}\right\} (A)$$

3\. *To express the sine of an angle of a spherical triangle, in terms of the sines of the sides of the triangle.*

By (α) we have,

$$\cos. A = \frac{\cos. a - \cos. b, \cos. c}{\sin. b \sin. c}$$

$$\therefore 1 + \cos. A = \frac{\cos. a - \cos. b \cos. c + \sin. b \sin. c}{\sin. b \sin. c}$$

$$= \frac{\cos. a - (\cos. b \cos. c - \sin. b \sin. c)}{\sin. b \sin. c}$$

$$= \frac{\cos. a - \cos. (b + c)}{\sin. b \sin. c}$$

$$= \frac{2 \sin. \frac{a+b+c}{2} \sin. \frac{b+c-a}{2}}{\sin. b \sin. c} \text{ (Plane Trig. Ch. II.)}$$

$$\text{Let } s = \frac{a+b+c}{2}$$

$$\therefore s - a = \frac{b+c-a}{2}$$

$$s - b = \frac{a+c-b}{2}$$

$$s - c = \frac{a+b-c}{2}$$

$$\therefore 1 + \cos. A = \frac{2 \sin. s \sin. (s-a)}{\sin. b \sin. c} \quad \text{- - - - - - - (1.)}$$

Again, resuming the expression for cos. A,

$$1 - \cos. A = \frac{\cos. b \cos. c + \sin. b \sin. c - \cos. a}{\sin. b \sin. c}$$

$$= \frac{\cos. (b - c) - \cos. a}{\sin. b \sin. c}$$

$$= \frac{2 \sin. \frac{a+b-c}{2} \sin. \frac{a+c-b}{2}}{\sin. b \sin. c}$$

$$= \frac{2 \sin. (s - c) \sin. (s - b)}{\sin. b \sin. c} \quad \text{- - - - (2.)}$$

Multiplying equations (1) and (2)

$$1 - \cos.^2 A = \frac{4 \sin. s \sin. (s - a) \sin. (s - b) \sin. (s - c)}{\sin.^2 b \sin.^2 c.}$$

$$\left.\begin{aligned}
&\therefore \sin. A = \frac{2}{\sin. b \sin. c} \sqrt{\sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)} \\
&\text{Similarly,} \\
&\sin. B = \frac{2}{\sin. a \sin. c} \sqrt{\sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)} \\
&\sin. C = \frac{2}{\sin. a \sin. b} \sqrt{\sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)}
\end{aligned}\right\} (\gamma.\ 1)$$

Now, by equation (1) we have,

$$1 + \cos A = \frac{2 \sin. s \sin. (s - a)}{\sin. b \sin. c}$$

or,

$$2 \cos.^2 \frac{A}{2} = \frac{2 \sin. s \sin. (s - a)}{\sin. b \sin\ c}$$

$$\therefore \cos. \frac{A}{2} = \sqrt{\frac{\sin. s \sin. (s - a)}{\sin. b \sin. c.}}$$

Similarly,

$$\left.\begin{aligned} \cos. \frac{B}{2} &= \sqrt{\frac{\sin. s \sin. (s - b)}{\sin. a \sin. c}} \\ \cos. \frac{C}{2} &= \sqrt{\frac{\sin. s \sin. (s - c)}{\sin. a \sin. b}} \end{aligned}\right\} (\gamma. 2)$$

Next, by equation (2),

$$1 - \cos. A = \frac{2 \sin. (s - b) \sin. (s - c)}{\sin. b \sin. c}$$

or,

$$2 \sin.^2 \frac{A}{2} = \frac{2 \sin. (s - b) \sin. (s - c)}{\sin. b \sin. c}$$

$$\therefore \sin. \frac{A}{2} = \sqrt{\frac{\sin. (s - b) \sin. (s - c)}{\sin. b \sin. c}}$$

Similarly,

$$\left.\begin{aligned} \sin. \frac{B}{2} &= \sqrt{\frac{\sin. (s - a) \sin. (s - c)}{\sin. a \sin. c}} \\ \sin. \frac{C}{2} &= \sqrt{\frac{\sin. (s - a) \sin. (s - b)}{\sin. a \sin. b}} \end{aligned}\right\} (\gamma. 3$$

Finally, dividing the expressions (γ. 3 by those γ. 2), we obtain,

$$\left.\begin{aligned} \tan. \frac{A}{2} &= \sqrt{\frac{\sin. (s - b) \sin. (s - c)}{\sin. s \sin. (s - a)}} \\ \tan. \frac{B}{2} &= \sqrt{\frac{\sin. (s - a) \sin. (s - c)}{\sin. s \sin. (s - b)}} \\ \tan. \frac{C}{2} &= \sqrt{\frac{\sin. (s - a) \sin. (s - b)}{\sin. s \sin. (s - c).}} \end{aligned}\right\} (\gamma. 4)$$

4. *To express the sine of a side of a spherical triangle, in terms of the sines and cosines of the angles.*

By (β) we have,

$$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C}$$

$$\therefore 1 + \cos. a = \frac{\cos. A + \cos. B \cos. C + \sin. B \sin. C}{\sin. B \sin. C}$$

$$= \frac{\cos. A + \cos. (B - C)}{\sin. B \sin. C}$$

$$= \frac{2 \cos. \frac{A + B - C}{2} \cos. \frac{A + C - B}{2}}{\sin. B \sin. C} \text{ (Plane Trig. Ch. II.)}$$

Let

and

$$s' - C = \frac{A + B - C}{2};$$

Hence,

$$1 + \cos. a = \frac{2 \cos. (s' - C) \cos. (s' - B)}{\sin. B \sin. C} \quad \text{- - - - - -} \quad (1.)$$

Resuming expression for cos. a,

$$1 - \cos. a = - \frac{\cos. B \cos. C - \sin. B \sin. C + \cos. A}{\sin. B \sin. C}$$

$$= - \frac{\cos. (B + C) + \cos. A}{\sin. B \sin. C}$$

$$= - \frac{2 \cos. \dfrac{A + B + C}{2} \cos. \dfrac{B + C - A}{2}}{\sin. B \sin. C}$$

$$= - \frac{2 \cos. s' \cos. (s' - A)}{\sin. B \sin. C} \quad \text{- - - - - - -} \quad (2.)$$

Multiplying Equations (1.) and (2.)

$$1 - \cos.^2 a = - \frac{4 \cos. s' \cos. (s' - A) \cos. (s' - B) \cos. (s' - C)}{\sin.^2 B \sin.^2 C}$$

$$\left.\begin{aligned} \therefore \sin. a &= \frac{2}{\sin. B \sin. C} \sqrt{-\cos. s' \cos. (s' - A) \cos. (s' - B) \cos. (s' - C)} \\ \text{Similarly,} & \\ \sin. b &= \frac{2}{\sin. A \sin. C} \sqrt{-\cos. s' \cos. (s - A) \cos. (s' - B) \cos. (s' - C)} \\ \sin. c &= \frac{2}{\sin. A \sin. B} \sqrt{-\cos. s' \cos. (s' - A) \cos. (s' - B) \cos. (s' - C)} \end{aligned}\right\} (\delta.\ 1.)$$

By Equation (1) we have,

$$1 + \cos. a = \frac{2 \cos. (s' - B) \cos. (s' - C)}{\sin. B \sin. C}.$$

$$\therefore 2 \cos.^2 \frac{a}{2} = \frac{2 \cos. (s' - B) \cos. (s' - C)}{\sin. B \sin. C}$$

$$\left.\begin{aligned} \therefore \cos. \frac{a}{2} &= \sqrt{\frac{\cos. (s' - B) \cos. (s' - C)}{\sin. B \sin. C}} \\ \text{Similarly,} & \\ \cos. \frac{b}{2} &= \sqrt{\frac{\cos. (s' - A) \cos. (s' - C)}{\sin. A \sin. C}} \\ \cos. \frac{c}{2} &= \sqrt{\frac{\cos. (s' - A) \cos. (s' - B)}{\sin. A \sin. B}} \end{aligned}\right\} (\delta.\ 2.)$$

By Equation (2.)

$$1 - \cos. a = - \frac{2 \cos. s' \cos. (s' - A)}{\sin. B \sin. C}$$

$$\therefore 2 \sin.^2 \frac{a}{2} = - \frac{2 \cos. s' \cos. (s' - A)}{\sin. B \sin. C}$$

$$\left.\begin{aligned}
\therefore\ \sin.\frac{a}{2} &= \sqrt{\frac{-\cos.s'\ \cos.(s'-A)}{\sin.B\ \sin.C}}\\
\sin.\frac{b}{2} &= \sqrt{\frac{-\cos.s'\ \cos.(s'-B)}{\sin.A\ \sin.C}}\\
\sin.\frac{c}{2} &= \sqrt{\frac{-\cos.s'\ \cos.(s'-C)}{\sin.A\ \sin.B}}
\end{aligned}\right\}\ (\delta.\ 3.)$$

Finally, dividing the expressions (δ. 3.) by the expressions (δ. 2.)

$$\left.\begin{aligned}
\tan.\frac{a}{2} &= \sqrt{\frac{-\cos.s'\ \cos.(s'-A)}{\cos.(s'-B)\ \cos.(s'-C)}}\\
\tan.\frac{b}{2} &= \sqrt{\frac{-\cos.s'\ \cos.(s'-B)}{\cos.(s'-A)\ \cos.(s'-C)}}\\
\tan.\frac{c}{2} &= \sqrt{\frac{-\cos.s'\ \cos.(s'-C)}{\cos.(s'-A)\ \cos.(s'-B)}}
\end{aligned}\right\}\ (\delta.\ 4.)$$

It is to be remarked, that although the expressions (δ. 1.), (δ. 3.), (δ. 4.), appear under an impossible form, they are in reality always possible.

For by Prop. IX. of Spherical Geometry, the sum of the angles of a spherical triangle, is always greater than two right angles, and less than six right angles.

$$\therefore\quad A+B+C \quad > \quad 180^\circ \text{ and } < 540^\circ$$

$$\therefore\quad \frac{A+B+C}{2} \text{ or } s' \quad > \quad 90^\circ \text{ and } < 270^\circ$$

Hence, cosine s' is always negative, and $\therefore$ $-\cos.s'$ is always positive.

Again, if a', b', c', be the three sides of the polar triangle, since the sum of any two sides of a spherical triangle is greater than the third side:

$$\begin{aligned}
b' + c' &> a'\\
\therefore\quad 180^\circ - B + 180^\circ - C &> 180^\circ - A\\
\therefore\quad B + C - A &< 180^\circ\\
\frac{B + C - A}{2} &< 90^\circ
\end{aligned}$$

$\therefore$ $\cos.(s'-A)$ is always positive, and in like manner $\cos.(s'-B)$, $\cos.(s'-C)$, are always positive; hence the above expressions are in every case possible.

5. *The sines of the angles of a spherical triangle are to each other as sines of the two sides opposite to them.*

Taking the expressions (γ. 1.), and calling the common radical quantity for the sake of brevity:

$$\sin.A = \frac{2N}{\sin.b\ \sin.}$$

$$\sin.B = \frac{2N}{\sin.a\ \sin.c}$$

Dividing the first of these by the second:

Similarly,

$$\left.\begin{aligned}\frac{\sin. A}{\sin. B} &= \frac{\sin. a \sin. c}{\sin. b \sin. c} = \frac{\sin. a}{\sin. b}\\ \frac{\sin. A}{\sin. C} &= \frac{\sin. a \sin. b}{\sin. c \sin. b} = \frac{\sin. a}{\sin. c}\\ \frac{\sin. B}{\sin. C} &= \frac{\sin. b \sin. a}{\sin. c \sin. a} = \frac{\sin. b}{\sin. c}\end{aligned}\right\} (\varepsilon.)$$

6. *To express the tangent of the sum and difference of two angles of a spherical triangle, in terms of the sides opposite to these angles, and the third angle of the triangle.*

By (α) we have,

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c} \quad \ldots\ldots\ldots\ldots (1.)$$

And,

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}$$

$$\therefore \cos. c = \cos. a \cos b + \sin. a \sin. b \cos. C \quad \ldots\ldots\ldots\ldots (2.)$$

Substituting this value of cos. c in Equation (1.):

$$\cos. A = \frac{\cos. a - \cos. a \cos.^2 b - \cos. b \sin. a \sin. b \cos. C}{\sin. b \sin. c}$$

$$= \frac{\cos. a\ (1 - \cos.^2 b) - \cos. b \sin. a \sin. b \cos. C}{\sin. b \sin. c}$$

$$= \frac{\cos. a \sin. b - \cos. b \sin. a \cos. C}{\sin. c} \quad \ldots\ldots\ldots\ldots (3.)$$

In like manner, substituting the value of cos. c in Equation (2), in the expression for cos. B, we shall find,

$$\cos. B = \frac{\cos. b \sin. a - \cos. a \sin. b \cos. C}{\sin c.} \quad \ldots\ldots\ldots\ldots (4.)$$

Adding Equations (3) and (4):

$$\cos. A + \cos. B = \frac{\sin. a \cos. b + \sin. b \cos. a - (\sin. a \cos. b + \sin. b \cos. a) \cos. C}{\sin. c}$$

$$= \frac{\sin. (a + b) - \sin. (a + b) \cos. C}{\sin. c}$$

$$= \frac{\sin. (a + b)\ (1 - \cos. C)}{\sin. c} \quad \ldots\ldots\ldots\ldots (5.)$$

Again, by Equation (ε) we have,

$$\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}$$

$$\therefore \frac{\sin. A + \sin. B}{\sin. B} = \frac{\sin. a + \sin. b}{\sin. b}$$

$$\therefore \sin. A \pm \sin. B = (\sin. a \pm \sin. b) \frac{\sin. B}{\sin. b}$$

$$= (\sin. a \pm \sin. b) \frac{\sin. C}{\sin. c} \quad \ldots\ldots\ldots\ldots (6.)$$

Dividing Equation (6) by Equation (5), and taking first the positive sign:

$$\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \frac{\sin. a + \sin. b}{\sin. (a+b)} \cdot \frac{\sin. C}{1 - \cos. C}$$

$$\therefore \frac{2 \sin. \frac{A+B}{2} \cos. \frac{A-B}{2}}{2 \cos. \frac{A+B}{2} \cos. \frac{A-B}{2}} = \frac{2 \sin. \frac{a+b}{2} \cos. \frac{a-b}{2}}{2 \sin. \frac{a+b}{2} \cos. \frac{a+b}{2}} \cdot \cot. \frac{C}{2}$$

$$\therefore \quad \tan. \frac{A+B}{2} = \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}} \cdot \cot. \frac{C}{2}$$

Again, dividing Equation (6) by Equation (5), and taking the negative sign.

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \frac{\sin. a - \sin. b}{\sin. (a+b)} \cdot \frac{\sin. C}{1 - \cos. C}$$

$$\frac{2 \sin. \frac{A-B}{2} \cos. \frac{A+B}{2}}{2 \cos. \frac{A-B}{2} \cos. \frac{A+B}{2}} = \frac{2 \sin. \frac{a-b}{2} \cos. \frac{a+b}{2}}{2 \sin. \frac{a+b}{2} \cos. \frac{a+b}{2}} \cdot \frac{\sin. C}{1 - \cos. C}$$

$$\therefore \quad \tan. \frac{A-B}{2} = \frac{\sin. \frac{a-b}{2}}{\cos. \frac{a+b}{2}} \cot. \frac{C}{2}$$

We have thus obtained the required expression, viz.

Similarly,

$$\left.\begin{aligned}
\tan. \frac{A+B}{2} &= \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}} \cot. \frac{C}{2} \\
\tan. \frac{A-B}{2} &= \frac{\sin. \frac{a-b}{2}}{\sin. \frac{a+b}{2}} \cot. \frac{C}{2} \\
\tan. \frac{B+C}{2} &= \frac{\cos. \frac{b-c}{2}}{\cos. \frac{b+c}{2}} \cot. \frac{A}{2} \\
\tan. \frac{B-C}{2} &= \frac{\sin. \frac{b-c}{2}}{\sin. \frac{b+c}{2}} \cot. \frac{A}{2} \\
\tan. \frac{A+C}{2} &= \frac{\cos. \frac{a-c}{2}}{\cos. \frac{a+c}{2}} \cot. \frac{B}{2} \\
\tan. \frac{A-C}{2} &= \frac{\sin. \frac{a-c}{2}}{\sin. \frac{a+c}{2}} \cot. \frac{B}{2}
\end{aligned}\right\} (r.)^*$$

* These Equations when converted into proportions, form what are called, from the inventor, *Napier's first and second Analogies.*

7. *To express the tangent of the sum and difference of two sides of a spherical triangle, in terms of the angles opposite to them and the third side of the triangle.*

Let A, B, C, a, b, c, be the sides and angles of a spherical triangle, A′, B′, C′, a', b', c', the corresponding parts of the polar triangle; then by expression (r),

$$\tan.\frac{A'+B'}{2} = \frac{\cos.\frac{a'-b'}{2}}{\cos.\frac{a'+b'}{2}}\cot.\frac{C'}{2}$$

$$\therefore \tan.\frac{180^\circ - a + 180^\circ - b}{2} = \frac{\cos.\frac{(180^\circ-A)-(180^\circ-B)}{2}}{\cos.\frac{(180^\circ-A)+(180^\circ-B)}{2}}\cot.\frac{(180^\circ-c)}{2}$$

$$\tan.\left(180^\circ - \frac{a+b}{2}\right) = \frac{\cos.\left(-\frac{A-B}{2}\right)}{\cos.\left(180^\circ-\frac{A+B}{2}\right)}\cot.\left(90^\circ-\frac{c}{2}\right)$$

$$\therefore \tan.\frac{a+b}{2} = \frac{\cos.\frac{A-B}{2}}{\cos.\frac{A+B}{2}}\tan.\frac{c}{2}$$

Again,

$$\tan.\frac{A'-B'}{2} = \frac{\sin.\frac{a'-b'}{2}}{\sin.\frac{a'+b'}{2}}\cot.\frac{C'}{2}$$

$$\therefore \tan.\frac{(180^\circ-a)-(180^\circ-b)}{2} = \frac{\sin.\frac{(180^\circ-A)-(180^\circ-B)}{2}}{\sin.\frac{(180^\circ-A)+(180^\circ-B)}{2}}\cot.\frac{(180^\circ-c)}{2}$$

$$\therefore \tan.\frac{a-b}{2} = \frac{\sin.\frac{A-B}{2}}{\sin.\frac{A+B}{2}}\tan.\frac{c}{2}$$

We shall thus obtain another group of formulæ analogous to the last.

$$\left.\begin{aligned}
\frac{a+b}{2} &= \frac{\cos.\frac{A-B}{2}}{\cos.\frac{A+B}{2}}\tan.\frac{c}{2} \\
\tan.\frac{a-b}{2} &= \frac{\sin.\frac{A-B}{2}}{\sin.\frac{A+B}{2}}\tan.\frac{c}{2}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\tan.\frac{b+c}{2} &= \frac{\cos.\frac{B-C}{2}}{\cos.\frac{B+C}{2}}\tan.\frac{a}{2} \\
\tan.\frac{b-c}{2} &= \frac{\sin.\frac{B-C}{2}}{\sin.\frac{B+C}{2}}\tan.\frac{a}{2}
\end{aligned}\right\}(\zeta)$$

$$\left.\begin{aligned}
\tan.\frac{a+c}{2} &= \frac{\cos.\frac{A-C}{2}}{\cos.\frac{A+C}{2}}\tan.\frac{b}{2} \\
\tan.\frac{a-c}{2} &= \frac{\sin.\frac{A-C}{2}}{\sin.\frac{A+C}{2}}\tan.\frac{b}{2}
\end{aligned}\right\}$$

8. *To express the cotangent of an angle of a spherical triangle, in terms of the side opposite one of the other sides and the angle contained between these two sides.*

By (α)

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c} \quad \text{--------------------} \quad (1)$$

and,

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}.$$

Hence,

$$\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C.$$

Substituting this value of cos. c in Equation (1), it becomes

$$\cos. A = \frac{\cos. a - \cos. a \cos.^2 b - \sin. a \sin. b \cos. b \cos. C}{\sin. b \sin. c}$$

$$= \frac{\cos. a\,(1 - \cos.^2 b) - \sin. a \sin. b \cos. b \cos. C}{\sin. b \sin. c}$$

$$\therefore \cos. A = \frac{\cos. a\,(1 - \cos.^2 b) - \sin. a \sin. b \cos. b \cos. C}{\sin. b \sin. c.}$$

$$\therefore \cos. A \sin. c = \cos. a \sin. b - \sin. a \cos. b \cos C$$

But,

$$\sin. c = \frac{\sin. C}{\sin. A} \sin. a, \text{ by } (\epsilon),$$

$$\therefore \cos. A \frac{\sin. C}{\sin. A} \sin. a = \cos. a \sin. b - \sin. a \cos. b \cos. C,$$

$$\therefore \cot. A = \cot\ a \sin. b \operatorname{cosec.} C - \cos. b \cot. C.$$

In which the cotangent of A is expressed in the required manner.

If in Equation (1), instead of substituting for cos. c, we had substituted for cos. b, the value derived from the Equation

$$\cos. B = \frac{\cos. b - \cos. a \cos. c}{\sin. a \sin. c},$$

we should have found a value for cot. A in terms of, a, c, B, or

$$\cot. A = \cot. a \sin. c \operatorname{cosec.} B - \cos. c \cot. B.$$

Proceeding in like manner for the other angles, we shall obtain similar results, and presenting them at one view we have,

$$\left.\begin{array}{rcl} \cot. A & = & \cot. a \sin. b \operatorname{cosec.} C - \cos. b \cot. C \\ & = & \cot. a \sin. c \operatorname{cosec.} B - \cos. c \cot. B \\ \cot. B & = & \cot. b \sin. a \operatorname{cosec.} C - \cos. a \cot. C \\ & = & \cot. b \sin. c \operatorname{cosec.} A - \cos. c \cot. A \\ \cot. C & = & \cot. c \sin. a \operatorname{cosec.} B - \cos. a \cot. B \\ & = & \cot. c \sin. b \operatorname{cosec.} A - \cos. b \cot. A \end{array}\right\} (\eta)$$

9. *To express the cotangent of a side of a spherical triangle, in terms of the opposite angle, one of the other angles, and the side interjacent to these two angles.*

Let A, B, C, a, b, c, be the angles and sides of a spherical triangle, and A′, B′, C′, a', b', c', the corresponding parts in the polar triangle.

Then by (η)

$$\cot. A' = \cot. a' \sin. b' \operatorname{cosec.} C' - \cos. b' \cot. C'$$

$$\therefore \cot. (180° - a) = \cot. (180° - A) \sin. (180° - B) \operatorname{cosec.} (180° - c) - \cos. (180° - B) \cot. (180° - c)$$

$$-\cot. a = -\cot. A \sin. B \operatorname{cosec.} c - \cos. B \cot. c$$

$$\therefore \cot. a = \cot. A \sin. B \operatorname{cosec.} c + \cos. B \cot. c.$$

Applying the same process to each of the expressions in (η), we shall obtain analogous results, and thus have a new set of formulæ :

$$\left.\begin{array}{rcl} \cot. a & = & \cot. A \sin. B \operatorname{cosec.} c + \cos. B \cot. c \\ & = & \cot. A \sin. C \operatorname{cosec.} b + \cos\ C \cot. b \\ \cot. b & = & \cot. B \sin. A \operatorname{cosec.} c + \cos. A \cot. c \\ & = & \cot. B \sin. C \operatorname{cosec.} a + \cos. C \cot. a \\ \cot. c & = & \cot. C \sin. A \operatorname{cosec.} b + \cos. A \cot. b \\ & = & \cot. C \sin. B \operatorname{cosec.} a + \cos. B \cot. a \end{array}\right\} (\theta)$$

By aid of the nine groups of formulæ marked, (α), (β), (γ), (δ), (ϵ), (σ), (ζ), (η), (θ), we shall be enabled to solve all the cases of spherical triangles, whether right-angled, or oblique-angled; and we shall proceed in the next chapter to apply them.

CHAPTER II.

ON THE SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

Spherical triangles, that have one right angle only, are the subject of investigation in this chapter; those that have two or three right angles are excluded.

A spherical triangle consists of 6 parts, the 3 sides and 3 angles, and any 3 of these being given, the rest may be found. In the present case, one of the angles is by supposition a right angle; if any other two parts be given, the other three may be determined. Now the combination of 5 quantities taken, 3 and 3 $= \frac{5.4.3}{1.2.3} = 10$; therefore ten different cases present themselves in the solution of right-angled triangles.

The manner in which each case may be solved individually, by applying the formulæ already deduced, will be pointed out at the conclusion of this chapter; but we shall in the first place explain two rules, by aid of which the computist is enabled to solve every case of right-angled triangles. These are known by the name of *Napier's Rules for Circular Parts;* and it has been well observed by the late Professor Woodhouse, that, in the whole compass of mathematical science, there cannot be found rules which more completely attain that which is the proper object of all rules, namely, facility and brevity of computation.

The rules and their description are as follows:

Description of the Circular Parts.

The right angle is thrown altogether out of consideration. The two sides, the complements of the two angles, and the complement of the hypotenuse, are called *the circular parts.* Any one of these circular parts may be called a *middle part* (M), and then the two circular parts immediately adjacent to the right and left of M are called *adjacent parts;* the other two remaining circular parts, each separated from M the middle part by an adjacent part, are called *opposite parts,* or, *opposite extremes.*

This being premised, we may now give

Napier's Rules.

1. *The product of* sin. M *and tabular radius* = *product of the tangents of the adjacent parts.*
2. *The product of* sin. M *and tabular radius** = *product of the cosines of the opposite parts.*

These rules will be clearly understood if we show the manner in which they are applied in various cases.

* See Plane Trigonometry, Chap. IV.

Let A B C be a spherical triangle, right-angled at C.
Let a be assumed as the middle part.
Then $(90° - B)$ and b are the adjacent parts,
And $(90° - c)$ and $(90° - A)$ are the opposite parts.
Then by rule (1)

$$R \times \sin. a = \tan. (90° - B) \tan. b$$
$$= \cot. B \tan. b \ldots\ldots (1)$$

By Rule (2)

$$R. \sin. a = \cos. (90° - A) \cos. (90° - c)$$
$$= \sin. A \sin. c, \ldots\ldots (2)$$

2. Let b be the middle part,
Then $(90° - A)$ and a are adjacent parts,
And $90° - c)$ and $(90° - B)$ are opposite parts.
Then Rule I,

$$\therefore R \sin. b = \tan. (90° - A) \tan. a$$
$$= \cot. A \tan. a \ldots\ldots (3).$$

And Rule II,

$$\text{and } R \sin b = \cos. (90° - B) \cos. (90° - c)$$
$$= \sin. B \sin. c \ldots\ldots (4).$$

3. Let $(90° - c)$ be the middle part,
Then $(90° - A)$, and $(90° - B)$ are adjacent parts,
And b and a are opposite parts.
Then,

$$R. \sin. (90° - c) = \tan. (90° - A) \tan. (90° - B)$$
$$R. \cos. c = \cot. A \cot. B \ldots\ldots (5)$$

And,

$$R. \sin. (90° - c) = \cos. a \cos. b.$$
$$R. \cos. c. = \cos. a \cos. b \ldots\ldots (6)$$

4. Let $(90° - A)$ be the middle part.
Then $(90° - c)$ and b are adjacent parts,
And $(90° - B)$ and a are opposite parts.
Then Rule I.

$$R. \sin. (90° - A) = \tan. (90° - c) \tan. b,$$
$$\therefore R \cos. A = \cot. c \tan. b \ldots\ldots (7)$$

And Rule II.

$$R. \sin. (90° - A) = \cos. (90° - B) \cos. a,$$
$$\therefore R \cos. A = \sin. B \cos. a. \ldots\ldots (8)$$

5. Let $(90° - B)$ be the middle part.
Then $(90° - c)$ and a are the adjacent parts,
And $(90° - A)$ and b are the opposite parts.
Then Rule I

$$\cos. B = \tan. (90° - c), \tan. a,$$
$$= \tan. a \cot. c \ldots\ldots (9)$$
$$\cos. B = \cos. (90° - A) \cos. b,$$
$$= \sin. A \cos. b \ldots\ldots (10)$$

Collecting the above results we shall have

$$
\begin{array}{lll}
* \sin. a & = \cot. B \tan. b \ldots\ldots\ldots\ldots & (1) \\
\sin. a & = \sin. A \sin. c \ldots\ldots\ldots\ldots & (2) \\
\sin. b & = \cot. A \tan. a \ldots\ldots\ldots\ldots & (3) \\
\sin. b & = \sin. B \sin c \ldots\ldots\ldots\ldots & (4) \\
\cos. c & = \cot. A \cot. B \ldots\ldots\ldots\ldots & (5) \\
\cos. c & = \cos. a \cos. b \ldots\ldots\ldots\ldots & (6) \\
\cos. A & = \tan. b \cot. c \ldots\ldots\ldots\ldots & (7) \\
\cos. A & = \sin. B \cos. a \ldots\ldots\ldots\ldots & (8) \\
\cos. B & = \tan. a \cot. c \ldots\ldots\ldots\ldots & (9) \\
\cos. B & = \sin. A \cos. b \ldots\ldots\ldots\ldots & (10)
\end{array}
$$

It now remains for us to show that these conclusions are accurate, and in accordance with the formulæ already deduced.

Now by (α).

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}$$

But when $C = 90°$, then $\cos. C = 0$.

$$\therefore 0 = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}$$

$\therefore \cos. c = \cos. a \cos. b$, which is formula (6) in the above table.

Again by (ϵ)

$$\frac{\sin. a}{\sin. c} = \frac{\sin. A}{\sin. C}$$

But when $C = 90°$ $\sin. C = 1$.

$\therefore \quad \sin. a = \sin. A \sin. c$. which is formula (2) above.

Similarly,

$$\frac{\sin. b}{\sin. c} = \frac{\sin. B}{\sin. C}$$

$\therefore \quad \sin. b = \sin. B \sin. c$, which is formula (4).

Next since by (α)

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}, \text{ substitute for } \cos. c \text{ its value in (6).}$$

$$= \frac{\cos. a - \cos. a \cos.^2 b}{\sin. b \sin. c}$$

$$= \frac{\cos. a \sin b}{\sin. c}, \text{ substitute for } \sin. c, \text{ its value as found in (2).}$$

$$= \frac{\cos. a \sin. b}{\frac{\sin. a}{\sin. A}}$$

$\therefore \sin. b = \cot. A \tan. a$, which is formula (3.)

Again,

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c} \text{ substitute for } \cos. a, \text{ its value in (6.)}$$

$$= \frac{\frac{\cos. c}{\cos. b} - \cos. b \cos. c}{\sin. b \sin. c}$$

* The number R need be introduced only when we have occasion to use tables, and will therefore be omitted in the investigation which follows.

$$= \frac{\cos.\ c \sin.\ b}{\sin.\ c \cos.\ b}$$

$$= \tan.\ b \cot.\ c, \text{ which is formula (7).}$$

Again, by (α).

$$\cos.\ B = \frac{\cos.\ b - \cos.\ a \cos.\ c}{\sin.\ a \sin.\ c} \text{ substitute for } \cos.\ c \text{ its value from (6)}$$

$$= \frac{\cos.\ b - \cos.\ b \cos.^2 a}{\sin.\ a \sin.\ c}$$

$$= \frac{\cos.\ b \sin.\ a}{\sin.\ c} \text{ substitute for } \sin\ c, \text{ its value from (4).}$$

$$= \frac{\cos.\ b \sin.\ a}{\frac{\sin.\ b}{\sin.\ B}}$$

$$\text{Sin. } a = \cot.\ B \tan.\ b, \text{ which is formula (1).}$$

Again,

$$\cos.\ B = \frac{\cos.\ b - \cos.\ a \cos.\ c}{\sin.\ a \sin.\ c}, \text{ substitute for } \cos.\ b, \text{ its value in (6).}$$

$$= \frac{\frac{\cos.\ c}{\cos.\ a} - \cos.\ a \cos.\ c}{\sin.\ a \sin.\ c}$$

$$= \frac{\cos.\ c \sin.\ a}{\sin.\ c \cos.\ a}$$

$$= \tan.\ a \cot.\ c, \text{ which is formula (9).}$$

Next by (β).

$$\cos.\ a = \frac{\cos.\ A + \cos.\ B \cos.\ C}{\sin.\ B \sin.\ C}$$

But $C = 90^{\circ}$ $\therefore$ $\cos.\ C = 0$, and $\sin.\ C = 1$.

$$\therefore \cos.\ a = \frac{\cos.\ A}{\sin.\ B}$$

$$\therefore \cos.\ A = \sin.\ B \cos.\ a, \text{ which is formula (8).}$$

Again,

$$\cos.\ b = \frac{\cos.\ B + \cos.\ A \cos.\ C}{\sin.\ A \sin.\ C} \text{ and when } C = 90^{\circ}.$$

$$= \frac{\cos.\ B}{\sin.\ A}$$

$$\therefore \cos.\ B = \sin.\ A \cos.\ b, \text{ which is formula (10).}$$

Lastly,

$$\cos.\ c = \frac{\cos.\ C + \cos.\ A \cos.\ B}{\sin.\ A \sin.\ B} \text{ and in this case,}$$

$$= \frac{\cos.\ A \cos.\ B}{\sin.\ A \sin.\ B}$$

$$= \cot.\ A \cot.\ B, \text{ which is formula (5).}$$

We have thus proved the truth of the results derived from the application of Napier's rules, and may therefore apply these rules without scruple to the solution of the various cases of right angled triangles.

Let us then take each combination of the two data, and determine in each case the other three quantities, adapting our formulæ to computation by tables.

1. Given A, B, required a, b, c.

$$R \cos. A = \sin B \cos. a \quad \therefore \quad \cos. a = R \frac{\cos. A}{\sin. B} \dots\dots (1)$$

$$R \cos. B = \sin. A \cos. b \quad \therefore \quad \cos. b = R \frac{\cos. B}{\sin. A} \dots\dots (2)$$

$$R \cos c = \cot. A \cot. B \dots\dots (3)$$

2. Given a, b, required A, B, c.

$$R \sin. a = \cot. B \tan. b \quad \therefore \quad \cot. B = R \sin. a \cot. b \dots\dots (4)$$

$$R \sin. b = \cot. A \tan. a \quad \therefore \quad \cot. A = R \sin. b \cot. a \dots\dots (5)$$

$$R \cos. c = \cos. a \cos. b \dots\dots (6)$$

3. Given a, c, required A, B, b.

$$R \sin. a = \sin. A \sin. c \quad \therefore \quad \sin. A = R \frac{\sin. a}{\sin. c} \dots\dots (7)$$

$$R \cos. c = \cos. a \cos. b \quad \therefore \quad \cos. b = R \frac{\cos. c}{\cos. a} \dots\dots (8)$$

$$R \cos. B = \tan. a \cot. c \dots\dots (9)$$

4. Given b, c, required A, B, a.

$$R \sin. b = \sin. B \sin. c \quad \therefore \quad \sin. B = R \frac{\sin. b}{\sin. c} \dots\dots (10)$$

$$R \cos. c = \cos. a \cos. b \quad \therefore \quad \cos. a = R \frac{\cos. c}{\cos. b} \dots\dots (11)$$

$$R \cos. A = \tan. b \cot. c \dots\dots (12)$$

5. Given A, c, required B, a, b.

$$R \cos. A = \tan. b \cot. c \quad \therefore \quad \tan. b = R \cos. a \tan. c \dots\dots (13)$$

$$R \cos. c = \cot. A \cot. B \quad \therefore \quad \cot. B = R \tan. A \cos. c \dots\dots (14)$$

$$R \sin. a = \sin. A \sin. c \dots\dots (15)$$

6. Given B, c, required A, a, b.

$$R \cos. B = \cot. c \tan. a \quad \therefore \quad \tan. a = R \cos. B \tan. c \dots (16)$$

$$R \cos. c = \cot. A \cot. B \quad \therefore \quad \cot. A = R \tan. B \cos. c \dots (17)$$

$$R \sin. b = \sin. B \sin. c \dots\dots (18)$$

7. Given A, b, required B, c, a.

$$R \cos. A = \cot. c \tan. b \quad \therefore \quad \cot. c = R \cos. A \cot. b \dots\dots (19)$$

$$R \sin. b = \cot. A \tan. a \quad \therefore \quad \tan. a = R \tan. A \sin. b \dots\dots (20)$$

$$R \cos. B = \sin. A \cos. b \dots\dots (21)$$

8. Given B, a, required A, c, b.

$$R \cos. B = \cot. c \tan. a \quad \therefore \quad \cot. c = R \cos. B \cot. a \dots\dots (22)$$

$$R \sin. a = \cot. B \tan. b \quad \therefore \quad \tan. b = R \tan. B \sin. a \dots\dots (23)$$

$$R \cos. A = \sin. B \cos. a \dots\dots (24)$$

9. Given A, a, required B, b, c.

$$R \cos. A = \sin. B \cos. a \quad \therefore \quad \sin. B = R \frac{\cos. A}{\cos. a} \dots\dots (25)$$

$$R \sin. a = \sin. A \sin. c \quad \therefore \quad \sin. c = R \frac{\sin. a}{\sin. A} \dots\dots (26)$$

$$R \sin. b = \cot. A \tan. a \dots\dots (27)$$

10. Given B, b, required A, a, c.

$$R \cos. B = \sin. A \cos. b \quad \therefore \quad \sin. A = R \frac{\cos. B}{\cos. b} \quad \text{.........} \quad (28)$$

$$R \sin. b = \sin. B \sin. c \quad \therefore \quad \sin. c = R \frac{\sin. b}{\sin. B} \quad \text{..........} \quad (29)$$

$$R \sin. a = \cot. B \tan. b \quad \text{..........} \quad (30)$$

CHAPTER III.

ON THE SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

The different cases which present themselves are contained in the following enumeration.

1. When two sides and the included angle are given.
2. When two angles and the side between them are given.
3. When two sides and the angle opposite to one of them are given.
4. When two angles and the side opposite to one of them are given.
5. When three sides are given.
6. When three angles are given.

I. When two sides and the included angle are given.

The remaining angles may be determined from the formula (σ).

Thus, let a, b, C, be given, A, B, c, required.

$$\tan. \frac{A+B}{2} = \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}} \cot. \frac{C}{2}$$

$$\tan. \frac{A-B}{2} = \frac{\sin. \frac{a-b}{2}}{\sin. \frac{a+b}{2}} \cot. \frac{C}{2}$$

Whence $\frac{A+B}{2}$ and $\frac{A-B}{2}$ are known from the tables.

$$\text{Let} \quad \frac{A+B}{2} = \theta$$

$$\frac{A-B}{2} = \phi$$

$$\therefore \quad A = \theta + \phi$$

$$B = \theta - \phi$$

A and B being known, c may be obtained from (ϵ).

$$\text{For} \quad \frac{\sin. c}{\sin. a} = \frac{\sin. C}{\sin. A}$$

$$\therefore \quad \sin. c = \sin. a \frac{\sin. C}{\sin. A}$$

And, in like manner, if any two other sides and the included angle be given, the remaining parts may be determined.

II. When two angles and the side between them are given.

The remaining sides may be determined from the formula (ζ).

Thus, let A, B, c; be given, a, b, C; required.

$$\tan. \frac{a+b}{2} = \frac{\cos. \frac{A-B}{2}}{\cos. \frac{A+B}{2}} \tan. \frac{c}{2}$$

$$\tan. \frac{a-b}{2} = \frac{\sin. \frac{A-B}{2}}{\sin. \frac{A+B}{2}} \tan. \frac{c}{2}$$

Whence $\frac{a+b}{2}$ and $\frac{a-b}{2}$ are known from the tables.

Let
$$\frac{a+b}{2} = \theta'$$
$$\frac{a-b}{2} = \phi'$$
$$\therefore \quad a = \theta' + \phi'$$
$$b = \theta' - \phi'$$

a and b being known, C may be obtained by (ι).

For
$$\frac{\sin. C}{\sin. A} = \frac{\sin. c}{\sin. a}$$
$$\therefore \quad \sin. C = \sin. A \frac{\sin. c}{\sin. a}$$

And, in like manner, if any two other angles and the included side are given the remaining parts may be determined.

III. When two sides and the angle opposite to one of them are given.

The angle opposite to the other side may be found from formula (ι).

Thus, let a, b, A be given, B, C, c; required.

$$\frac{\sin. B}{\sin. A} = \frac{\sin. b}{\sin. a}$$
$$\therefore \quad \sin. B = \sin A \frac{\sin. b}{\sin. a}$$

The angle B being determined, the remaining angle C will be found from (ε).

For
$$\tan. \frac{A+B}{2} = \frac{\cos. \frac{a-b}{2}}{\cos. \frac{a+b}{2}} \cot. \frac{C}{2}$$
$$\cot. \frac{C}{2} = \frac{\cos. \frac{a+b}{2}}{\cos. \frac{a-b}{2}} \tan. \frac{A+B}{2}$$

The angle C being determined, the remaining side c will be found from (ι).

For
$$\frac{\sin. c}{\sin. a} = \frac{\sin. C}{\sin. A}$$
$$\therefore \quad \sin. c = \sin. a \frac{\sin. C}{\sin. A}$$

or c may be found from (ζ).

And, in like manner, if any other two sides and the angle opposite to one of them be given, the remaining parts may be determined.

IV. When two angles and the side opposite to one of them are given.

The side opposite to the other angle may be found from formula (ε).

Thus, let A, B, a; be given, b, c, C; required.

$$\frac{\sin. b}{\sin. a} = \frac{\sin. B}{\sin. A}$$

$$\therefore \qquad \sin. b = \sin. a \frac{\sin. B}{\sin. A}$$

The side b being determined, the remaining side c will be found from (ζ)

$$\text{For} \qquad \tan. \frac{a+b}{2} = \frac{\cos. \frac{A-B}{2}}{\cos. \frac{A+B}{2}} \tan. \frac{c}{2}$$

$$\therefore \qquad \tan. \frac{c}{2} = \frac{\cos. \frac{A+B}{2}}{\cos. \frac{A-B}{2}} \tan. \frac{a+b}{2}$$

The side c being determined, the remaining angle C will be found from (ε).

$$\text{For} \qquad \frac{\sin. C}{\sin. A} = \frac{\sin. c}{\sin. a}$$

$$\therefore \qquad \sin. C = \sin. A \frac{\sin. c}{\sin. a}$$

or c may be found from (σ).

And, in like manner, any other two sides being given and the angle opposite to one of them, the remaining parts may be determined.

V. When three sides are given.

The three angles may be immediately determined from any one of the groups of formulæ (γ 1), (γ 2), (γ 3), (γ 4).

The choice of the formula, which it will be advantageous to employ in practice, will depend upon the consideration already noticed in the solution of the analogous case in plane trigonometry.

VI. When three angles are given.

The three sides may be immediately determined from any of the groups of formulæ (δ 1), (δ 2), (δ 3), (δ 4).

CHAPTER IV.

ON THE USE OF SUBSIDIARY ANGLES.

We have already explained in Plane Trigonometry, the meaning of Subsidiary Angles, and the purposes for which they are introduced; we shall now proceed to point out under what circumstances they may be employed with advantage, in Spherical Trigonometry.

In the solution of case I. where two sides and the included angle were given, we first determined the two remaining angles, and having found these, we were enabled to find the side also. It frequently happens, however, that the side alone is the object of our investigations, and it is therefore convenient to have a method of determining it, independently of the angle.

Thus, for example, let b, c, A, be given, and let it be required to determine a, independently of the angles B, C.

By (α), we have

$$\cos A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$$

Whence $\qquad \cos. a = \cos. A \sin. b \sin. c + \cos. b \cos. c.$

From which equation a is determined, but the expression is not in a form adapted to logarithmic computation; we can, however, effect the necessary transformation by the introduction of a subsidiary angle.

Add and subtract sin. b sin. c on the right hand side of the equation.

$$\begin{aligned}
\text{Then } \cos. a &= \cos. A \sin. b \sin. c + \cos. b \cos. c + \sin. b \sin. c - \sin. b \sin. c \\
&= \cos. b \cos. c + \sin. b \sin. c + \sin. b \sin. c \cos. A - \sin. b \sin. c \\
&= \cos. (b - c) - \sin. b \sin. c \text{ vers. } A
\end{aligned}$$

$$\begin{aligned}
1 - \cos. a &= 1 - \cos. (b - c) + \sin. b \sin. c \text{ vers. } A \\
\text{vers. } a &= \text{vers. } (b - c) + \sin. b \sin. c \text{ vers. } A \\
&= \text{vers. } (b - c) \left\{ 1 + \frac{\sin. b \sin. c \text{ vers. } A}{\text{vers. } (b - c)} \right\}
\end{aligned}$$

$$\text{Let } \tan.^2 \theta = \frac{\sin. b \sin. c \text{ vers. } A}{\text{vers. } (b - c)}$$

$$\begin{aligned}
\therefore \text{ vers. } a &= \text{vers. } (b - c) \{1 + \tan.^2 \theta\} \\
&= \text{vers. } (b - c) \sec.^2 \theta
\end{aligned}$$

from which a may be determined by the tables, θ being known from the equation.

$$\tan.^2 \theta = \frac{\sin. b \sin. c \text{ vers. } A}{\text{vers. } (b - c)}.$$

In like manner in case II, where two angles and the included side were given, we first determined the remaining sides, and then we were enabled to find the remaining angle. Now, let us suppose, that A, B, c, are given, and that we are required to find C independently of a and b.

$$\text{From } (\beta) \qquad \cos. c = \frac{\cos. C + \cos. A \cos. B}{\sin. A \sin. B}$$

$$\begin{aligned}
\therefore \qquad \cos. C &= \cos. c \sin. A \sin. B - \cos. A \cos. B \\
\therefore 1 - \cos. C &= 1 - \sin A \sin. B (1 - \text{vers. } c) + \cos. A \cos. B \\
&= 1 + \cos. (A + B) + \sin. A \sin. B \text{ vers. } c.
\end{aligned}$$

$$\begin{aligned}
\text{or} \qquad 2 \sin.^2 \frac{C}{2} &= 2 \cos.^2 \frac{A + B}{2} + \sin. A \sin. B \text{ vers. } c \\
&= 2 \cos.^2 \frac{A + B}{2} \left\{ 1 + \frac{\sin. A \sin. B \text{ vers. } c}{2 \cos.^2 \frac{A + B}{2}} \right\}
\end{aligned}$$

$$\therefore \qquad \sin.^2 \frac{C}{2} = \cos.^2 \frac{A + B}{2} \sec.^2 \theta$$

If we assume

$$\tan.^2 \theta = \frac{\sin. A \sin. B \text{ vers. } c}{2 \cos.^2 \frac{A + B}{2}}$$

In case III, where two sides and the angle opposite to one of them were given we first determined the angle opposite to the other side, and then the remaining angles and the remaining side in succession. Now, let us suppose, that a, b, A, are given, and that we are required to determine the angle C and the side c, independently of the angle B and of each other, under a form adapted for logarithmic computation.

To find C, we have from (n).

$$\cot. A = \cot. a \sin. b \operatorname{cosec}. C - \cos. b \cot. C$$

or $$\cot. A \sin. C = \cot. a \sin. b - \cos. b \cos. C$$

or $$\sin. C = \cot. a \sin. b \tan. A - \cos. b \cos. C \tan A.$$

$$\therefore \sin. C + \cos. C \cos. b \tan. A = \cot. a \sin. b \tan. A$$

Let $$\cos. b \tan. A = \tan. \theta = \frac{\sin. \theta}{\cos. \theta}$$

$$\therefore \quad \sin. C + \frac{\sin. \theta}{\cos. \theta} \cos. C = \cot. a \sin. b \tan. A$$

$$\therefore \sin. C \cos. \theta + \cos. C \sin. \theta = \cot. a \sin. b \tan. A \cos. \theta$$

$$\sin. (C + \theta) = \cot. a \sin. b \tan. A \frac{\sin. \theta}{\cos. b \tan. A}$$

$$= \cot. a \tan. b. \sin. \theta$$

whence C is known, θ being previously determined from equation.

$$\tan. \theta = \cos. b \tan. A.$$

To find c, we have from (α).

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$$

$$\therefore \quad \sin. c \sin. b \cos. A = \cos. a - \cos. b \cos. c$$

$$\sin. c \tan. b \cos. A = \frac{\cos. a}{\cos. b} - \cos. c$$

Let $$\tan. b \cos. A = \tan. \theta = \frac{\sin. \theta}{\cos. \theta}$$

$$\sin. c \frac{\sin. \theta}{\cos. \theta} + \cos. c = \frac{\cos. a}{\cos. b}$$

$$\cos. (c - \theta) = \frac{\cos. a \cos. \theta}{\cos. b}$$

whence c may be found, θ being previously determined from the equation.

$$\tan. \theta = \tan. b \cos. A.$$

In like manner, in case IV, when two angles and the side opposite to one of them were given, we first determined the side opposite to the other angle, then the remaining side and the remaining angle in succession. Now, let A, B, a, be given, and let it be required to determine c and C, independently of b and of each other, and under a form adapted to logarithmic computations. If we take the formula (θ).

$$\cot. a = \cot. A \sin. B \operatorname{cosec}. c + \cos. B \cot. c$$

or $$\cot. a \sin. c = \cot. A \sin. B + \cos. B \cos. c$$

or $$\sin. c = \cot. A \sin. B \tan. a + \cos. B \cos. c \tan. a$$

$$\therefore \quad \sin. c - \cos. c \cos. B \tan. a = \cot. A \sin. B \tan. a.$$

Let $$\cos. B \tan. a = \tan. \theta = \frac{\sin. \theta}{\cos. \theta}$$

$$\sin. c - \frac{\sin. \theta}{\cos. \theta} \cos. c = \cot. A \sin. B \tan a$$

$$\sin. (c - \theta) = \cot. A \sin. B \tan. a \cos. \theta$$

$$= \cot. A \sin. B \tan. a \frac{\sin. \theta}{\cos. B \tan. a}$$

$$= \cot. A \tan. B \sin. \theta$$

whence c may be determined, θ being previously known from equation

$$\tan. \theta = \cos. B \tan. a$$

To find C, we have from (β)

$$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C}$$

$$\therefore \quad \sin. B \sin. C \cos. a = \cos. A + \cos. B \cos. C$$

$$\therefore \quad \sin. C \tan. B \cos. a = \frac{\cos. A}{\cos. B} + \cos. C$$

$$\therefore \sin. C \tan. B \cos. a - \cos. C = \frac{\cos. A}{\cos. B}$$

Let $\tan. B \cos. a = \tan. \theta = \dfrac{\sin. \theta}{\cos. \theta}$

$$\sin. C \times \frac{\sin. \theta}{\cos. \theta} - \cos. C = \frac{\cos. A}{\cos. B}$$

$$\therefore \quad -\cos. (C + \theta) = \frac{\cos. A \cos. \theta}{\cos. B}$$

whence C may be found, θ being known from equation

$$\tan. \theta = \tan. B \cos. a.$$

In the fifth and sixth cases, any one of the angles or sides required, may be found independently of the rest by the formulæ referred to.

Examples in Spherical Trigonometry.

(1.) In the right-angled spherical triangle ABC, the hypothenuse AB is $65°5'$ and the angle A is $48°12'$; find the sides AC, CB, and the angle B.

Ans. $AC = 55° \; 7' \; 32''$
$BC = 42 \; 32 \; 19$
$\angle B = 64 \; 46 \; 14.$

(2.) In the oblique-angled spherical triangle A B C, given $AB = 76° \; 20'$, $BC = 119° \; 17'$, and $\angle B = 52° \; 5'$; to find AC and the angles A and C.

Ans. $AC = 66° \; 5' \; 36''$
$\angle A = 131 \; 10 \; 42$
$\angle C = 56 \; 58 \; 58.$

(3.) In an oblique spherical triangle the three sides are

$$a = 81°17', \; b = 114°3', \; c = 59°12';$$

required the angles A, B, C.

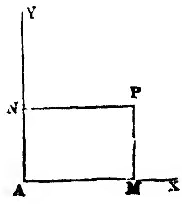
Ans. $A = 62° \; 39' \; 42''$
$B = 124 \; 50 \; 50$
$C = 50 \; 51 \; 42.$

ANALYTICAL GEOMETRY

TWO DIMENSIONS.

If we reflect on the nature of Geometrical Problems, we shall perceive that the greater number of them depend ultimately on finding the distance of one or more unknown points, from other points or straight lines, whose position is already known and determined. If, therefore, we have a method which enables us to determine analytically the position of a point, with reference to certain other points or straight lines whose position is known, we shall be in a state to resolve all kinds of geometrical problems.

Let there be two straight lines AX, AY, whose position is known and determined, situated in the same plane at right angles to each other, and let P be any point in the same plane whose position we are required to determine.



From the point P let fall PM, PN, perpendiculars on AX and AY. Then it is manifest that the point P will be determined, if we know the length of the sides AM, AN, of the rectangle AP. For these sides are the distances of the point P from the two fixed straight lines AX, AY, so that, if we draw from the points M and N two straight lines, respectively parallel to AY and AX, the point where they intersect will be the point required.

The two fixed lines AX, AY are called *Axes.*

The distance AM or PN of the point P from the axis AY is called the *Abscissa* of the point P, and is usually designated algebraically by the letter x.

The distance AN or PM of the point P from the axis AX is called the *Ordinate* of the point P, and is usually designated algebraically by the letter y.

The two distances x and y are together denominated the *Co-ordinates* of the point P.

The two axes are distinguished from each other by calling the axis AX, along which the abscissas are reckoned, *the Axis of Abscissas*, or *the Axis of* x's; and in like manner the axis AY, along which the ordinates are reckoned, is called *the Axis of Ordinates*, or the Axis of y's.

The point A is called the *Origin of Co-ordinates*, since it is from this point that the distances are reckoned.

EQUATIONS OF A POINT.

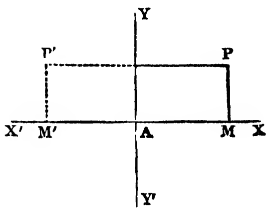
The characteristics of every point situated on the axis of y's is $x = 0$, since that equation indicates that the distance of the point in question from that axis is *nothing*.

Similarly the characteristic of every point situated on the axis of x's is $y = 0$

Hence the system of two equations, $x = 0$, $y = 0$, characterizes the point A the origin of co-ordinates, since these equations can hold good at the same time for no other point.

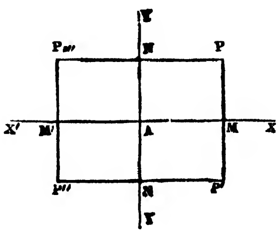
In general the two equations $x = a$, $y = b$, when considered together characterize a point situated at a distance a from the axis of y's, and at a distance b from the axis of x's. The first of these equations, when considered separately, belongs to all the points of a straight line drawn parallel to the axis of y's, at a distance AM $= a$, and the second to all the points of a straight line drawn parallel to the axis of x's, at a distance AN $= b$. Hence the system of two equations together belongs to the point P, in which these lines intersect, and belongs to this point alone. These expressions are thus, as it were, the analytical representations of the point, and for this reason are called the *Equations of the point.*

We must always consider, in the expressions a and b, not only the *absolute* or numerical values of the distances of the point from the two axes, but likewise the *signs* by which they may be affected, according to the position of the point in the plane of the axes AX and AY. For, according to the conventions explained in the first chapter of Analytical Plane Trigonometry, if we agree to consider as *positive*, distances such as AM reckoned along AX *to the right* of the point A, we ought to consider as *negative*, distances such as AM′ reckoned *to the left* of the same point.



In like manner, if we consider as positive, distances such as AN reckoned along AY *upwards* from the point A, we must regard as negative, distances such as AN′ reckoned along AY *downwards* from the point A.

If, then, we exhibit the different signs with which a and b may be affected, we shall have four systems of equations to characterize the four different positions of the point P.



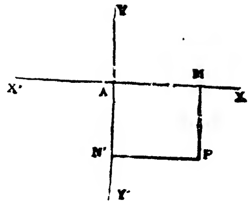
For P we have, ... $x = a, y = b$

P′ .. ... $x = a, y = -b$

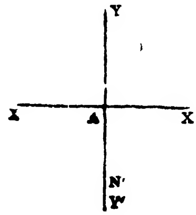
P″ .. ... $x = -a, y = -b$

P‴ .. ... $x = -a$ ·· $-\ b$

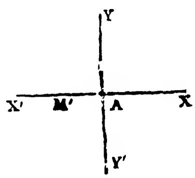
Thus, for example, the point whose equations are $x = +2$, $y = -3$, is situated in the angle XAY′, at the distance AM $= 2$ from the axis of y's, and at the distance AN′ $= 3$ from the axis of x's.



The point whose equations are $x = 0$, $y = -2$, is situated on the axis AY′, at a distance AN′ $= 2$ from the axis of x's

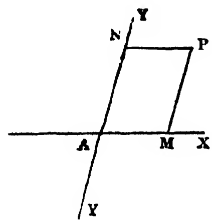


The point whose equation is $x = -4$, $y = 0$, is situated on the axis AX, to the left of A, at a distance AM′ $= 4$.



We have hitherto supposed that the axes are perpendicular to each other, because that position is the most simple, and most frequently employed; however, it is sometimes necessary to consider the axes as inclined at any given angle to each other.

In this case the co-ordinates are no longer straight lines drawn perpendicular to the axes, but are straight lines parallel to these axes; that is to say, the distances PM, PN, are reckoned parallel to AX, AY.



All the other remarks which we have made upon the supposition that the axes were rectangular, apply equally to the case in which they are oblique.

In order to complete our discussion on the equations to a point, let it be required

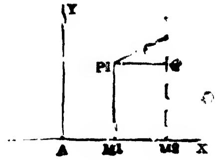
To determine the analytical expression for the distance between two given points which are situated in the same plane.

Let the co-ordinates of the first point P_1 be x', y', and of the second point P_2 be x'', y'', so that the equations to these points, whose positions we suppose known, are

$$\text{Of } P_1 \left\{ \begin{array}{l} x = x' \\ y = y' \end{array} \right\} \ldots\ldots(1) \quad \text{And of } P_2 \left\{ \begin{array}{l} x = x'' \\ y = y'' \end{array} \right\} \ldots\ldots(2)$$

It is required to express the distance $P_1 P_2$ of these points in terms of the given co-ordinates x', y', x'', y''.

Let the distance P_1P_2 be called R.



Draw the ordinates P_1M_1 P_2M_2, of the two points, and through P_1 draw P_1Q parallel to AX.

The right angled triangle P_1QP_2, gives us

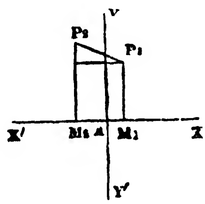
$$P_1P_2^2 \text{ or } R^2 = P_1Q^2 + P_2Q^2 \ldots\ldots\ldots\ldots(A)$$

$$\text{But } P_1Q = M_1M_2 = AM_2 - AM_1 = x'' - x'$$
$$\text{And } P_2Q = P_2M_2 - QM_2 = P_2M_2 - P_1M_1 = y'' - y'$$

Substituting these values of P_1Q and P_2Q, in (A) we have

$$R = \sqrt{(x' - x'')^2 + (y' - y'')^2}$$

This formula is quite general, and will apply equally well to the case in which the two points are situated on different sides of the axes.

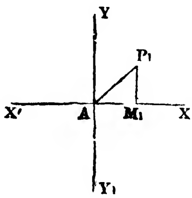


It will only be necessary, in this case, to introduce the changes in the signs which correspond to changes in position; thus, for example, to obtain the distance of two points, one of which is situated in the angle YAX, and the other P_2 in the angle YAX', we must change the sign of x'', which gives us

$$R = \sqrt{(x' + x'')^2 + (y' - y'')^2}$$

In fact, if we perform the calculations as in the former case, we find

$$P_2P_1 = P_2Q^2 + P_1Q^2$$
$$P_2Q = (x' + x'')$$
$$P_1Q = (y' - y'')$$
$$\text{And } \therefore \quad R = \sqrt{(x' + x'')^2 + (y' - y'')^2}$$



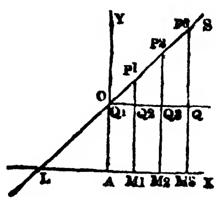
If one of the points, P_2 for example, is the origin of co-ordinates, in that case $x'' = 0$, $y'' = 0$, and the formula becomes

$$R = \sqrt{x'^2 + y'^2}$$

For here we have

$$AP_1^2 = AM_1^2 + P_1M_1^2$$
$$i.\,e. \quad R = \sqrt{x'^2 + y'^2}$$

TO FIND THE EQUATION TO A STRAIGHT LINE.



Let LOS be a straight line of indefinite length, and situated in a plane.

Draw in this plane two axes, AX and AY, at right angles to each other, and let the situation of the straight line with regard to these axes be any whatever.

In the straight line take any points P_1, P_2,

P_3, ... and from these points draw P_1M_1, P_2M_2, P_3M_3, ... perpendicular on AX, and through the point O, in which the straight line meets the axis AY, draw OQ parallel to AX.

The similar triangles P_1Q_1O, P_2Q_2O, P_3Q_3O, will give a series of equal ratios.

$$\frac{P_1Q_1}{Q_1O} = \frac{P_2Q_2}{Q_2O} = \frac{P_3Q_3}{Q_3O}, \text{ or, since } AO = Q_1M_1 = Q_2M_2 = \&c.$$

$$\frac{P_1M_1 - AO}{AM_1} = \frac{P_2M_2 - AO}{AM_2} = \frac{P_3M_3 - AO}{AM_3} = \&c.$$

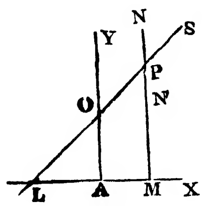
Which proves that *the difference between the ordinate of any point in a straight line, and the distance of the straight line from the origin, is in a constant ratio to the abscissa of the same point.*

Let us then call the co-ordinates of *any* point in the straight line x and y, and let us designate by b the distance AO; that is, the distance from the origin of the point in which the straight line cuts AY; let a be the constant ratio which we have just mentioned, we shall then have the relation.

$$\frac{y-b}{x} = a$$

Or, $$y = ax + b \dots\dots\dots\dots(1)$$

Now this relation holds good, as has been shown above, for every point in the straight line LOS, but it will not hold good for any point which is not situated in this straight line.



For let N be any other point taken either above or below the straight line LOS.

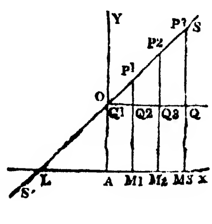
Now, since the ordinate NPM of that point is either greater or less than the ordinate PM of the straight line corresponding to the same abscissa AM, and since by hypothesis we have for the point P the relation

$$PM = a \,.\, AM + b$$

it follows that since NM is either $>$ or $<$ PM, we have for the co-ordinates of the point N,

$$y \gtrless a.\ AM + b$$

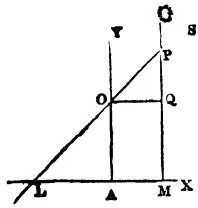
We thus perceive that the relation (1) is characteristic of every point in the straight line LOS, and that it does not hold good for any point without that line, and is therefore *the analytical representation* of that straight line; for if this relation be given in the first instance, we are enabled, by means of it, to determine the position of the straight line, and to trace it graphically.



For this purpose it is sufficient to give to x a series of values, which we measure along AX, such as AM_1, AM_2, &c. and drawing from these points straight lines M_1P_1, M_2P_2, ... parallel to AY, and making these straight lines equal to the corresponding values of y, found from equation (1), we shall in this manner determine the points P_1, P_2, ... situated in the required straight line.

For this reason the relation (1) is denominated *the Equation to the Straight Line* LOS.

The quantities x and y, which represent the coordinates of the different points in the straight line, are called the *Variables* in the equation, and the quantities a and b, which do not change for the same straight line, are called the *Constants* in the equation.



The constant b, it has been already shown, is the distance from the origin of the point in which the straight line cuts the line AY, or, this is the ordinate of the straight line at the origin; it remains, therefore, to examine the constant ratio which is expressed in the equation by a, now

$$a = \frac{PQ}{QO} = \frac{PM}{ML} = \tan. PLM.$$

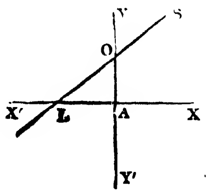
Thus, it appears, that the constant ratio, is the trigonometrical tangent of the angle which the straight line makes with the axis AX.

DISCUSSION OF THE EQUATION TO A STRAIGHT LINE.

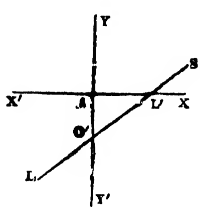
$$y = a\,x + b \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.)$$

The above equation we have seen, is the representation of a straight line, b being the distance of the point in which it cuts the axis AY from the origin, and a being the trigonometrical tangent of the angle which the straight line makes with the axis of x's. Now, the quantities a and b, are fixed and determined for all points of the same straight line, but when we come to consider different straight lines; we shall find that they are distinguished from each other, by the different values which the quantities a and b receive in the equations of each. For it is evident from the nature of the quantities a and b, that they are susceptible of all degrees of magnitude, since the first is a trigonometrical tangent, and the second expresses the distance of a fixed point A, from a point in the indefinite line YAY'.

Let us first consider the changes which may take place in b.



In deducing the equation $y = a\,x + b$, we supposed that the straight line intersected the axis AY in some point O, above A.



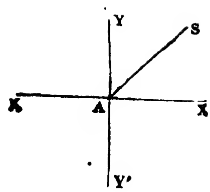
But, if we suppose the straight line in question to intersect the axis AY in some point, O' situated below A; then, from what has been said with regard to the signs of these quantities, it appears that b will have the negative sign, and consequently the equation to the straight line, will, in this case, become

$$y = ax - b \ldots\ldots \qquad (2)$$

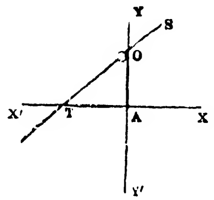
Again, let us suppose that the straight line passes through the origin A, then the distance from the origin of the point in which the line cuts YAY′, is O and $\therefore b = 0$.

The equation, therefore, to a straight line which passes through the origin, is,

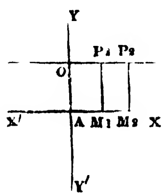
$$y = ax \text{ .... } (3)$$



Let us now consider the different portions of the straight line which will correspond with a change in a.



In determining the equation, we supposed the position of the straight line to be that represented in the figure, and that a was the tangent of the angle OTA.



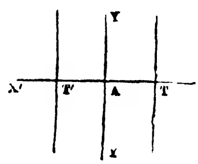
Now, suppose that the straight line is parallel to AX, then it is manifest that for all values of x, AM_1, AM_2, which we may assume; the value of y will always remain the same and be equal to AO; hence in this case, the equation assumes the form

$$y = b \text{ .... } (4)$$

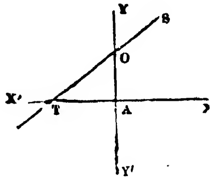
which therefore, represents a straight line drawn parallel to the axis AX, at a distance b from the origin.

Similarly, if the straight line be parallel to the axis AY, it is evident that for all values of y, the value of x will always remain the same; and hence the equation to a straight line in this position, will be

$$x = c \text{ .... } (5)$$

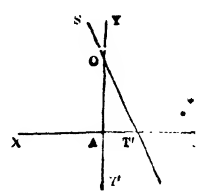


which is therefore, the equation to a straight line parallel to the axis AY, and which cuts the axis AX at a distance c from the origin.



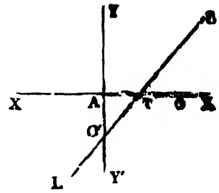
In deducing the equa $= ax + b$, we supposed the straight line to make an acute angle OTA, of which the tangent is a and positive; if however, we suppose the straight line to revolve until it comes into the position OT′; then it makes an angle OT′X with AX, which is greater than 90° and less than 180°, and whose tangent is consequently negative, hence the equation of a straight line in this position, is

$$y = -ax + b \text{ .... } (6)$$



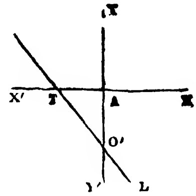
If we suppose the position of the straight line to be that represented in the adjacent figure, since the angle which it makes with AX is $>$ 180° and $<$ 270°, a is positive, and the equation is

$$y = ax - b$$



Lastly, the equation to a straight line in the position represented in the figure, is

$$y = - ax - b.$$



Thus we may have the following equations to a straight line, according to the different positions which it may assume.

$$y = ax.$$
$$y = - ax.$$
$$y = b.$$
$$x = c.$$
$$y = - ax + b.$$
$$y = ax - b.$$
$$y = - ax - b.$$
$$y = ax + b.$$

PROBLEMS CONNECTED WITH THE EQUATION TO A STRAIGHT LINE.

PROB. I.

To find the equation to a straight line which passes through a given point.

Let x', y', be the co-ordinates of the given point.
Let x, y, be the co-ordinates of any other point in the line.
Then the general equation to the straight line will be,

$$y = ax + b \text{ .... } (1)$$

And, since x', y', are points in this straight line, it must satisfy the equation.

$$y' = ax' + b \text{ .... } (2)$$

$\therefore$ Subtracting (2) from (1) we have,

$$y - y' = a (x - x')$$

which is the equation required.

PROB. II.

To find the equation to a straight line which passes through two given points.

Let the co-ordinates of the given points be, x', y' and x'', y''; and let x and y be the co-ordinates of any other point whatever in the straight line.

Then, in general, the equation to the straight line will be,

$$y = ax + b \quad \ldots\ldots (1)$$

But, since x', y', and x'', y'', are points in the straight line, it must satisfy the equations.

$$y' = ax' + b \quad \ldots\ldots (2)$$
$$y'' = ax'' + b \quad \ldots\ldots (3)$$

Subtract (3) from (2).

$$y' - y'' = a\,(x' - x'')$$

whence $$a = \frac{y' - y''}{x' - x''} \quad \ldots\ldots (4)$$

Again, subtract (2) from (1).

$$y - y' = a\,(x - x').$$

Substitute in this equation the value of a, obtained from (4), and we have

$$y - y' = \frac{y' - y''}{x' - x''}\,(x - x')$$

which is the equation required.

PROB. III.

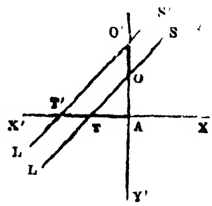
To find the equation to a straight line, parallel to a given straight line.

Let the equation to the given straight line ST, be

$$y = ax + b$$

where a is the tangent of the angle OTA.

Now, since the straight line S′T′, whose equation is required, is parallel to the given straight line; its inclination to the axis AX is the same, and therefore its equation will differ from that of the given straight line only in the quantity b, which expresses the distance of the origin from the point in which it cuts the axis AY.



The required equation will therefore, be

$$y = ax + b'$$

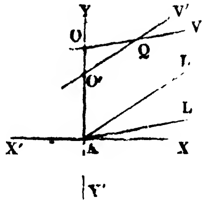
where $b' = $ AO′.

PROB. IV.

To find the tangent of the angle between two given straight lines, which intersect each other.

Let OV, O′V′, be the given straight lines intersecting the point Q, and let their respective equations be

$$y = ax + b$$
$$y' = a'x + b'$$



Draw through the origin two straight lines, AL, AL, respectively parallel to OV, O′V′; then it is manifest that the angle LAL′ = angle VQV′, and the equations to these two straight lines will be

$$y = ax$$
$$y = a'x.$$

Now the angle $\text{LAL}' = \angle \text{LAX} - \angle \text{L}'\text{AX}$

$\therefore \quad \tan.\ \text{LAL}' = \tan.\ (\text{LAX} - \text{L}'\text{AX})$

$$= \frac{\tan.\ \text{LAX} - \tan\ \text{L}'\text{AX}}{1 + \tan.\ \text{LAX}\ \tan.\ \text{L}'\text{AX}}$$

$$= \frac{a - a'}{1 + aa'}.$$

If the straight lines be parallel $\quad \frac{a - a'}{1 + aa'} = 0$ and $\therefore a = a'$

If the straight lines be at right angles $\frac{a - a'}{1 + aa'} = \infty$

$$\therefore \quad 1 + aa' = 0$$

$$\text{or} \quad a' = -\frac{1}{a}$$

Hence it appears, that if the equation to a straight line be

$$y = ax + b$$

the equation to a straight line perpendicular to it, will be

$$y = -\frac{1}{a}x + b.$$

PROB. V.

To find the equation to a straight line drawn through a given point, perpendicular to a given straight line.

Let the equation to the given straight line be

$$y = ax + b$$

Let the co-ordinates of the given point be x', y'.

Let the required equation be of the form

$$y = \text{A}x + \text{B}$$

where A and B are unknown.

Then, since the straight line $y = \text{A}x + \text{B}$, passes through the point x', y', its equation will be

$$y - y' = \text{A}(x - x')$$

and since it is perpendicular to the straight line whose equation is

$$y = ax + b$$

$$\therefore \quad \text{A} = -\frac{1}{a} \text{ by last problem.}$$

$$\therefore \quad y - y' = -\frac{1}{a}(x - x')$$

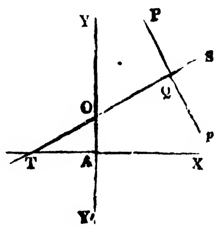
is the equation required.

PROB. VI.

To find the length of a straight line drawn from a given point, perpendicular to a given straight line.

Let the equation to the given straight line TS be

$y = ax + b$ .... (1)



Let the co-ordinates of the given point P be x', y'.

Then, since PQp is drawn through a point, x' y' perpendicular to a straight line, whose equation is $y = ax + b$; by the last Prob. the equation to Pp is

$y - y' = -\frac{1}{a}(x - x')$ .... (2)

Now, to obtain the length of PQ which is required, we must find the co-ordinates of the point Q in which Pp meets TS, and then substitute x' y' and the co-ordinates of Q in the general expression for the distance of the points, viz.

$$D = \sqrt{(x' - x'')^2 + (y' - y'')^2} \quad \ldots\ldots \quad (3)$$

Let us call the co-ordinates of the point Q, x'' y''; then, since Q is a point in the straight line TS, that straight line must satisfy the equation

$$y'' = ax'' + b$$

For the sake of convenience let us put this equation under the form

$$y'' - y' = a(x'' - x') - y' + ax' + b \quad \ldots\ldots \quad (4)$$

which is done by subtracting y' from each side of the equation, and adding ax' to, and subtracting it from the right hand side.

But, since x'' y'' is a point in the straight line Pp, also that straight line whose equation we have found (2), must satisfy the equation

$$y'' - y' = -\frac{1}{a}(x'' - x') \quad \ldots\ldots \quad (5)$$

Now for the point Q, equations (4) and (5) hold good together, therefore subtracting, we find

$$0 = \frac{1}{a}(x'' - x') - y' + ax' + b + a(x'' - x')$$

whence $x'' - x' = a\dfrac{y' - ax' - b}{1 + a^2}$

Substitute this value of $x'' - x'$ in equation (5), and we have

$$y'' - y' = -\frac{y' - ax' - b}{1 +}$$

Substituting these values of $x'' - x$ and $y'' - y'$ in the general expression (3), for the distance of two points

$$D = \sqrt{a^2\left(\frac{y' - ax' - b}{1 + a^2}\right)^2 + \left(\frac{y' - ax' - b}{1 + a^2}\right)^2}$$

$$= \pm\frac{y' - ax' - b}{1 + a^2} \cdot \sqrt{1 + a^2}$$

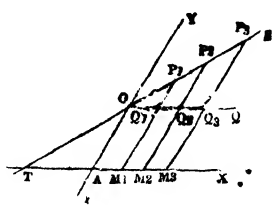
$$= \pm\frac{y - ax' - b}{\sqrt{1 + a^2}}$$

which is the length of the perpendicular required.

PROB. VII.

To find the equation to a straight line referred to obliq e axes.

Let SOT be a straight line of indefinite length situated in a plane.



Draw in this plane the axes AX, AY, inclined to each other at any given angle, and let the situation of the straight line with regard to these be any whatever.

In the straight line take any points, P_1, P_2, P_3, ... from these points draw P_1M_1, P_2M_2, P_3M_3, ... parallel to AY, and

through the point O in which the straight line meets the axis AY, draw OQ parallel to AX.

The similar triangles P_1Q_1O, P_2Q_2O, P_3Q_3O, ... will give a series of equal ratios as in the case of rectangular co-ordinates.

$$\frac{P_1Q_1}{Q_1O} = \frac{P_2Q_2}{Q_2O} = \frac{P_3Q_3}{Q_3O}, \text{ or, since } AO = Q_1M_1 = Q_2M_2 = \&c.$$

$$\frac{P_1M_1 - AO}{AM_1} = \frac{P_2M_2 - AO}{AM_2} = \frac{P_3M_3 - AO}{AM_3} = \&c.$$

Which proves, as in the former case, that the difference between the ordinate of any point in the straight line, and the distance of the straight line from the origin, is in a constant ratio to the abscissa of the same point.

Let us call x and y the co-ordinates of *any* point in the straight line, and let us designate by b the distance AO; that is, the distance from the origin of the point in which the straight line cuts AY; let a be the constant ratio which we have just mentioned, we shall then have the relations

$$\frac{y-b}{x} = a$$

or

$$y = ax + b.$$

In this equation b, as in the case of rectangular co-ordinates, expresses the distance from the origin of the point in which the given straight line cuts the axis AY.

Let us now examine the constant ratio a.

$$a = \frac{P_1Q_1}{Q_1O} = \frac{OA}{AT} = \frac{\sin. OTA}{\sin. TOA}$$

Thus it appears that the constant ratio a, is the ratio of the sines of the angles which the given straight line makes with the axes AX, AY, respectively.

Hence the general form of the equation to a straight line, whether it be referred to rectangular or oblique co-ordinates, is

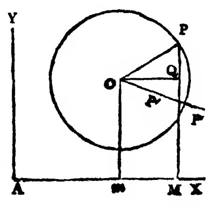
$$y = ax + b$$

observing that in the former case, a represents the trigonometrical tangent of the angle which the given straight line makes with the axis AX; and in the latter case, a represents the ratio of the sines of the angles which the given straight line makes with the axes AX, AY, respectively.

In both cases, b represents the distance from the origin of the point in which the given straight line cuts the axis AY.

PROB. VIII.

To find the equation to a circle.



Let PQ be a circle whose centre is O, and whose radius is OP.

Draw the axes AX, AY, at right angles to each other.

Let the co-ordinates of the point O be x' y', and of any point P in the circumference, x, y.

Then the expression for the distance of the two points O and P, whose co-ordinates are x' y' and x y, is

$$OP^2 \text{ or } R^2 = (x - x')^2 + (y - y')^2 \quad \text{.................................} \quad (1)$$

This relation characterizes all the points in the circumference, inasmuch as it is evidently satisfied by the co-ordinates of each of these points, and can be satisfied by these only.

For example, let P′ be any point taken either within or without the circumference, calling x_1 and y_1 the co-ordinates of that point, we have

$$OP'^2 = (x_1 - x')^2 + (y_1 - y')^2.$$

But OP′ is evidently $>$ OP when P′ is without the circle and $<$ OP when P′ is within the circle, whence we have

$$OP'^2 > \text{ or } < (x - x')^2 + (y - y')^2$$

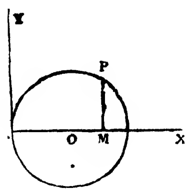
Hence the equation (1) cannot be verified for any point which is not on the circumference of the circle.

This equation then, is, *the Equation to the Circle.*

The constant quantities x', y', r, which enter into this equation are the co-ordinates of the centre and of the radius; and we know, that, when the centre of a circle is given, and the length of its radius, the magnitude of the circle is completely determined.

The above equation (1) assumes a form more or less simple according to the position of the point which we assume as the origin of co-ordinates.

1. Let us assume some point A in the circumference as the origin of co-ordinates, and let the axis of x's be a diameter.



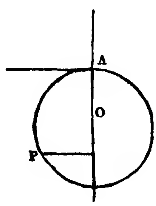
In this case, since the centre is situated on the axis AX, $y' = 0$ and $x' = r$, therefore the equation

$$r^2 = (x - x')^2 + (y - y')^2$$

becomes $r^2 = x^2 - 2rx + r^2 + y^2$

or $\quad y^2 = 2rx - x^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

2. If we assume one of the diameters as the axis of y', as in the annexed figure, we shall have



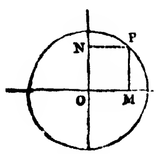
$$x' = 0, \qquad y' = r$$

and equation (1) becomes

$$r^2 = x^2 + y^2 - 2ry + r^2$$

or $\quad x^2 = 2ry - y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

3. If we make the centre of the circle the origin of co-ordinates, then



$$x' = 0, \qquad y' = 0$$

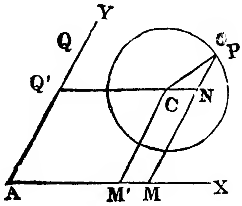
and equation (1) becomes

$$r^2 = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

It may be remarked, that equations (2) and (4) are those which are most generally employed.

Prob. ix.—*To find the equation to the circle, the axes of co-ordinates being inclined at any angle.*

Let the straight lines AX, AY, which are inclined to each other at a given angle ϕ, be assumed as axes.



Take P any point in the circle, and let the co-ordinates of P be called x and y.

Let C be the centre of the circle, and let the co-ordinates of the point C be x' y'.

Draw PM, CM′, parallel to AY; and PQ. CQ′ parallel to AX; produce Q′C to meet PM in N; join C, P;

$$CP = r,\ AM = x,\ MP = y,\ AM' = x',\ CM' = y',\ \angle e\ YAX = \phi$$

$$CP^2 = NP^2 + CN^2 - 2NP \,.\, CN \cos. CNP.$$

Now $NP = MP - MN = MP - CM' \quad = y - y'$

$CN = MM' \quad = AM - AM' \quad = x - x'$

$\angle e\ CNP = \angle e\ PMA \quad = (180^\circ - YAX) = (180^\circ - \phi).$

Hence the above equation becomes

$$r^2 = (y - y')^2 + (x - x')^2 + 2(x - x')(y - y')\cos.\phi$$

which is the equation required.

Before proceeding farther with this subject, it may not be improper to make some general observations on the nature of equations to lines, and on the use which we may make of them.

We have seen that the position of a straight line and of a circle is fixed upon a plane by means of an equation between the co-ordinates of each of its points, and a certain number of constant quantities; the knowledge of which enables us to determine its position geometrically.

Suppose then, that x and y being considered to denote the distance of a point from two rectangular or oblique axes, the resolution of some problem has led to an equation of the form

$$f(x, y) = 0.$$

If we wish to fix the position of the point which verifies this equation, we shall find that there are an infinite number of such points, and that this series of points constitutes a line which is either straight or curved according to the nature of the equation. For since there is only *one* equation between x and y we may give any value we please to one of these variables, and then the equation will give the corresponding value of the other variable.

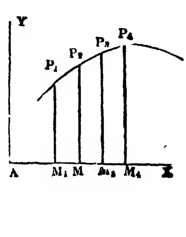
Let us, for example, give the abscissa x the series of values

$$x = a_1, a_2, a_3, a_4, \ldots .$$

If y enters in the equation in the simple power only, we shall derive from the equation a succession of corresponding values of that variable;

$$y = b_1, b_2, b_3, b_4 \ldots .$$

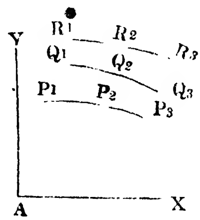
If in AX we take AM_1, AM_2, AM_3, AM_4, ... equal to a_1, a_2, a_3, a_4, ... and from the points M_1, M_2, M_3, M_4, ... raise perpendiculars M_1P_1, M_2P_2, M_3P_3, M_4P_4, ... equal respectively to b_1, b_2, b_3, b_4, ... we shall find a series of points P_1, P_2, P_3, P_4; all of which will equally satisfy the conditions of the equation.



Now, since we may give to x a series of values

which differ very little from each other, and since, in that case, the successive values of y will, generally speaking, likewise differ very little from each other; the points P_1, P_2, P_3, P_4, ... will be very near to each other, and we shall be thus enabled to unite all these points with each other by means of a continuous line, P_1, P_2, P_3, P_4, ... *all the points of which* will be so many solutions of the question, since all the intermediate points of this line comprised between those constructed in the manner described, may be supposed to correspond to the values of x and y derived from the equation of the problem.

The form of the curve will be determined with more accurate precision in proportion, as the points P_1, P_2, P_3, P_4, ... are nearer to each other.



Let us now suppose that y enters into the proposed equation in some power higher than the first. Since in this case, for each value of x, there will be two or more corresponding values of y, according to the degree of the equation; it follows that the curve will be composed of two or more branches, P_1, P_2, P_3, ... Q_1, Q_2, Q_3, ... R_1, R_2, R_3, ...

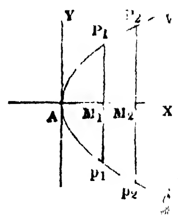
Let it be required, for example, to construct the curve whose equation is

$$y^2 = 2x$$

Solving the equation for y, we have

$$y = \pm\sqrt{2x}.$$

Which proves, in the first place, that for each value of x there are two corresponding values of y equal to each other, but with opposite signs; and in the second place, that for all negative values of x the corresponding values of y are imaginary, that is to say, that the curve can have no point situated to the left of the origin AY.



This being established, let us make $x = 0$, then $y = 0$, which shows that the origin of co-ordinates is placed on some point in the curve; or, in other words, that the curve passes through the origin.

Next, let $x = 1 \therefore y = \pm\sqrt{2} = \pm 1.4$ . . . .

Take therefore upon AX a distance AM_1 = the linear unit; and from M_1 draw a perpendicular to AX, and on each side of AX take M_1P_1, M_1p_1 = 1.4 . . . .; then P and p will be two points in the required curve.

Next, let $x = 2 \therefore y = \pm\sqrt{4} = \pm 2$

Constructing these values as before, we shall find P_2, and p_2, for two new points in the curve.

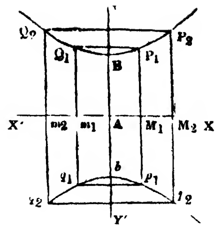
Continuing in this manner to give a succession of values to x, and constructing the corresponding value of y, we shall obtain a curve of the form VAv, which consists of two branches AV, and Av, which extend indefinitely to the *right* of AY, since for all positive values of x, the corresponding values of y are real.

For a second example, let us take the equation

$$y^2 - x^2 = 4$$

whence $$y = \pm\sqrt{4 + x^2}$$

We perceive, in the first place, that, for the same value x, there are two equal values of y with contrary signs; and, in the second place, that, whatever value



we giv to x, whether positive or negative, we shall always obtain real values for y. Hence we can conclude at once, that the curve extends indefinitely both above and below the axis AP, and both to the right and left of AY. Let us now make some particular suppositions.

$$\text{Let } x = 0 \therefore y = \pm\sqrt{4} = \pm 2.$$

Take on AY two distances AB, Ab, each $= 2$, the points B and b belong to the curve.

$$\text{Next, let } x = 1 \therefore y = \pm\sqrt{5} = \pm 2.2\ldots.$$

On AX take $AM_1 = 1$ through M_1 draw a straight line $P_1 p_1$ parallel to AY, and make $M_1 P_1 = M_1 p_1 = 2.2 \ldots.$ then P_1 and p_1 are two new points in the curve,

$$\text{Let } x = 2 \therefore y = \pm\sqrt{8}, = \pm 2.8 \ldots.$$

Constructing this value of y in the same manner, we obtain P_2 and p_2 for two other points in the curve, and so on for the other points to the right of AY.

In order to obtain the points to the left of AY, since the values of x, which are numerically the same but taken with different signs, correspond to the same values of y; it will be sufficient to take Am_1, Am_2, ... equal to AM_1, AM_2, ... and through the points m_1, m_2, ... to draw straight lines parallel to AY, and through the points P_1, p_1, P_2, p_2, straight lines parallel to AX, and we shall thus determine the points Q_1, q_1, Q_2, q_2, ... belonging to the curve, which will evidently be composed of two branches distinct and opposite, P_2BQ_2 $p_2\,b\,q_2$

The curve represented by an equation between x and y, is called *the Geometrical Locus* of the equation.

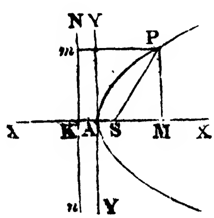
Reciprocally, if a curve be traced upon a plane, and if by any means founded upon the definition or upon some characteristic property of the curve, we can arrive at a relation which exists between the co-ordinates x and y of all points in that curve, and exists for these points alone; the relation thus obtained is called the *Equation to the Curve.*

We shall now proceed in this manner to obtain the equations to the most important curves

PROB. X.

To find the equation to the parabola.

DEFINITION.—A Parabola is the locus of a point whose distance from a given fixed point, and from a straight line given in position, is always the same.



Let S be the given fixed point, and Nn the straight line given in position;

Draw SK perpendicular to Nn, and bisect SK in A;

Then by definition A is a point in the parabola.

Take P any point in the curve and join S, P;

From P draw Pm perpendicular to Nn;

From A draw AY perpendicular to ASX.

Let A be the origin and AX, AY, the axes of coordinates.

From P draw PM perpendicular on AX.

Then, let $AM = x$, $PM = y$, $SP = r$, $AS = m$.

Then we have

$$\begin{aligned} r^2 &= SP^2 \\ &= Pm^2 \text{ by definition} \\ &= (AK + AM)^2 \\ &= (m + x)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1) \end{aligned}$$

Again $$\begin{aligned} r^2 &= PM^2 + SM^2 \\ &= PM^2 + (AM - AS)^2 \\ &= y^2 + (x - m)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \end{aligned}$$

Equating these two values of r^2 we obtain a relation between x and y.

$$y^2 + (x - m)^2 = (x + m)^2$$

or, $$y^2 = 4mx$$

which is the equation to the parabola.

In order to find the value of the ordinate passing through the focus

Let $x = m \therefore y^2 = 4m^2$

$\therefore y = \pm 2m$

which shows that $4m$ is the double ordinate passing through the focus, or the *Latus Rectum* of the parabola.

Solving the equation for y

$$y = \pm 2\sqrt{mx}$$

For all negative values of x, y is impossible, which shows that there is no point of the curve to the left of the origin A.

When $x = 0$, then $y = 0$ also.

Which shows that the curve passes through the origin, as is evident from other considerations.

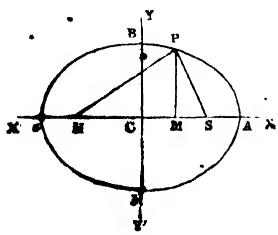
Giving a succession of positive values to x, we perceive that as x increases, y increases also, and that for each value of x there will be two equal values of y with opposite signs.

Hence the curve extends indefinitely to the right of A, and is symmetrically situated with regard to AX.

PROB. XI.

To find the equation to the Ellipse.

Definition.—An Ellipse is the locus of a point, whose distance from two given fixed points is equal to a constant quantity.



Let S and H be the two given fixed points;

Join S, H; bisect SH in C;

Let P be any point in the curve, join S, P; H, P;

Draw PM perpendicular to CX;

Draw CY perpendicular to HS, let C be the origin, and CX, CY, the axes of co-ordinates.

Let the quantity, to which the sum of SP and HP is always equal, be $2a$.

Then, let CM $= x$, MP $= y$, SP $= r$, HP $= r'$, SH $= 2c$.

Then we have

$$r^2 = y^2 + (c - x)^2 \text{ ......} (1)$$
$$r'^2 = y^2 + (c + x)^2 \text{ ......} (2)$$
$$r + r' = 2a \text{ ......} (3)$$

If, therefore, we eliminate r and r' between these three equations, we shall obtain a relation between x and y, which will be the required equation to the curve.

Subtracting (1) from (2)

$$r'^2 - r^2 = 4cx$$

or, $(r' + r)(r' - r) = 4cx$

or $r' - r = \frac{2cx}{a} \quad \because r + r' = 2a$

and $r + r' = 2a$

$\therefore$ adding and subtracting

$$r' = a + \frac{cx}{a} \quad \therefore r'^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$
$$r = a - \frac{cx}{a} \quad r^2 = a^2 - 2cx + \frac{c^2x^2}{a^2}$$
$$\therefore \quad r^2 + r'^2 = 2a^2 + \frac{2c^2x^2}{a^2}$$

again, adding (1) and (2),

$$r^2 + r'^2 = 2y^2 + 2c^2 + 2x^2.$$

Equating these values of $r^2 + r'^2$ we obtain

$$2y^2 + 2c^2 + 2x^2 = 2a^2 + \frac{2c^2x^2}{a^2}$$

or, $a^2y^2 + (a^2 - c^2)\, x^2 = a^2(a^2 - c^2)$.

Since r and r' or $2a$ is always $>$ SH or $2c$, $\therefore$ a is always $> c$, and $\therefore$ the quantity $a^2 - c^2$ is essentially positive.

Let $a^2 - c^2 = b^2$

Then $a^2y^2 + b^2x^2 = a^2b^2$ ...... (A)

which is the most simple form of the equation to the ellipse.

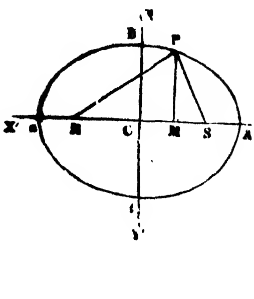
Solving the equation for y and x in succession, we obtain

$$\left.\begin{aligned} y &= \pm \frac{b}{a}\sqrt{a^2 - x^2} \\ x &= \pm \frac{a}{b}\sqrt{b^2 - y^2} \end{aligned}\right\} \text{ ......} (B)$$

When $y = 0 \; x = \pm a$

.... $x = 0 \; y = \pm b$.

Hence it appears that the curve cuts the axis of x's at the points A, a, where CA $=$ Ca $= a$, and cuts the axis of y's at the points B, b, where CB $=$ Cb $= \sqrt{a^2 - c^2} = b$.



Hence it appears that the quantities a and b in the equation (A), are the semi-major and semi-minor axes of the ellipse; and for this reason the equation (A) is called the *Equation to the Ellipse referred to its axes.*

Resuming the equation (B), it is evident that

the curve is situated symmetrically with regard both to the axes CX and CY. For, taking the first of the two equations, we perceive that for each assumed value of x, we shall obtain two equal values of y with opposite signs, which shows that the curve is situated symmetrically with regard to CX. And in the same manner, taking the second equation, we perceive that for each assumed value of y, we shall obtain two equal values of x with opposite signs.

The distance $CS = CH$ is called the *Eccentricity* of the ellipse, and the ratio of c to a is usually denoted by the symbol e.

$$\text{Thus,} \qquad \frac{c}{a} = e$$

$$c = ae$$

$$\therefore \quad c^2 = a^2e^2$$

$$\text{but} \quad a^2 - b^2 = c^2$$

$$\therefore \quad a^2 - b^2 = a^2e^2$$

$$b = \pm a\sqrt{1 - e^2}.$$

It is necessary to observe these equations, since the quantity e is very frequently introduced in calculations where the equation to the ellipse is employed

PROB. XII.

To find the equation to the ellipse, referred to the vertex as origin.

In order to transport the origin from C to a, since the new origin a is situated in the old axis of x's at a distance $= -a$, we have only to substitute* $x - a$ for x in equation (A), which then becomes

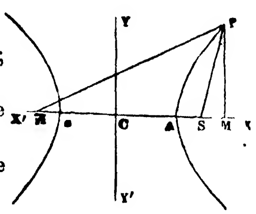
$$y^2 = \frac{b^2}{a^2}(2ax - x^2) \quad \text{(C)}$$

which is the equation required.

PROB. XIII.

To find the equation to the Hyperbola.

DEFINITION.—The Hyperbola is the locus of a point, the difference of whose distance from two given fixed points is equal to a constant quantity.



Let S and H be the two given fixed points.

Join S, H; and bisect SH in C.

Let P be any point in the curve, join S, P; H, P;

Draw PM perpendicular to CX.

Draw CY perpendicular to CX, and let C be the origin, and CX, CY, the axes of co-ordinates.

Let the constant quantity to which the difference of SP and HP is always equal be $2a$.

Let $CM = x$, $MP = y$, $SP = r$, $HP = r'$, $SH = 2c$

$$\text{Then } r^2 = y^2 + (x - c)^2 \quad \text{(1)}$$

$$r'^2 = y^2 + (x + c)^2 \quad \text{(2)}$$

$$r' - r = 2a \quad \text{(3).}$$

If we eliminate r and r' between these three equations, we shall arrive at an equation between x and y, which will be the equation to the curve.

* See chapter on the "Transformation of Co-ordinates."

Subtract (1) from (2);

$$r'^2 - r^2 = 4cx$$

or, $$(r' + r)(r' - r) = 4cx$$

$\therefore$ $$r' + r = \frac{2cx}{a} \therefore r' - r = 2a$$

and $$r' - r = 2a$$

$\therefore$ $$r = \frac{cx}{a} - a \therefore r^2 = \frac{c^2x^2}{a^2} - 2cx + a^2$$

$$r' = \frac{cx}{a} + a \quad r'^2 = \frac{c^2x^2}{a^2} + 2cx + a^2$$

$$r'^2 + r^2 = \frac{2c^2x^2}{a^2} + 2a^2$$

But, adding (1) and (2)

$$r'^2 + r^2 = 2y^2 + 2x^2 + 2c^2$$

Equating these equal values of $r'^2 + r^2$

$$y^2 + x^2 + c^2 = a^2 + \frac{c^2x^2}{a^2}$$

or, $$a^2y^2 + (a^2 - c^2)\,x^2 = a^2\,(a^2 - c^2).$$

Now $2a$ must always be $\angle$ $2c$, and therefore a always $\angle$ c.

Hence $a^2 - c^2$ is essentially negative.

Assuming therefore $a^2 - c^2 = -b^2$; the above equation becomes

$$a^2y^2 - b^2x^2 = -a^2b^2 \quad \text{.........................} \quad (D)$$

which is the most simple form of the equation to the hyperbola.

Solving the equation for y and x in succession, we obtain

$$\left.\begin{aligned} y &= \pm \frac{b}{a}\sqrt{(x^2 - a^2)} \\ x &= \pm \frac{a}{b}\sqrt{y^2 + b^2} \end{aligned}\right\} \text{.........} \quad (E)$$

Let $y = 0 \therefore x = \pm a$.

From which it appears that the curve cuts the axis of x, at the points A, a; where CA $=$ Ca $= a$.

Let $x = 0 \therefore y = \pm b\sqrt{-1}$, an impossible result.

From which it appears that the curve does not meet the axis of y's.

We may, however, take two points B, b, in this axis on different sides of C, making CB $=$ Cb $= \sqrt{c^2 - a^2}$.

In order to fix the position of these points, from the point A as centre with radius equal to CS $=$ CH, describe a circle which will cut the line YY$'$ in the two points required; for we have

$$\text{CB} = \sqrt{\text{CS}^2 - \text{CA}^2} = \sqrt{c^2 - a^2}.$$

Hence it appears, that the quantities a and b are the semi-major and semi-minor axes of the hyperbola, and hence the equation (D) is called the *equation to the Hyperbola referred to its axes.*

Resuming the equation (E), we perceive that if x be $\angle$ a, the values of y are impossible; and hence we conclude, that there is no point in the curve situated between A and a.

When x is $>$ a, then as x increases y increases also; and for each value of x, there will be two equal values of y with opposite signs.

Hence, it is evident from the equation, that the hyperbola consists of two opposite branches, one extending indefinitely to the right of A, and the other indefinitely to the left of a, and both symmetrically situated with regard to XX$'$.

The distance CS = CH = c, is called the eccentricity of the hyperbola, and the ratio c to a is usually designated by the symbol e; hence we have

$$\frac{c}{a} = e$$

$$c = ae$$

$$c^2 = a^2e^2$$

$$c^2 - a^2 = a^2(e^2 - 1)$$

or, $$b = \pm\, a\sqrt{e^2 - 1}.$$

If we wish to obtain the equation to the hyperbola referred to the vertex A, as the origin of co-ordinates, since this new origin is situated on the axis of x's, at a distance $+ a$ from the former origin; if we substitute $(x + a)$ for x in equation (D), we obtain

$$y^2 = \frac{b^2}{a^2}(2ax + x^2) \text{ .... (F)}$$

which is the equation required.

On the Transformation of Co-ordinates.

When we reflect upon the equations to the straight line and circle, and consider the different forms which these equations assume according to the different positions of these lines with regard to the axes of co-ordinates; we perceive that the same line may be represented by different equations which will be more or less simple, according as the position of the line is more or less simple relatively to the axes, and according as the axes themselves are rectangular or oblique.

Thus, the most general equation to a straight line being

$$y = ax + b.$$

The equation to a straight line passing through the origin, is

$$y = ax;$$

a having in each of the above equations a different signification when the axes are oblique, from that which is attributed to it when the axes are rectangular.

In like manner, the most general equation to a circle when referred to oblique axes, is

$$(x - x')^2 + (y - y')^2 + 2(x - x')(y - y') \cos.\, \theta = r^2$$

which becomes

$$x^2 + y^2 = r^2$$

when the circle is referred to rectangular co-ordinates, and the centre is the origin.

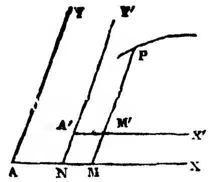
It is easy to conceive, that, when the position of a curve upon a plane is fixed by means of an equation, if we perceive that the position of the curve with regard to two new straight lines, is more simple than with regard to the axes to which it is referred by the equation in question; it would greatly facilitate our investigations respecting the properties of the curve, if we could deduce an equation to the curve, referred to these new straight lines as axes, from that equation to the curve which we actually possess.

Such then is the object of the problem which is proposed in the transformation of co-ordinates, which may be enunciated in its most general terms, as follows:—

Given an equation to a curve referred to any two axes whatever, to find the equation to the same curve when referred to two new axes.

Before however proceeding to solve the problem in its most general form, we shall consider one or two particular cases which are of most frequent occurrence.

I. *Let the new axes be parallel to the former ones.*



Let AX, AY, be the original axes;

Let AX′, AY′, be the new axes parallel to the former.

Let x, y, be the co-ordinates of a point P, referred to the old axes;

Let x', y', be the co-ordinates of a point P, referred to the new axes.

Let α, β, be the co-ordinates of the new origin A′.

Draw PM and A′N, parallel to AY.

Then, AM $= x$, MP $= y$, A′M′ $= x'$, PM′ $= y'$, AN $= \alpha$, A′N $= \beta$.

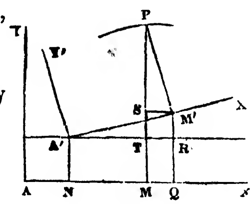
$$\text{AM or } x = \text{AN} + \text{NM}$$
$$= \text{AN} + \text{A'M'}$$
$$= \alpha + x' \quad \ldots\ldots\ldots\ldots \quad (1)$$
$$\text{MP or } y = \text{MM'} + \text{M'P}$$
$$= \beta + y' \quad \ldots\ldots\ldots\ldots \quad (2)$$

If therefore, in the equation to the proposed curve, we substitute $x' + \alpha$ for x, and $y' + \beta$ for y; we shall obtain a relation between x' and y', which will be the equation to the curve referred to the new axes.

Cor. If the new origin be on the axis AX, then $\beta = 0$:

If the new origin be on the axis AY, then $\alpha = 0$.

II. *To pass from one system of rectangular axes, to another also rectangular*



Let AX, AY, be the original system, and AX′, AY′, the new;

Let the notation be the same as in the last case;

Let the inclination of A′X′ to AX, be denoted by the symbol (xx').

Then, AM or x = AN + NM

$$= \text{AN} + \text{NQ} - \text{MQ}$$
$$= \text{AN} + \text{A'R} - \text{M'S}$$
$$= \alpha + x' \cos.(xx') - y' \sin.(xx') \quad \because \text{PM'S} = 90^\circ - (xx').$$
$$\text{MP or } y = \text{MT} + \text{TS} + \text{SP}$$
$$= \text{A'N} + \text{M'R} + \text{SP}$$
$$= \beta + x' \sin.(xx') + y' \cos.(xx').$$

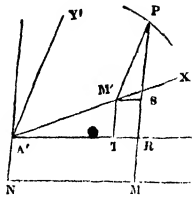
Substituting therefore these values of x and y in the equation proposed, we shall obtain a relation between x' and y', which will be the equation to the curve referred to the axes AX′, AY′.

Cor. If the new origin be coincident with the old, then $\alpha = 0$, $\beta = 0$, and the above equations become

$$x = x' \cos.(xx') - y' \sin.(xx')$$
$$y = x' \sin.(xx') + y' \cos.(xx').$$

III. **We may now proceed to the solution of the general problem, viz.—*To pass from one system of axes inclined at any given angle to another system; also inclined at any given angle.***

Let AX, AY, be the original system; A′X′, A′Y′, the new.
In addition to the former notation,
Let the inclination of A′X′ to AX be called (xx')
Let the inclination of A′Y′ to AX be called (xy')
Let the inclination of AY to AX be called (xy).
&c. &c.



Then, $AM \text{ or } x = AN + NM$

$$= AN + A'T + M'S \quad \dots\dots\dots (1)$$

$$MP \text{ or } y = MR + RS + SP$$

$$= A'N + M'T + SP \quad \dots\dots\dots (2)$$

Now $\dfrac{A'T}{A'M'} = \dfrac{\sin.\ A'M'T}{\sin.\ A'TM'} = \dfrac{\sin.(x'y)}{\sin.(xy)}$ $\quad \therefore A'T = x' \dfrac{\sin.(x'y)}{\sin.(xy)}$

$\dfrac{M'S}{M'P} = \dfrac{\sin.\ M'PS}{\sin.\ M'SP} = \dfrac{\sin.(yy')}{\sin.(xy)}$ $\quad \therefore M'S = y' \dfrac{\sin.(yy')}{\sin.(xy)}$

$\dfrac{M'T}{A'M'} = \dfrac{\sin.\ M'A'T}{\sin.\ A'TM'} = \dfrac{\sin.(xx')}{\sin.(xy)}$ $\quad \therefore M'T = x' \dfrac{\sin.(xx')}{\sin.(xy)}$

$\dfrac{SP}{M'P} = \dfrac{\sin.\ SM'P}{\sin.\ M'SP} = \dfrac{\sin.(xy')}{\sin.(xy)}$ $\quad \therefore SP = y' \dfrac{\sin.(xy')}{\sin.(xy)}$

Substituting these values in (1) and (2)

$$x = \alpha + \frac{x' \sin.(x'y) + y' \sin.(yy')}{\sin.(xy)}$$

$$y = \beta + \frac{x' \sin.(xx') + y' \sin.(xy')}{\sin.(xy.)}.$$

Such is the most general formula for the transformation of co-ordinates, from which it is easy to deduce the formulas corresponding to all positions of a new origin, and to the different inclinations of the new axes compared with the old ones, by giving proper values either positive or negative to α and β, and any value to the angles (xx'), (xy'), from 0 up to 90°.

As to the angle xy, it is always given *a priori*, since it is the angle contained by the original axes.

We can easily deduce from the general formula, the results already obtained in cases I. II.

REMARKS.

(1). In general we distinguish between two different species of the transformation of co-ordinates; The change in the position of the origin, and the change in the direction of the axes. When the problem proposed requires this double transformation, it is frequently more advantageous to execute them in succession than at first.

(2). Since we have frequently occasion, in the same question, to effect several transformations of co-ordinates, it is convenient to suppress the accents of x', y', in the second member of the formulas which relate to these transformations; that is to say, we may designate both the old and new co-ordinates by x and y, although their values are different, but the circumstance of using the different formulæ in succession will be sufficient to point out, that the curve after having

been referred to one system of axes, is afterwards referred to a second, to a third, and so on.

Thus, in order to pass from a rectangular or oblique system to another system parallel to it, we may, in the equation to the curve, substitute $x + \alpha$ for x, and $y + \beta$ for y, and the x and y of the second equation will represent the co-ordinates referred to the new axes, the co-ordinates of whose origin, referred to the former origin, are α, β. In like manner we may proceed in all other cases, and thus simplify our calculations by avoiding the use of numerous accents.

(3). The quantities α, β, (xx'), (xy'), &c. which enter into the above formulas, are constants whose value fixes the position of the new origin and the direction of the new axes with reference to the original axes, whose inclination to each other is expressed by (xy). The quantities α, β, (xx'), (xy'), &c. must be regarded as known and given *a priori*, whenever we wish to refer the curve to new axes whose position with regard to the proposed curve has been discovered to be more simple than that of the old axes.

It frequently happens, however, that we perform a transformation of co-ordinates when our object in so doing is to make some specific change in the form of the equation to the curve, for example, to make certain terms disappear. In this case α, β, (xx'), (xy'), &c. are constants which are, for the time being, indeterminate; and whose values we afterwards endeavour to calculate in such a manner as to simplify the equation in the manner required. With regard to the angle (xy) we cannot employ it in this manner, since it is the angle contained by the old axes, and is in every case supposed to be known *a priori*.

The number of terms which it is our wish to remove from the equation, will indicate the number of indeterminate quantities which we must introduce into our calculation, and therefore the system of formulas which we must employ.

These remarks will be better understood when applied to particular examples.

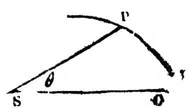
ON POLAR CO-ORDINATES.

We have hitherto supposed the position of a curve upon a plane to be determined by means of an equation between variables, expressing the distances of each of the points in the curve from two fixed straight lines, the distances being reckoned parallel to these lines. There is, however, another method for determining the position of a point or of a series of points which in certain cases is more convenient.

To explain this mode of representing curves analytically, let us consider any curve Pp.

Let SO be a given straight line in the plane of the curve, and S a given point in that line.

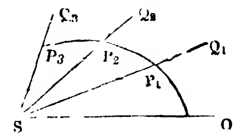
From S draw a straight line SP to any point P in the curve.



Let SP be called r, and the angle between SP and SO be θ,

It is evident that, if we can obtain a relation between r and θ which holds good for every point in the curve, the curve will be entirely determined, for

If we give to θ a succession of values θ_1, θ_2, θ_3, &c. we shall obtain from the equation between r and θ a series r_1, r_2, r_3, &c. of corresponding values of r.



Making therefore at the point S the angle Q_1SO, Q_2SO, Q_3SO, &c. respectively equal to θ_1, θ_2, θ_3, &c. and taking SP_1, SP_2, SP_3, &c. equal to the corresponding values of r, we shall obtain the points P_1, P_2, P_3, &c. which belong to the curve.

The variable quantities r and θ are called *Polar Co-ordinates*, the point S is called the *Pole*, r the *Radius Vector*, and the relation between r and θ is termed the *Polar Equation* to the curve.

A curve being traced upon a plane, we may, from some known property of the curve, determine the polar equation at once, more usually, however, we have the position of the curve determined by an equation between rectilinear co-ordinates, and it is required to deduce the equation between polar co-ordinates. This can be easily effected by a transformation of co-ordinates, which we shall now proceed to explain.

Let us begin with two of the most simple and useful cases:

1. Let Pp be the curve whose equation is given in terms of rectangular co-ordinates, AX and AY being the axes. Let it be required to determine the polar equation, S being the pole and SO parallel to AX.

Let the co-ordinates of the point S referred to the axes AX, AY, be α, β.

Take any point P in the curve, draw PM perpendicular to AX, join SP, draw SN perpendicular to AX.

Then, $\quad AM = x,\ MP = y,\ SP = r,\ PSO = \theta,\ AN = \alpha,\ SN = \beta$

$$\left.\begin{aligned} AM \text{ or } x &= AN + NM \\ &= \alpha + r\cos.\,\theta \\ MP \text{ or } y &= MR + RP \\ &= \beta + r\sin.\,\theta. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

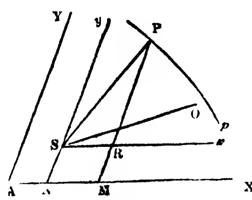
Substituting $\therefore$ these values of x and y in the equation to the curve, we shall obtain a relation between r and θ which will be the polar equation required.

2. Let SO coincide with AX, the point S with A, in this case, α, $\beta = 0$, and the above formula becomes

$$\left.\begin{aligned} x &= r\cos.\,\theta \\ y &= r\sin.\,\theta \end{aligned}\right\} (b).$$

The general problem is, given the equation to a curve referred to any system of axes, to find the polar equation; the position of the pole being any whatever.

Let Pp be curve referred to the axes AX, AY.



Take any point S as the pole, and let SO be the fixed straight line, and let the co-ordinates of S referred to AX, AY, be α, β.

Through S draw Sx, Sy parallel to AX, AY.

Draw PM parallel to AY, join S, P; draw SN parallel to AY.

Then, $AM = x$, $MP = y$, $AN = \alpha$, $SN = \beta$, SP $= r$, $PSO = \theta$.

Let the angle between the axes AX, AY, be denoted by $(x, y,)$ the angle between the fixed line SO and the axis AX being ϕ.

Then, AM or $x = AN + NM$

$= AN + SR$ .... (1)

MP or $y = MR + RP$

$= SN + RP$ .... (2)

Now, $\frac{SR}{SP} = \frac{\sin. SPR}{\sin. PRS} = \frac{\sin. \{(xy) - \theta - \phi\}}{\sin. (xy)}$ $\therefore SR = r \frac{\sin. \{(xy) - \theta - \phi\}}{\sin. (xy)}$

And $\frac{RP}{SP} = \frac{\sin. RSP}{\sin. SRP} = \frac{\sin. (\theta + \phi)}{\sin (xy)}$ $\therefore RP = r \frac{\sin. (\theta + \phi)}{\sin. (xy)}$

Substituting these values of SR, RP in equations (1) and (2), we have

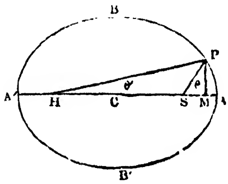
$$\left.\begin{aligned} x &= \alpha + r \frac{\sin. \{(xy) - \theta - \phi\}}{\sin. (xy)} \\ y &= \beta + r \frac{\sin. (\theta + \phi)}{\sin. (xy)} \end{aligned}\right\} (c).$$

These equations will be found to agree with those already found (*a*), (*b*), for in the former cases $xy = 90°$, $\phi = 0$; hence we have

$$x = \alpha + r \sin. \theta$$
$$y = \beta + r \sin. \theta.$$

PROB. I.

To find the polar equation to the ellipse, the focus being the pole.



Let S be the pole, P any point in the curve;
Join S, P; draw PM perpendicular to AA'.
Assume

$$SP = r \qquad ASP = \theta$$
$$CS = x \qquad MP = y.$$

Then we have seen in deducing the equation to the ellipse, that the distance of the point S from any point in the curve, is

$$r = a - \frac{cx}{a} \text{ .... (1)}$$

Where $x = CM$
$= CS + SM$
$= c + r \cos. \theta.$

Substituting this value of x in equation (1)

$$r = a - \frac{c^2 + cr \cos. \theta}{a}$$

$$ar = a^2 - c^2 - cr \cos. \theta$$

Whence $$r = \frac{a^2 - c^2}{a + c \cos. \theta}$$

$$= \frac{a^2 - a^2e^2}{a + ae \cos. \theta} \because c = ae$$

$$= \frac{a(1 - e^2)}{1 + e \cos. \theta}$$

which is the polar equation to the ellipse usually employed.

If we take the other focus H for the pole, we have the distance

$$HP \text{ or } r' = a + \frac{cx}{a}$$

Where $\quad x = \mathrm{CM} = \mathrm{HM} - \mathrm{CH} = r' \cos.\, \theta' - c$

$$r' = a + \frac{cr' \cos.\, \theta' - c^2}{a}$$

$$\therefore \quad r' = \frac{a\,(1 - e^2)}{1 - e \cos.\, \theta'}$$

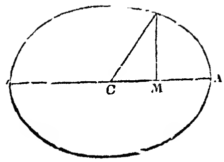
which is the polar equation in this case.

PROB. II.

To find the polar equation to the ellipse, the centre being the pole.

Let P be any point in the curve, C the centre;
Join C, P; draw PM perpendicular to CA.
Assume

$$\mathrm{CP} = r \qquad \mathrm{PCA} = \theta$$
$$\mathrm{CM} = x \qquad \mathrm{MP} = y.$$



Then from the right angled triangle PCM, we have

$$r^2 = x^2 + y^2$$

$$= x^2 + \frac{b^2}{a^2}(a^2 - x^2), \text{ substituting for } y \text{ its value}$$

derived from the equation to the curve.

But $\qquad x = r \cos.\, \theta$, substituting $\therefore$ this value of x in the above equation

$$a^2 r^2 = a^2 r^2 \cos.^2 \theta + b^2 a^2 - b^2 r^2 \cos.^2 \theta$$

$$\therefore \; r^2 (a^2 - a^2 \cos.^2 \theta + b^2 \cos.^2 \theta) = a^2 b^2$$

$$r^2 \left\{a^2 (1 - \cos.^2 \theta) + b^2 \cos.^2 \theta\right\} = a^2 b^2$$

$$r^2 (a^2 \sin.^2 \theta + b^2 \cos.^2 \theta) = a^2 b^2$$

$$r = \frac{\pm\, ab}{\sqrt{a^2 \sin.^2 \theta + b^2 \cos.^2 \theta}}$$

which is the polar equation required.

If in the above equation we substitute for b, its value $a\sqrt{1 - e^2}$ the equation becomes

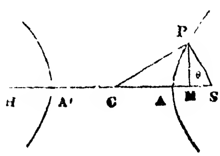
$$r = \frac{a\sqrt{1 - e^2}}{\sqrt{1 - e^2 \cos.^2 \theta}}$$

PROB. III.

To find the polar equation to the hyperbola, the focus being the pole.

Let S be the pole, P any point in the curve;
Join S, P; draw MP perpendicular to CA.
Assume

$$\mathrm{SP} = r \qquad \mathrm{ASP} = \theta$$
$$\mathrm{CM} = x \qquad \mathrm{MP} = y.$$



Then we have already seen that the distance tween S and any point in the curve, is

$$r = \frac{cx}{a} - a \text{ ....} (1)$$

$$x = \mathrm{CM}$$
$$= \mathrm{CS} - \mathrm{MS}$$
$$= c - r \cos.\, \theta$$

Substituting $\therefore$ this value of x in equation (1) we find

$$r = \frac{c^2 - cr\cos.\theta}{a} - a$$

$$ar = c^2 - cr\cos.\theta - a^2$$

$$r = \frac{c^2 - a^2}{a + c\cos.\theta}$$

$$= \frac{a^2e^2 - a^2}{a + ae\cos.\theta} \because c = ae$$

$$= \frac{a(e^2 - 1)}{1 + e\cos.\theta}$$

which is the equation required.

If we take the other focus H for the pole, we have

$$\text{HP or } r' = \frac{cx}{a} + a$$

$$\text{But CM or } x = \text{HM} - \text{CH}$$

$$= r'\cos.\theta' - c$$

$$\therefore \quad r' = \frac{cr'\cos.\theta' - c^2}{a} + a$$

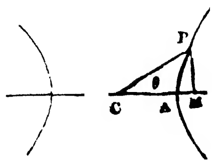
$$\therefore \quad r' = -\frac{a(e^2 - 1)}{1 - e\cos.\theta'}$$

PROB. IV.

To find the polar equation to the hyperbola, the centre being the pole.

Let C be the centre, P any point in the curve;
Join C, P; draw PM perpendicular to CA.
Assume

$$\text{CP} = r \qquad \text{PCA} = \theta$$
$$\text{CM} = x \qquad \text{MP} = y.$$



From the right angled triangle CPM, we have $r^2 = x^2 + y^2$, substituting for y^2 its value derived from the equation to curve

$$= x^2 + \frac{b^2}{a^2}(x^2 - a^2)$$

But $\qquad x = r\cos.\theta.$

Substituting therefore this value of x.

$$a^2r^2 = a^2r^2\cos.^2\theta + b^2r^2\cos.^2\theta - a^2b^2$$

$$r^2(a^2 - a^2\cos.^2\theta - b^2\cos.^2\theta) = -a^2b^2$$

$$r^2\{a^2(1 - \cos.^2\theta) - b^2\cos.^2\theta\} = -a^2b^2$$

$$r^2(a^2\sin.^2\theta - b^2\cos.^2\theta) = -a^2b^2$$

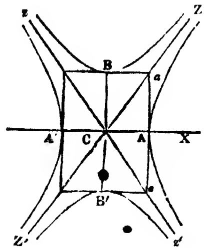
$$r = \frac{\pm ab}{\sqrt{b^2\cos.^2\theta - a^2\sin.^2\theta}}$$

which is the equation required.

If we substitute for b^2 its value $a^2(e^2 - 1)$ the above becomes

$$r = \frac{\pm a\sqrt{e^2 - 1}}{\sqrt{e^2\cos.^2\theta - 1}}$$

To find the equation to the hyperbola referred to its asymptotes as axes of co-ordinates.



It has been already shown that the asymptotes of the hyperbola are diameters of the curve, and that they are the diagonals of the rectangle ABA′B′, whose sides are equal the major and minor axes of the curve.

Now C being the origin of co-ordinates, the equation to the straight line CZ is of the form

$$y = ax$$

where a is the tangent of ZCX.

But $$\tan. \mathrm{ZCX} = \frac{\mathrm{A}a}{\mathrm{AC}}$$

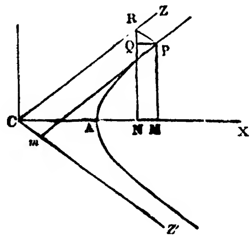
∴ the equation to the asymptote ZCZ′

$$= \frac{b}{a} x \ldots\ldots\ldots\ldots\ldots\ldots \text{(M)}$$

And in like manner the equation to the asymptote $z\mathrm{C}z'$ is

$$y = -\frac{b}{a} . x \ldots\ldots\ldots\ldots\ldots\ldots \text{(N)}$$

This being premised,



Let PA p be a hyperbola whose equation referred to its axes is

$$a^2y^2 - b^2x^2 = -a^2b^2 \ldots\ldots\ldots\ldots \text{(O)}$$

It is required to transform this equation to another in which the curve shall be referred to the asymptotes CZ, CZ′, as axes of co-ordinates.

We might solve the problem directly by making use of the formula for passing from a rectangular to an oblique system of axes; we prefer however, in this case, to perform the operation independently.

Let CZ be assumed as the new axis of y's, and CZ′ as the new axis of x's, let the angle ZCX which they make with CX be called φ.

Take any point P in the curve, draw PM perpendicular to CX; Pm parallel to CZ; PR parallel to CZ′; RN perpendicular to CX; PQ parallel to CX.

Assume $\mathrm{CM} = x, \ \mathrm{MP} = y$

$\mathrm{C}m = \mathrm{X}, \ m\mathrm{P} = \mathrm{Y}, \text{ angle ZCX} = \varphi$

Now, $\mathrm{CM} \text{ or } x = \mathrm{CN} + \mathrm{NM}$

$= \mathrm{Y} \cos. \varphi + \mathrm{X} \cos. \varphi$

$\mathrm{MP} \text{ or } y = \mathrm{NR} - \mathrm{QR}$

$= \mathrm{Y} \sin. \varphi - \mathrm{X} \sin. \varphi$

Substituting these values of x and y in equation (O), it becomes

$$a^2 (\mathrm{Y} - \mathrm{X})^2 \sin.^2 \varphi - b^2 (\mathrm{Y} + \mathrm{X})^2 \cos.^2 \varphi = -a^2b^2.$$

Now we have seen above, that

$$\tan. \mathrm{ZCX} \text{ or } \varphi = \frac{b}{a}, \therefore \cos.^2 \varphi = \frac{a^2}{a^2 + b^2} \text{ and } \sin.^2 \varphi = \frac{b^2}{a^2 + b^2}$$

Substituting ∴ for $\sin.^2 \varphi$ and $\cos.^2 \varphi$ those values thus derived,

$$\frac{a^2b^2}{a^2 + b^2}(\mathrm{Y} - \mathrm{X})^2 - \frac{a^2b^2}{a^2 + b^2}(\mathrm{Y} + \mathrm{X})^2 = -a^2b^2$$

Which equation reduced becomes

$$-\frac{4XY}{a^2+b^2} = . -1$$

Or, changing the large letters, which we no longer require for distinction,

$$xy = \frac{a^2+b^2}{4}$$

which is the equation to the hyperbola referred to its asymptotes.

Exercises in Analytical Geometry.

(1.) Construct the equations $5y - 3x - 2 = 0$, $8y - 6x + 5 = 0$, $y + 3x = 6$ $y^2 - 5x^2 = 0$, and $y^2 - 7y + 12 = 0$; the axes of co-ordinates being rectangular.

(2.) Find the equation to the straight line which passes through the two points (2, 3) and (4, 5). Ans. $y = x + 1$.

(3.) Describe the circle whose equation is $y^2 + x^2 + 4y - 4x = 8$.

(4.) Find the co-ordinates x' y' of the centre, and R the radius of the circle whose equation is

$$y^2 + x^2 - 6y + 8x - 11 = 0.$$

Ans. $y' = 3$, $x' = -4$ and $R = 6$.

(5.) Prove that the perpendiculars drawn from the angles to the opposite sides of a triangle pass through the same point.

(6.) Prove that the straight lines drawn from the angles of a triangle, to bisect the opposite sides, pass through the same point.

(7.) Given the base $= a$, and the sum of the squares of the sides $= s^2$, to determine the locus of the vertex of the triangle.

(8.) Given the base of a triangle $= a$, and the ratio of the sides $m : n$, to find the locus of the vertex.

(9.) Given the base and the vertical angle, to determine the locus of the vertex of the triangle.

(10.) From a given point A, either within or out of a given circle, let a straight line AC be drawn to the circumference, in which take AB, so that AB.AC may always be equal to a given space; find the locus of the point B.

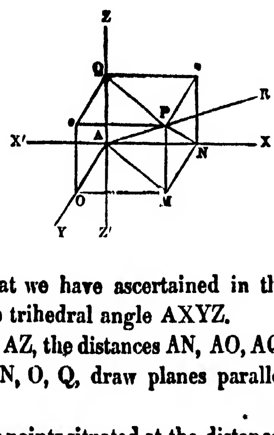
ANALYTICAL GEOMETRY

OF

THREE DIMENSIONS.

EQUATIONS OF A POINT.

We have seen that the position of a point in a plane is determined when we know its distances from two straight lines drawn in that plane; in like manner we shall now proceed to show that the position of a point in space, is determined by its distances from three planes.



Let there be three planes YAZ, XAZ, XAY, which we shall suppose to be perpendicular to each other, and whose intersections are the three straight lines AZ, AY, AX, each of which is perpendicular to the other two according to the principles established in the Geometry of Planes. Let us call the distances of a point in space from these three planes a, b, c, and let us suppose these distances are known, then the position of the point will be completely determined, provided that we have ascertained in the first instance, that the point is situated within the trihedral angle AXYZ.

For, take on the three straight lines AX, AY, AZ, the distances AN, AO, AQ, respectively, equal to a, b, c; through the points N, O, Q, draw planes parallel to the given planes.

Since the two first parallel planes have all their points situated at the distances a and b, respectively, from the planes YAZ, XAZ, it follows that all the points of the straight line PM, which is the common intersection of these two planes have exclusively the property of being at the same distances from the planes YAZ, XAZ. Hence the point sought must be situated in the straight line PM. Again, the point sought must be situated somewhere in the third plane PnQo which is parallel to XAY, since all the points in this plane have exclusively the property of being at the distance c from the plane XAY. Hence the point sought must be the point P in which the third plane cuts the common intersection of the two first, and thus its position is altogether determined.

We may designate by x the distance of a point from the plane YAZ reckoned along AX;

We may designate by y the distance of a point from the plane XAZ reckoned along AY;

We may designate by z the distance of a point from the plane XAY reckoned along AZ.

So that AX, AY, AZ, the intersections of the three planes, two and two, will be the axes of x's, of y's, and of z's. They are called conjointly *Axes of Co-ordinates*, the three planes *the Co-ordinate Planes*, and the three distances the *Co-ordinates of a point.* These terms are all analogous to those already employed in Analytical Geometry of two dimensions.

The plane YAZ perpendicular to the axis of x's, is called the plane yz;

The plane XAZ perpendicular to the axis of y's, is called the plane xz;

The plane XAY perpendicular to the axis of z's, is called the plane xy.

This last plane is usually represented in a horizontal position, and the two others in a vertical position.

It follows from what has been said above that the equations

$$x = a, \; y = b, \; z = c$$

(a, b, c being known quantities) are sufficient to determine the position of a point in space, they are for that reason called the *Equations of a point in space.*

We must remark, that, since the three co-ordinate planes when prolonged indefinitely determine eight trihedral angles, viz. four formed above the plane of xy, and four formed below the same plane; it is necessary for us to express analytically in which of these eight angles the point is situated. It is sufficient for this purpose, to extend to planes the principles which have been applied to distances from points and straight lines, that is to say, *if we regard as* POSITIVE *distances reckoned along* AX *to the right of* A, *we must regard as* NEGATIVE *distances reckoned along* AX *to the left of* A, that is to say, *in the direction* AX, the remark applies to the two other co-ordinate axes.

We must therefore consider in the quantities a, b, c, not only the numerical value of these quantities, but also the signs with which they are affected, in order that we may be enabled to determine in which of the eight trihedral angles about the point A the required point is situated.

According to this principle we have, in order to express completely the position of a point in space, the following combinations:

$x = +a, \; y = +b, \; z = +c$, point situated in the angle AXYZ,
$x = -a, \; y = +b, \; z = +c$, point situated in the angle AX'YZ,
$x = +a, \; y = -b, \; z = +c$, point situated in the angle AXY'Z,
$x = +a, \; y = +b, \; z = -c$, point situated in the angle AXYZ',
$x = -a, \; y = -b, \; z = +c$, point situated in the angle AX'Y'Z,
$x = -a, \; y = +b, \; z = -c$, point situated in the angle AX'YZ',
$x = +a, \; y = -b, \; z = -c$, point situated in the angle AXY'Z',
$x = -a, \; y = -b, \; z = -c$, point situated in the angle AX'Y'Z',

in all, *eight* combinations, viz. *two* systems in which the signs are the same, *three* in which one sign is negative and the two others positive, and *three* in which one sign is positive and the two others negative.

There are also some particular positions of the point which it is proper to notice. For example, in order to express that a point is situated in the plane xy, we must write that its distance from that plane is nothing, and we shall have for the equation of such a point

$$x = a, \; y = b, \; z = 0.$$

Similarly, a point situated on the axis of x's, whose distances from the planes xz and xy are nothing at the same time, will have for its equation

$$x = a, \ y = 0, \ z = 0$$

and so for other points situated on the planes or on the co-ordinate axes.

The planes parallel to the three co-ordinate planes, and which have served to determine the position of the point P, constitute, along with these, a rectangular parallelopipedon of which the twelve edges, which are equal, taken four and four, are the three co-ordinates x, y, z, of the point P.

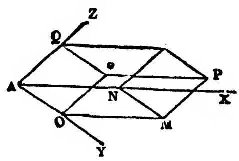
Definition.—If from any point in space, a straight line be drawn perpendicular to a given plane, the foot of the perpendicular is called the *projection* of the given point upon the given plane.

In like manner, if from every point of any line in space, whether straight or curved, perpendiculars be drawn to any given plane, the line traced out by the feet of the perpendiculars upon the given plane, is called the *projection* of the given line upon the given plane.

If we suppose that $x = a$, $y = b$, $z = c$ are the equations of the point P,
the co-ordinates of the point M are .... $x = a, \ y = b$,
the co-ordinates of the point n are .... $x = a, \ z = c$,
which gives for the co-ordinates of the point o .... $y = b, \ z = c$.

From which it appears, that if the projection of a point P upon two of the co-ordinate planes be known, the third projection will also necessarily be known.

When the co-ordinates are not at right angles to each other, in which case the axes AX, AY, AZ, make with each other any angle whatever, and are called *oblique axes*, the equations of a point P are still



$$x = a, \ y = b, \ z = c.$$

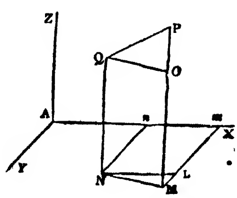
But in this case, a, b, c, express distances reckoned parallel to these axes, and the projections of the point P are obtained by the straight lines PM, Pn, Po, respectively, parallel to AX, AY, AZ.

In other respects, every thing that has been said with regard to rectangular axes, is applicable to oblique axes also.

In what follows we shall always suppose the axes rectangular, unless the contrary is specified.

Proposition.—*To find an expression for the distance between two points space, whose co-ordinates are known.*

Let P and Q be the given points whose co-ordinates referred to the rectangular axes AX, AY, AZ, are respectively x', y', z'; and x'', y'', z''.



From P and Q let fall PM, QN, perpendicular on the plane xy;

From M and N draw Mm, Nn, parallel to AY;

We then have

$Am = x \quad Mm = y' \quad An = x'' \quad Nn = y''$.

Join NM which determines a trapezium PQMN, in the plane of this trapezium draw QO parallel to MN, and in the plane xy, draw NL parallel to AX.

The right angled triangles PQO, MNL, give

$$PQ^2 = PO^2 + QO^2 = PO^2 + MN^2$$

And $$MN^2 = NL^2 + ML^2 = mn^2 + ML^2$$

$$\therefore \quad PQ^2 = mn^2 + ML^2 + PO^2$$

But $$mn = x' - x'', \ ML = y' - y'', \ PO = z' - z''$$

$$\therefore \quad \delta^2 \text{ or } PQ^2 = (x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2$$

$$\delta = \sqrt{(x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2}$$

the expression required.

If one of the given points be the origin, then

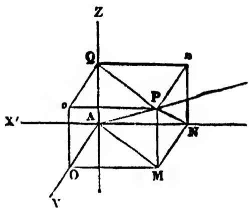
$$x'' = 0 \quad y'' = 0 \quad z'' = 0$$

and the above expression becomes

$$\delta = \sqrt{x^2 + y^2 + z^2}.$$

The last formula may be derived directly from the figure at the beginning of the chapter, as follows:

Join A, P; A, M; then from the right angled triangles AMP, ANM



$$AP^2 = AM^2 + PM^2$$

$$AM^2 = AN^2 + MN^2$$

$$\therefore \quad AP^2 = AN^2 + MN^2 + PM^2$$

But $\quad AN = x, NM = AO = y, PM = z$

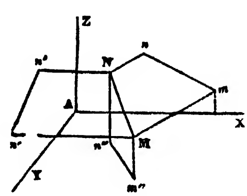
$$\therefore \quad AP^2 \text{ or } \delta^2 = x^2 + y^2 + z^2$$

from which it appears, that *the square of the diagonal of a rectangular parallelopipedon is equal to the sum of the squares of the three edges.*

To find the equation to a straight line in space.

The projections of a straight line on two planes is sufficient to determine its position, and hence it follows that a straight line will be determined analytically, if we know the equations of its projections upon two of the three co-ordinate planes, We generally consider the projections of the straight line on the planes of xz, and yz; and since these two planes have AZ for their common axis, this line is regarded in each of the planes as the axis of abscissas; AX is, therefore, the axis of ordinates in the plane of xz, and AY is the axis of ordinates in the plane of yz.

Let MN be any straight line in space, and mn, $m'n'$ its projections on the planes xz, yz then the equations of these two projections will be of the form



$$\left.\begin{array}{ll} x = az + \alpha & \dots\dots\dots (1) \\ y = bz + \beta & \dots\dots\dots (2) \end{array}\right\}$$

a, b, are constants denoting the tangents of the angles which mn, $m'n'$ form with the axis AZ, and α, β, express the distances from the origin to the points in which these straight lines cut the axes AX, AY.

It is to be observed that the equation

$$x = az + \alpha$$

expresses not only the relation between the co-ordinates of any point in the

straight line mn, but also the relation between the co-ordinates of any point in the plane MNnm drawn through MN perpendicular to the plane of xz. In like manner, the equation

$$y = bz + \beta$$

belongs not only to the straight line $m'n'$, but likewise to all the points of the projecting plane $m'n'$NM drawn perpendicular to the plane of yz through the straight line MN.

It appears then, that this system of equations holds for all the points of the straight line MN, which is the intersection of the two planes perpendicular to the planes of xz and yz, and holds good for the points of this straight line alone. These equations therefore are, in this sense, *the equations to the straight line itself*, although, in the first instance, we established them separately as the equations of the projections.

It follows from this, that the elimination of the variable z between these two equations, gives rise to a third equation between x and y, viz.

$$b(x - \alpha) = a(y - \beta)$$

which represents the straight line $m''n''$ the projection of MN on the plane of xy; or, more generally, this equation belongs to all the points of the projecting plane MN$n''m''$ drawn through MN perpendicular to the plane of xy.

When the straight line passes through the origin, its projections will also pass through the origin, in this case the distances α, β, are nothing, and the straight line is represented by the system of equations

$$\left.\begin{aligned} x &= az \\ y &= bz \end{aligned}\right\}$$

The straight line may be situated in one of the co-ordinate planes, for example in the plane of xz. In this case, for all points in this straight line

$$y = 0$$

and the system of equations representing the straight line becomes

$$\left.\begin{aligned} x &= az + \alpha \\ y &= 0 \end{aligned}\right\}$$

that is to say, in this case we shall have $b = 0, \beta = 0$, which is evident from the figure, for the projection of the straight line on the plane of yz will coincide with AZ.

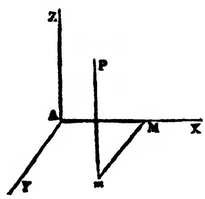
When the constants a, b, α, β, are given *a priori*, the position of the straight line is completely determined. In order to obtain its different points we must give a succession of particular values to one of the variables, z for example, in each of the equations $x = az + \alpha$, $y = bz + \beta$, by means of which we shall obtain corresponding values for the two other variables x, y. Then let $z = z'$ then

$$x = az' + \alpha = p \text{ a known quantity,}$$
$$y = bz' + \beta = q \text{ a known quantity.}$$

Take in AX a distance AM $= p$;

From p draw Mm parallel to AY and $= q$;

From m draw mP perpendicular to xy and $= z'$;



The point P thus determined belongs to the straight line, and in the same manner we may obtain all the other points.

It may be required however, to determine the constants a, b, α, β, conformably to certain conditions, which gives rise to a

series of problems in Analytical Geometry of three dimensions, analogous to those which we have already considered in reference to a straight line traced upon a plane.

To find the equations to a straight line in space, which passes through a given point.

Let the equations to the straight line required be of the form

$$\left.\begin{aligned} x &= az + \alpha \quad \ldots\ldots (1) \\ y &= bz + \beta \quad \ldots\ldots (2) \end{aligned}\right\}$$

Let the co-ordinates of the given point be x', y', z';

Then since the straight line passes through this point it must satisfy the equations

$$\left.\begin{aligned} x' &= az' + \alpha \quad \ldots\ldots (3) \\ y' &= bz' + \beta \quad \ldots\ldots (4) \end{aligned}\right\}$$

And, since these equations hold good together for the straight line required, subtracting (3) from (1) and (4) from (2) we have

$$\left.\begin{aligned} x - x' &= a\,(z - z') \quad \ldots\ldots (5) \\ y - y' &= b\,(z - z') \quad \ldots\ldots (6) \end{aligned}\right\}$$

the equations required.

To find the equations to a straight line passing through two given points.

Let the co-ordinates of the given points be x', y', z'; x'', y'', z'';

Let the equations of the straight line required be of the form

$$\left.\begin{aligned} x &= az + \alpha \quad \ldots\ldots (1) \\ y &= bz + \beta \quad \ldots\ldots (2) \end{aligned}\right\}$$

where a, b, α, β, are quantities supposed to be unknown, and which we must determine from the data of the problem.

Since the points x', y', z'; x'', y'', z''; belong by hypothesis to the straight line whose equations are sought, it must satisfy the two systems of equations

$$\left.\begin{aligned} x' &= az' + \alpha \quad \ldots\ldots (3) \\ y' &= bz' + \beta \quad \ldots\ldots (4) \end{aligned}\right\}$$

$$\left.\begin{aligned} x'' &= az'' + \alpha \quad \ldots\ldots (5) \\ y'' &= bz'' + \beta \quad \ldots\ldots (6) \end{aligned}\right\}$$

If therefore, we eliminate a, b, α, β, between these six equations in such a manner as to obtain two equations which involve only x, y, z, and the known quantities x', y', z'; x'', y'', z''; the problem will be solved.

Subtracting (3) from (1), and (4) from (2)

$$\left.\begin{aligned} x - x' &= a\,(z - z') \qquad \text{whence } a = \frac{x - x'}{z - z'} \\ y - y' &= b\,(z - z') \qquad \text{whence } b = \frac{y - y'}{z - z'} \end{aligned}\right\} \ldots\ldots (7)$$

Again, subtracting (5) from (1), and (6) from (2)

$$x - x'' = a\,(z - z'')$$
$$y - y'' = b\,(z - z'')$$

Substituting in these last equations the value of a and b, found in (7), we have

$$\left.\begin{array}{ll} x - x'' = \dfrac{x - x'}{z - z'}(z - z'') \ldots\ldots\ldots\ldots (8) \\ y - y'' = \dfrac{y - y'}{z - z'}(z - z'') \ldots\ldots\ldots\ldots (9) \end{array}\right\}$$

the equations required.

In finding the equation to a straight line drawn through one given point the constants a, b were left undetermined, because a straight line may be drawn through *one* point, *in any direction whatever*. But when a straight line is drawn through *two* given points, its direction is determined, and hence, in this case, it became necessary to eliminate the arbitrary quantities a, b.

Through a given point without a straight line in space, to draw a straight line parallel to the former.

Let x', y', z', be the co-ordinates of the given point.

Let the equations of the given straight line be

$$x = az + \alpha$$
$$y = bz + \beta$$

The equations of the straight line required will be of the form

$$x - x' = \mathrm{A}(z - z')$$
$$y - y' = \mathrm{B}(z - z')$$

where A and B are quantities which it is required to determine.

Since the straight lines are parallel, the planes which project them respectively upon the planes of xz and yz, must be parallel, hence the intersection of these parallel planes with the co-ordinate planes must be parallel, that is to say, the projection of the straight lines must be parallel, hence we have necessarily the relations

$$\mathrm{A} = a, \ \mathrm{B} = b$$

which gives for the equations of the straight line required

$$x - x' = a(z - z')$$
$$y - y' = b(z - z')$$

Given the equations to two straight lines in space, to determine the relation which must exist between the constants, in order that the two lines may intersect.

Let

$$\left.\begin{array}{l} x = az + \alpha \ldots\ldots (1) \\ y = bz + \beta \ldots\ldots (2) \end{array}\right\} \quad \left.\begin{array}{l} x = a'z + \alpha' \ldots\ldots (3) \\ y = b'z + \beta' \ldots\ldots (4) \end{array}\right\}$$

be the equations to the two given straight lines

If these straight lines cut one another, these equations must hold good together at the point of intersection; eliminating, therefore, x, y, z, we find the relation

$$(\alpha - \alpha')(b - b') = (\beta - \beta')(a - a')$$

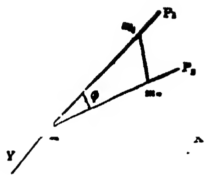
Unless this equation of condition be satisfied, the lines do not intersect, if it holds good, then the point of intersection has for its co-ordinates

$$z = \frac{\alpha' - \alpha}{a - a'} = \frac{\beta' - \beta}{b - b'}, \ y = \frac{b\beta' - b'\beta}{b - b'}, \ x = \frac{a\alpha' - a'\alpha}{a - a'}$$

Given two straight lines in space which intersect, to find the angle contained between them.

Let the equations to the given straight lines be

$$\left.\begin{array}{l} x = a_1 z + \alpha_1 \\ y = b_1 z + \beta_1 \end{array}\right\} \text{ and } \left.\begin{array}{l} x = a_2 z + \alpha_2 \\ y = b_2 z + \beta_2 \end{array}\right\}$$



Let A be the origin of co-ordinates.

Draw through A two straight lines AP_1, AP_2, parallel to the given straight lines.

Then the angle P_1AP_2 (φ) will be the required angle.

The equations to AP_1 and AP_2 will be respectively

$$\left.\begin{array}{l} x = a_1 z \\ y = b_1 z \end{array}\right\} \text{ and } \left.\begin{array}{l} x = a_2 z \\ y = b_2 z \end{array}\right\}$$

In AP_1 take any point m_1 whose co-ordinates are $x_1y_1z_1$;

AP_2 .. .. m_2 .. .. .. $x_2y_2z_2$;

Join $m_1 m_2$; let $Am_1 = r_1$, $Am_2 = r_2$

Then

$$\overline{m_1m_2}^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos.\varphi$$

Also

$$\overline{m_1m_2}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$\therefore r_1^2 + r_2^2 - 2r_1r_2 \cos.\varphi = x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2x_1x_2 - 2y_1y_2 - 2z_1z_2$$

But

$$r_1^2 = x_1^2 + y_1^2 + z_1^2, \text{ and } r_2^2 = x_2^2 + y_2^2 + z_2^2$$

$$\therefore \cos.\varphi = \frac{x_1x_2 + y_1y_2 + z_1z_2}{r_1r_2}.$$

Now, since $x_1y_1z_1$ is a point in AP_1, and $x_2y_2z_2$ is a point in AP_2, the following equations hold good.

$$\begin{array}{ll} \left.\begin{array}{l} x_1 = a_1 z \\ y_1 = b_1 z \end{array}\right\} & \left.\begin{array}{l} x_2 = a_2 z \\ y_2 = b_2 z \end{array}\right\} \\ \therefore \quad x_1^2 = a_1^2 z^2 & x_2^2 = a_2^2 z^2 \\ y_1^2 = b_1^2 z^2 & y_2^2 = b_2^2 z^2 \\ z_1^2 = z_1^2 & z_2^2 = z_2^2 \end{array}$$

$$\therefore x_1^2 + y_1^2 + z_1^2 = z_1^2 (1 + a_1^2 + b_1^2), \text{ and } x_2^2 + y_2^2 + z_2^2 = z_2^2 (1 + a_2^2 + b_2^2)$$

$$\therefore z_1 = \frac{r_1}{\sqrt{1 + a_1^2 + b_1^2}} \qquad z_2 = \frac{r_2}{\sqrt{1 + a_2^2 + b_2^2}}$$

$$a_1z_1 \text{ or } x_1 = \frac{a_1r_1}{\sqrt{1 + a_1^2 + b_1^2}} \qquad x_2 = \frac{a_2r_2}{\sqrt{1 + a_2^2 + b_2^2}}$$

$$b_1z_1 \text{ or } y_1 = \frac{b_1r_1}{\sqrt{1 + a_1^2 + b_1^2}} \qquad y_2 = \frac{b_2r_2}{\sqrt{1 + a_2^2 + b_2^2}}.$$

Substituting the value in these equation for cos. φ we have

$$\cos.\varphi = \frac{r_1r_2a_1a_2 + r_1r_2b_1b_2 + r_1r_2}{r_1r_2\sqrt{1 + a_1^2 + b_1^2}.\sqrt{1 + a_2^2 + b_2^2}}$$

$$\therefore \quad \cos.\varphi = \frac{1 + a_1a_2 + b_1b_2}{\sqrt{1 + a_1^2 + b_1^2}.\sqrt{1 + a_2^2 + b_2^2}}$$

Given a straight line in space, to find the angle which it makes with each of the axes of co-ordinates.

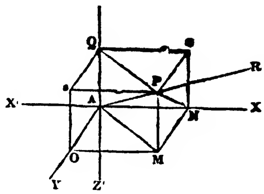
Let the equations to the given straight line be

$$\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\}$$

Through the origin A draw a straight line AR parallel to the given straight line, then the equations to this line are

$$\left.\begin{array}{l} x = az \\ y = bz \end{array}\right\}$$

Take any point P in the straight line, whose co-ordinates are x, y, z, and let AP $= r$.



Let the angles which r makes with the axes AX, AY, AZ, respectively be called (rx), (ry), (rz).

Then,

$$x = r \cos.(rx)$$
$$y = r \cos.(ry)$$
$$z = r \cos.(rz)$$

But,

$$r^2 = x^2 + y^2 + z^2$$
$$= z^2(1 + a^2 + b^2)^*$$

$$\therefore \quad r = z\sqrt{1 + a^2 + b^2}$$

$$\therefore \cos.(rx) = \frac{x}{r} = \frac{az}{z\sqrt{1 + a^2 + b^2}} = \frac{a}{\sqrt{1 + a^2 + b^2}}$$

$$\therefore \cos.(ry) = \frac{y}{r} = \frac{bz}{z\sqrt{1 + a^2 + b^2}} = \frac{b}{\sqrt{1 + a^2 + b^2}}$$

$$\cos.(rz) = \frac{z}{r} = \frac{z}{z\sqrt{1 + a^2 + b^2}} = \frac{1}{\sqrt{1 + a^2 + b^2}}$$

Cor. Since

$$x = r \cos.(rx)$$
$$y = r \cos.(ry)$$
$$z = r \cos.(rz)$$

$$\therefore \quad x^2 + y^2 + z^2 = r^2 \left\{\cos.^2 (rx) + \cos.^2 (ry) + \cos.^2 (rz)\right\}$$

But

$$x^2 + y^2 + z^2 = r^2$$

$$\therefore \quad r^2 \left\{\cos.^2 + (rx) + \cos.^2 (ry) + \cos.^2 (rz)\right\} = r^2$$

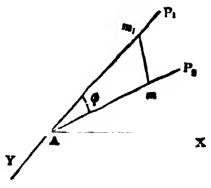
$$\therefore \quad \cos.^2 (rx) + \cos.^2 (ry) + \cos.^2 (rz) = 1$$

A very remarkable result, which shows that *the sum of the squares of the cosines of the angles which a straight line in space forms with the three axes is equal to unity.*

* For $x = az \qquad y = bz$

$$\therefore \quad \begin{array}{l} x^2 = a^2z^2 \\ y^2 = b^2z^2 \\ z^2 = z^2 \\ \hline x^2 + y^2 + z^2 = z^2(1 + a^2 + b^2) \end{array}$$

Given two straight lines in space which intersect each other, to find the angle contained between them in terms of the angles which each of the straight lines makes with the axes of co-ordinates.



Let the angles which r_1 makes with the axes AX, AY, AZ, be called (r_1x), (r_1y), (r_1z).

Let the angles which r_2 makes with the axes AX, AY, AZ, be called (r_2x), (r_2y), (r_2z).

Now as before

$$r_1r_2 \cos. \phi = x_1x_2 + y_1y_2 + z_1z_2$$

But

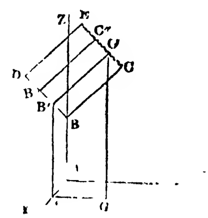
$$x_1 = r \cos. (r_1x) \qquad x_2 = r_2 \cos. (r_2x)$$
$$y_1 = r_1 \cos. (r_1y) \qquad y_2 = r_2 \cos. (r_2y)$$
$$z_1 = r_1 \cos. (r_1z) \qquad z_2 = r_2 \cos. (r_2z)$$

$$\therefore \cos. \phi = \cos. (r_1x) \cos. (r_2x) + \cos. (r_1y) \cos. (r_2y) + \cos. (r_1z) \cos. (r_2z)$$

To find the equation to a plane.

Definition.—A plane is a surface generated by a straight line of indefinite length, which moves parallel to itself along another straight line, also of indefinite length.

Let BC, BD be the intersections of any plane BCDE, with the co-ordinate planes xz, yz. BC, BD are called the *traces* of the plane BCDE, on the planes xz, yz.



The plane BCDE may be conceived to be generated by the straight line BC, moving parallel to itself, along the straight line BD.

Since the straight BC lies wholly in the plane of xz, its equations will be

$$y = o, \ z = Ax + C \ldots\ldots\ldots\ldots\ldots (1)$$

where A is the tangent of the angle which BC makes with the axis AX, and where C = AB.

In like manner, the equation to the straight line BD, which lies wholly in the plane yz, will be

$$x = 0, \ z = By + C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where B is the tangent of the angle which BD makes with the axis AY.

Now let B′C′ be any position of the generating line, then B′C′ is parallel to BC, its projection on the plane xy will be parallel to AX, and its projection on the plane xz will be parallel to BC, hence the equations to B′C′ will be

$$y = \alpha, \ z = Ax + \beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where α, β are quantities constant for all points of the same position B′C′ of the generating line, but variable for any other position, such as B″ C″.

It remains for us to express analytically that the generating line BC intersects in all its positions the line BD.

In order that this may be the case the equations (2) and (3) must hold good

at the same line; eliminating, therefore, x, y, z, between these four equations, we arrive at the equation of condition:

$$\beta = B\alpha + C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

which must hold, in order that the straight lines BC, BD, may always intersect. Now the equations (3) and (4) must hold good together, for each position of the generating line. If, therefore, we eliminate the indeterminate quantities α, β, between these equations, we conclude that the resulting equation

$$z = Ax + By + C$$

is that of the plane, since x, y, z, are the co-ordinates of the different points of the generating line in any position whatever.

The meaning of the constants in this equation is manifest from what has been said above. A and B are the tangents of the angles which the traces of the plane on the co-ordinate planes of xz, yz, make with the axes AX, AY; and C, is the distance from the origin of the point in which the plane meets the axis AZ.

If BD be parallel to AY, then the plane is perpendicular to the plane of xz, and its equation becomes

$$z = Ax + C$$

the same as that of its trace.

To find the equations of the projection on the co-ordinate planes of the intersections of two given planes.

Let the equations to the two planes be

$$z = Ax + By + C, \quad z = A'x + B'y + C'$$

Since these equations hold good together for the straight line which is their common intersection, if we eliminate z we shall have the equation to the projection on the plane xy, that is

$$(A - A')\,x + (B - B')\,y + C - C' = 0$$

In like manner, eliminating x or y, we find

$$(A - A')\,z + (AB' - A'B)\,y + AC' - A'C = 0$$

$$(B' - B)\,z + (A'B - AB')\,x + BC' - B'C = 0$$

the equations of the projections on the planes of yz and xz.

To find the equation to a plane passing through one, two, or three given points.

The equation will be of the form

$$z = Ax + By + C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

And since it passes through a point whose co-ordinates are x', y', z', it must satisfy the equation

$$z' = Ax' + By' + C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Subtracting (2) from (1)

$$z - z' = A\,(x - x') + B\,(y - y') \text{ the equation required.}$$

The constants A and B being arbitrary, the problem will be indeterminate, and, in point of fact, we know that any number of planes may be drawn through a given point.

If the plane be required to pass through a second point x'', y'', z'', then it must satisfy the equation

$$z'' = Ax'' + By'' + C \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and two of the constants may be eliminated between equations (1), (2), (3); one however will still remain, and the problem will be indeterminate, for any number of planes may be drawn through two given points.

If the plane be required to pass through a third point x''', y''', z''', it must satisfy the equation

$$z''' = Ax''' + By''' + C \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

the three constants may then be eliminated between equations (1), (2), (3), (4), and the result will be the equation of the only plane that can pass through the three given points. For we know that three points suffice to determine the position of a plane.

To find the conditions that must hold good, in order that a straight line and a plane may coincide, or be parallel.

Let the equations to the plane and the straight line be

$$z = Ax + By + C$$
$$x = az + \alpha, \; y = bz + \beta$$

Substituting $az + \alpha$ and $bz + \beta$ for x and y in the equation to the plane, we have

$$(A\alpha + B\beta + C) + (Aa + Bb - 1)z = 0$$

If the straight line and plane have only one point in common, we should thus be able to determine its co-ordinates, but if the straight line be altogether situated in the given plane, the above equation of condition must hold good *whatever may be the value of z*: hence the two parts of the equation must be independent of each other, and we shall have

$$A\alpha + B\beta + c = 0, \qquad Aa + Bb - 1 = 0$$

the equations of condition required.

If the straight line be merely parallel to the plane, if we move them in a direction parallel to their original position until they reach the origin, the plane and the straight line will coincide, hence the above equations must be satisfied on the supposition that α and β and C are each $= 0$, therefore

$$Aa + Bb - 1 = 0$$

will be the equation of condition which must be satisfied, in order that a straight line and a plane may be parallel.

To find the conditions requisite in order that a straight line may be perpendicular to a plane.

If a straight line be perpendicular to a plane, the projection of the straight line and the trace of the plane upon any of the co-ordinate planes will be perpendicular to one another.

Let the equation to the given plane and straight line be

$$z = Ax + By + C$$

$$\left.\begin{array}{l} x = az + \alpha \quad \dots\dots (1) \\ y = bz + \beta \quad \dots\dots (2) \end{array}\right\}$$

where it must be remembered that (1) is the projection of the given straight line on xz, and (2) its projection on yz.

The trace of the plane on xz is

$$z = Ax + C, \quad \text{or,} \quad x = \frac{1}{A} \cdot z - \frac{C}{A} \quad \dots\dots (3)$$

and on yz

$$z = By + C, \quad \text{or,} \quad y = \frac{1}{B} \cdot z - \frac{C}{B} \quad \dots\dots (4)$$

Hence the straight line (3) is perpendicular to (1), and (4) is perpendicular to (2).

$$\therefore A + a = 0 \qquad \text{and} \qquad B + b = 0$$

which are the equations of condition required.

To find the equations to a straight line which passes through a given point, and is perpendicular to a given plane.

Let the equation to the plane be

$$z = Ax + By + C$$

The equations to the required line, since it passes through a point $(x', y', z',)$ must be of the form

$$\left.\begin{array}{l} x - x' = a(z - z') \\ y - y' = b(z - z') \end{array}\right\}$$

and, since it is perpendicular to the plane

$$a = -A, \; b = -B,$$

and therefore the equations required are

$$x - x' + A(z - z') = 0, \qquad y - y' + B(z - z') = 0.$$

To find the distance (δ) of the given point, in the last problem, from the plane.

The equations to the straight line are

$$x - x' + A(z - z') = 0 \dots\dots (1) \quad y - y' + B(z - z') = 0 \dots\dots (2)$$

and the equation to the plane is

$$z = Ax + By + C \quad \dots\dots (3)$$

which may be put under the form

$$z - z' = A(x - x') + B(y - y') + C + Ax' + By' - z'$$

or $\quad z - z' = A(x - x') + B(y - y') + M$

where $\quad M = C + Ax' + By' - z'$

Now, if we suppose x, y, z, to be the co-ordinates of the point in which the perpendicular meets the plane, equations (1), (2), and (3), will hold good together, and we shall have

$$z - z' = \frac{M}{1 + A^2 + B^2}, \quad y - y' = \frac{-MB}{1+A^2+B^2}, \quad x - x' = \frac{-MA}{1 + A^2 + B^2}$$

But the distance (r) of the two points whose co-ordinates are x, y, z; x', y', z', is

$$r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\delta = \frac{M}{\sqrt{1 + A^2 + B^2}}.$$

To find the distance (Δ) from a point in space to a straight line.

Let the co-ordinates of the point be (x', y', z',) and let the equations of the straight line be

$$\left.\begin{array}{ll} x = az + \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & (1) \\ y = bz + \beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & (2) \end{array}\right\}$$

The equation to a plane passing through the point x', y', z', and perpendicular to the given straight line, will be

$$(z - z') + a\,(x - x') + b\,(y - y') = 0 \quad \ldots\ldots\ldots\ldots \quad (3)$$

Now, if we suppose x, y, z; to be the co-ordinates of the point in which the plane meets the given straight line, the equations (1), (2), (3), will hold good together, and if we find values of $(x - x')$, $(y - y')$, $(z - z')$, from these equations and substitute the values thus obtained in the general expression for the distance of two given points in space, viz.

$$r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

we shall solve the problem.

In order to effect this, let us put the equations (1) and (2) under the form

$$(x - x') = a\,(z - z') + \alpha - x' + az' \quad \ldots\ldots\ldots\ldots \quad (4)$$
$$(y - y') = b\,(z - z') + \beta - y' + bz' \quad \ldots\ldots\ldots\ldots \quad (5)$$

Substitute these values of $(x - x')$ and $(y - y')$ in equation (3), which will then become

$$(z - z')\,(1 + a^2 + b^2) + a\,(\alpha - x' + az') + b\,(\beta - y' + bz') = 0$$

whence we find

$$(z - z') = \frac{a\,(x' - \alpha) + b\,(y' - \beta) + z'}{1 + a^2 + b^2} - z'$$

$$= \frac{N}{1 + a^2 + b^2} - z' \quad \text{if} \quad N = a\,(x' - \alpha) + b\,(y' - \beta) + z'.$$

Substitute this value of $z - z'$ in equations (4) and (5), and we find the corresponding values of $(x - x')$ and $(y - y')$ to be

$$x - x' = \frac{Na}{1 + a^2 + b^2} - (x' - \alpha)$$

$$y - y' = \frac{Nb}{1 + a^2 + b^2} - (y' - \beta)$$

squaring these quantities and adding, we find

$$\Delta^2 = \frac{N^2(1+a^2+b^2)}{(1+a^2+b^2)^2} + (x'-\alpha)^2 + (y'-\beta)^2 + z'^2 - 2N.\frac{z'+a(x'-\alpha)+b(y'-\beta)}{1+a^2+b^2}$$

$$= (x'-\alpha)^2 + (y'-\beta)^2 + z'^2 - \frac{N^2}{1+a^2+b^2}.$$

To determine the angle between two given planes.

Let the equations to the planes be

$$z = Ax + By + C \text{ .... } (1); \quad z = A'x + B'y + C' \text{ .... } (2)$$

If we let fall from the origin two straight lines perpendicular on these planes, the angle contained by the straight lines will be the same as the angle contained by the planes, let the equation to these straight lines be

$$\left.\begin{array}{l} x = az \\ y = bz \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} x = a'z \\ y = b'z \end{array}\right\}$$

the angle between them is known from the expression

$$\cos.\varphi = \frac{1 + aa' + bb'}{\sqrt{(1+a^2+b^2)(1+a'^2+b'^2)}}$$

But, in order that the straight lines may be perpendicular to the given planes, we must have

$$A + a = 0, \; B + b = 0, \; A' + a' = 0, \; B' + b' = 0$$

Substituting therefore, the values of a, b, a', b', derived from these equations, we find that the expression of the cosine of the angle between the two planes,

$$\cos.\varphi = \frac{1 + AA' + BB'}{\sqrt{(1+A^2+B^2)(1+A'^2+B'^2)}}$$

In order to find the angle which any plane makes with the co-ordinate planes, we have only to suppose that one of the above planes assumes in succession the position of the different co-ordinate planes, thus let us suppose, that (2) is the plane of xz, then its equation becomes

$$y = 0, \text{ so that, } A' = 0, \; C' = 0$$

and therefore, if we denote by the symbols (xz), (yz), (xy), the angles which the given plane makes with the planes xz, yz, xy, we have

$$\cos.(xz) = \frac{B}{\sqrt{1+A^2+B^2}}$$

$$\cos.(yz) = \frac{A}{\sqrt{1+A^2+B^2}}$$

$$\cos.(xy) = \frac{1}{\sqrt{1+A^2+B^2}}$$

and have

$$\cos.^2(xy) + \cos.^2(yz) + \cos.^2(xz) = 1$$

and

$$\cos.\varphi = \cos.(xz)\cos.(x'z') + \cos.(xy)\cos.(x'y') + \cos.(yz)\cos.(y'z')$$

To find the angle (θ) contained by a plane and straight line in space.

The angle sought is that which the straight line makes with its projection on the plane. If from any point in the given straight line we let fall a perpendi-

cular upon the plane, the angle contained between these two straight lines will be the complement of the required angle.

Let the equation to the given plane be

$$z = Ax + By + C \text{ .... } (1)$$

The equation to the given straight line

$$\left.\begin{array}{l} x = az + \alpha \text{ .... } (2) \\ y = bz + \beta \text{ .... } (3) \end{array}\right\}$$

The equations of the line let fall perpendicular on the plane will be of the form

$$\left.\begin{array}{l} x = a'z + \alpha' \text{ .... } (4) \\ y = b'z + \beta' \text{ .... } (5) \end{array}\right\}$$

But in order that this may be perpendicular to the given plane, we must have

$$A + a' = 0 \qquad\qquad B + b' = 0$$

Now, the cosine of the angle contained by the two straight lines, is

$$\cos. \varphi = \frac{1 + a'a + b'b}{\sqrt{(1 + a^2 + b^2)(1 + a'^2 + b'^2)}}$$

It appears from what has been said above, that, in the present case $\varphi = 90° - \theta$, and $\therefore \cos. \varphi = \sin. \theta$. Substituting therefore for a', b', these values in terms of A and B, we find

$$\sin. \theta = \frac{1 - Aa - Bb}{\sqrt{(1 + a^2 + b^2)(1 + A^2 + B^2)}}$$

DIFFERENTIAL CALCULUS.

CHAPTER I.

DEFINITIONS.

In considering the relations which exist between different quantities, those which during the whole of any investigation are supposed to retain the same value are called *constant quantities,* those to which different values may be assigned are called *variable quantities.*

Constant quantities are usually represented by the first letters of the alphabet, a, b, c, &c. variable quantities by the letters u, x, y, z, &c.

When two or more variable quantities are connected in such a manner, that the value of one of them is determined by the value assigned to the other, the former is said to be a *function* of the other variables.

Thus in the equation

$$y = Ax + Bx^2 + C$$

where the value of y depends upon the value assigned to x, y is said to be a function of x.

In like manner if we have

$$y = Az^2 + Bx^2 + Cx^3 + D$$

where the value of y depends upon the values assigned to x and z, y is said to be a function of x and z.

The words "function of x," are usually expressed by the symbols, $f\ (x)$, $\phi\ (x)$, $\psi\ (x)$, or similar abbreviations, and the above equations expressed in general terms would be written

$$y = f(x)$$
$$y = f(x, z)$$

If $y = f\ (x)$, and a change takes place in the value of $f\ (x)$ such that x becomes $x + h$, x being quite indeterminate, and h any quantity whatever, either positive or negative, a corresponding change must take place in the value of y, which may then be represented by y'. If the quantity $f\,(x + h)$ be now developed in a series of the form

$$f(x) + Ah + Bh^2 + Ch^3 + \ldots$$

in which the first term is the original function $f\ (x)$, and the other terms ascend regularly by positive and integral powers of h, and A, B, C, &c., are independent of h;* then the co-efficient of the simple power of h in this series is

* We shall, in the mean time, take for granted that $f\ (x + h)$ can always be developed in a series of the above form, (showing, however, as we advance, that this is actually the case for all the particular functions which fall under our notice) and defer the general demonstration of this principle until we proceed in Chapter V. to the discussion of Taylor's theorem.

called *the first differential co-efficient of y or f(x).* This is the fundamental definition of the differential calculus.

Then let y be a function of x such that

$$y = ax^2$$

Let x become $x + h$ and y become y'

$$y' = a(x + h)^2$$

expanding $$= ax^2 + 2ax \cdot h + ah^2$$

This we at once perceive is a series of the required form, the first term ax^2 is the original function y, and the other terms ascend by integral and positive powers of h; hence, according to our definition, $2ax$ the co-efficient of the simple power of h in this series is the first differential co-efficient of y or $f(x)$.

Again, let

$$y = x^3$$

Let x become $x + h$ and y become y'

$$y' = (x + h)^3$$

expanding $$= x^3 + 3x^2 \cdot h + 3x \cdot h^2 + h^3$$

Here again we perceive that the series is of the required form, and, therefore, $3x^2$ the co-efficient of the simple power of h is the first differential co-efficient of x^3.

Again, let

$$y = ax^3 + bx^2 + cx + d$$

Let x become $(x + h)$ and y become y'

$$\therefore \quad y' = a(x + h)^3 + b(x + h)^2 + c(x + h) + d$$

expanding $= ax^3 + 3ax^2 h + 3axh^2 + h^3 + bx^2 + 2bxh + bh^2 + cx + ch + d$

arranging according to powers of h

$$= (ax^3 + bx^2 + cx + d) + (3ax^2 + 2bx + c)h + (3ax + b)h^2 + h^3$$

a series of the required form, for the first term is $ax^3 + bx^2 + cx + d$, the original function, and the succeeding terms ascend regularly by powers of h.

Hence, $3ax^2 + 2bx + c$ the co-efficient of the simple power of h in the developement of y' is the first differential co-efficient of y or $ax^3 + bx^2 + cx + d$.

If*

$$y = f(x)$$

the first differential co-efficient of y is denoted by the symbol $\frac{dy}{dx}$, thus in the above examples

$$y = ax^2$$

$$\frac{dy}{dx} = 2ax$$

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

* In this treatise the principles of Lagrange have been almost exclusively adopted, but although that writer has with great propriety denominated this branch of Analysis "The Calculus of Functions," yet it has been thought expedient to retain in the present work the nomenclature and notation of the Differential Calculus, since it is employed almost universally in the scientific publications both of this country and of the Continent.

$$y = ax^3 + bx^2 + cx + d$$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

in like manner if $u = f(z)$ the first differential co-efficient of u or $f(z)$ will be represented by $\frac{du}{dz}$.

We might obtain the first differential co-efficient of any function presented to us by following a process analogous to that exhibited above, but we shall materially abridge the labour of our operations by establishing certain general rules, which will enable us at once to determine the first differential co-efficient of any variable quantity, without the necessity of having recourse to the substitution of $x + h$ for x, and the subsequent expansion. The investigation of these will form the subject of the two following chapters.

Note.—Since a constant quantity is not susceptible of change, it is manifest that it can have no differential co-efficient, or if $y = a$, $\frac{dy}{dx} = 0$.

CHAPTER II.

ON FINDING THE FIRST DIFFERENTIAL CO-EFFICIENT OF SIMPLE FUNCTIONS OF ONE VARIABLE.

1. *To find the first differential co-efficient of any power of a simple Algebraic quantity.*

Let $y = x^n$

Let x become $x + h$

$\therefore y' = (x + h)^n$

Expanding by the binomial

$$= x^n + nx^{n-1}h + \frac{n(n-1)}{1.2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}h^3 + \&c$$

$$\frac{dy}{dx} = nx^{n-1}$$

From this it is manifest that

The first differential co-efficient of any power of a simple Algebraic quantity is found by multiplying the quantity by the index of the power, and then diminishing the exponent by unity.

Ex. 1. $y = x^7$ $\qquad \frac{dy}{dx} = 7x^6$

2. $y = ax^{p+q}$ $\qquad \frac{dy}{dx} = a(p+q)x^{p+q-1}$

3. $y = a + x^{-q}$ $\qquad \frac{dy}{dx} = -qx^{-(q+1)}$

4. $y = a + 7x^{-\frac{m}{n}}$ $\qquad \frac{dy}{dx} = -7\frac{m}{n}x^{-(\frac{m}{n}+1)}$

2. *To find the first differential co-efficient of a^x.*

Let $y = a^x$

Let x become $x + h$ and y become y'

$$y' = a^{x+h}$$
$$= a^x \times a^h$$
$$= a^x . (1 + \frac{ph}{1} + \frac{p^2 h^2}{1.2} + \frac{p^3 h^3}{1.2.3} + \ldots) \text{ See p. 334.}$$
$$= a^x + a^x p . h + a^x \frac{p^2}{1.2} h^2 + \ldots$$

where $p = (a - 1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots = \log. a$

$$\therefore \quad \frac{dy}{dx} = a^x . p = \log. a . a^x$$

In the system of logarithms whose base is ϵ, $p = 1$.

$\therefore$ If $y = \epsilon^x$

$$\frac{dy}{dx} = \epsilon^x$$

3. *To find the first differential co-efficient of log. x.*

Let $y = \log. x$

Let x become $x + h$ and y become y'

$$y' = \log(x + h)$$
$$= \log. x\left(1 + \frac{h}{x}\right)$$
$$= \log. x + \log. \left(1 + \frac{h}{x}\right)$$
$$= \log. x + \frac{1}{p}\left(\frac{h}{x} - \frac{1}{2} . \frac{h^2}{x^2} + \frac{1}{3} . \frac{h^3}{x^3} - \ldots\right) \text{See Art. 205, p. 342.}$$
$$= \log. x + \frac{1}{p} . \frac{1}{x} . h - \frac{1}{p} . \frac{1}{2x^2} . h^2 + \frac{1}{p} . \frac{1}{3x^3} . h^3 \ldots$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{p} . \frac{1}{x} = \frac{1}{\log. a} . \frac{1}{x}$$

If the logarithms be taken in the system whose base is ϵ, then $p = 1$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}$$

4. *To find the first differential co-efficient of sin. x.*

Let $y = \sin. x$

Let x become $x + h$ and y become y'

$$y' = \sin. (x + h)$$
$$= \sin. x \cos. h + \sin. h \cos. x$$

Substituting for sin. h and cos. h their developements as found in p. 587.

$$= \sin. x\left(1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \ldots\right) + \cos. x\left(\frac{h}{1} - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \ldots\right.$$

arranging according to powers of h

$$= \sin. x + \cos. x \,.\, h - \frac{\sin. x}{1\,.\,2} h^2 - \frac{\cos. x}{1\,.\,2\,.\,3} h^3 + \ldots$$

$$\therefore \quad \frac{dy}{dx} = \cos. x.$$

From this it appears that the first differential co-efficient of the sine of an angle is its cosine.

5. *To find the first differential co-efficient of cos. x.*

Let $y = \cos. x$.

Let x become $x + h$, and y become y'.

$$y' = \cos. (x + h)$$

$$= \cos. x \cos. h - \sin. x \sin. h.$$

Substituting for cos. h and sin. h, their developements

$$= \cos. x \left(1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \ldots\right) - \sin. x \left(\frac{h}{1} - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \right)$$

arranging according to powers of h

$$= \cos. x - \sin. x \,.\, h - \frac{\cos. x}{1.2} h^2 + \frac{\sin. x}{1.2.3} h^3 + \ldots.$$

$$\therefore \frac{dy}{dx} = - \sin. x$$

Hence it appears, that the first differential co-efficient of the cosine of an angle is its sine with an opposite sign.

We may recapitulate the results of this chapter as follows:

If $\quad y = ax^n \quad$ then $\quad \dfrac{dy}{dx} = anx^{n-1}$ whatever be the value of n.

$$y = a^x \qquad \frac{dy}{dx} = p \,.\, a^x = \log. a \,.\, a^x$$

$$y = \log. x \qquad \frac{dy}{dx} = \frac{1}{p} \,.\, \frac{1}{x}$$

$$y = \sin. x \qquad \frac{dy}{dx} = \cos. x$$

$$y = \cos. x \qquad \frac{dy}{dx} = - \sin. x$$

$$y = e^x \qquad \frac{dy}{dx} = e^x$$

$$y = \log._e x^* \qquad \frac{dy}{dx} = \frac{1}{x}$$

$$y = \sin. mx \qquad \frac{dy}{dx} = m \cos. mx$$

$$y = \cos. mx \qquad \frac{dy}{dx} = - m \sin. mx$$

* The expression $\log._e x$ signifies the log. of x to the base e, or the Napierian log.

6. If any of the preceding functions be increased or diminished by a constant quantity, the differential coefficient will remain unchanged, because the constant can only appear in the first term of the expansion, and the other terms, involving the variable and its powers, remain unchanged. Also, if any of these functions be multiplied by a constant, the differential coefficient will be multiplied by the same constant, because every term of the expanded series is multiplied by the constant. Hence, if

$y = x^n \pm b$	then $\frac{dy}{dx} = nx^{n-1}$	$y = cx^n$	then $\frac{dy}{dx} = ncx^{n-1}$
$y = a^x \pm b$	$\frac{dy}{dx} = \log. a . a^x$	$y = ca^x$	$\frac{dy}{dx} = c \log. a . a$
$y = e^x \pm b$	$\frac{dy}{dx} = e^x$	$y = c . e^x$	$\frac{dy}{dx} = c . e^x$
$y = \log. x \pm b$	$\frac{dy}{dx} = \frac{1}{\log. a} . \frac{1}{x}$	$y = c \log. x$	$\frac{dy}{dx} = \frac{c}{\log. a} . \frac{1}{x}$
$y = \log._e x \pm b$	$\frac{dy}{dx} = \frac{1}{x}$	$y = c \log._e x$	$\frac{dy}{dx} = \frac{c}{x}$
$y = \sin x \pm b$	$\frac{dy}{dx} = \cos x$	$y = c \sin x$	$\frac{dy}{dx} = c \cos x$
$y = \cos x \pm b$	$\frac{dy}{dx} = - \sin x$	$y = c \cos x$	$\frac{dy}{dx} = - c \sin x$

CHAPTER III.

ON FINDING THE FIRST DIFFERENTIAL CO-EFFICIENTS OF COMPOUND FUNCTIONS OF ONE VARIABLE.

1. *To find the first differential co-efficient of the sum of any number of simple functions.*

Let
$$y = f(x) + \phi(x) + \psi(x) + \ldots$$
where $f(x)$, $\phi(x)$, $\psi(x)$, are all simple functions of x.

Let x become $x + h$ and y become y'
$$y' = f(x+h) + \phi(x+h) + \psi(x+h) + \ldots.$$
Let the expansion of $f(x+h)$ when developed in a series ascending regularly according to powers of h be
$$f(x+h) = f(x) + Ah + Bh^2 + \ldots.$$
And in like manner
$$\phi(x+h) = \phi(x) + A'h + B'h^2 + \ldots.$$
$$\psi(x+h) = \psi(x) + A''h + B''h^2 + \ldots.$$
Then $y' = f(x) + Ah + Bh^2 + \ldots + \phi(x) + A'h + B'h^2 + \ldots + \psi(x) + A''h + B''h^2 + \ldots$

Collecting those terms which involve the simple power of h

$$=\{f(x)+\phi(x)+\psi(x)\}+(\mathrm{A}+\mathrm{A}'+\mathrm{A}''+\ldots)h+\mathrm{P}h^2+\mathrm{Q}h^3+\ldots$$

$$\therefore \frac{dy}{dx}=\mathrm{A}+\mathrm{A}'+\mathrm{A}''+\ldots$$

But on inspecting the above series it will be seen that A is the first differential co-efficient of $f(x)$, A$'$ of $\phi(x)$, A$''$ of $\psi(x)$, &c. hence we conclude that

The first differential co-efficient of the sum of any number of simple functions is equal to the sum of the first differential co-efficients of each of the functions considered separately.

Ex. (1.)

$$\text{Let } y=x^n+a^x+\log. x+\sin x+\cos x$$

$$\frac{dy}{dx}=nx^{n-1}+p\,a^x+\frac{1}{p}\cdot\frac{1}{x}+\cos x-\sin x$$

(2.)

$$y=ax^7+bx^6-7x^5+4x^4+3x^3-2x^2-x+1$$

$$\frac{dy}{dx}=7ax^6+6bx^5-35x^4+16x^3+9x^2-4x-1$$

(3.)

$$y=m\sin x-2\cos x+4x^{-p}-6x^{-q}+10$$

$$\frac{dy}{dx}=m\cos x+2\sin x-4px^{-(p+1)}+6qx^{-(q+1)}$$

2. *To find the first differential co-efficient of the product of any number of simple functions.*

Let us take, in the first place, the product of two simple functions only.

Let $y'=f(x)\,\phi(x)$

Let x become $x+h$ and y become y'

$y=f(a+h)\,\phi\,(x+h)$

Let $f(x+h)$ and $\phi(x+h)$ be developed in series ascending regularly by powers of h

$$y'=\{f(x)+\mathrm{A}h+\mathrm{B}h^2+\ldots\}\ \{\phi(x)+\mathrm{A}'h+\mathrm{B}'h^2+\ldots.\}.$$

Performing the multiplication indicated, and arranging according to powers of h.

$$=f(x)\,\phi(x)+\{\mathrm{A}'f(x)+\mathrm{A}\,\phi(x)\}\,h+\mathrm{P}h^2+\mathrm{Q}h^3+\ldots$$

$$\therefore \frac{dy}{dx}=\mathrm{A}'f(x)+\mathrm{A}\,\phi(x)$$

But A is the first differential co-efficient of $f(x)$, and A$'$ is the first differential co-efficient of $\phi(x)$, hence it appears that

To obtain the first differential co-efficient of the product of any two simple functions, we must multiply each of the functions by the first differential co-efficient of the other, and add together the two products.

Let us now take the product of any number of simple functions.

Let $\quad y=f(x)\,\phi(x)\,\psi(x)\ldots$

Let x become $x + h$ and y become y'

$$y' = f(x+h)\,\varphi(x+h)\,\psi(x+h)\ldots$$

Let $f(x+h)$, $\varphi(x+h)$, $\psi(x+h)\ldots$ be developed in regular series ascending according to powers of h.

$$= \{f(x) + Ah + Bh^2 + \ldots\}\ \{\varphi(x) + A'h + B'h^2 + \ldots\}$$
$$\{\psi(x) + A''h + B''h^2 + \ldots\}\ldots$$

Performing the multiplications indicated, and arranging the product according to powers of h.

$$= f(x)\,\varphi(x)\,\psi(x) + \{A\,\varphi(x)\,\psi(x)\ldots + A' f(x)\,\psi(x)\ldots + A'' f(x)\varphi(x)\ldots\}h$$
$$+ Ph^2 + Qh^3 + \ldots$$

$$\therefore\ \frac{dy}{dx} = A\,\varphi(x)\,\psi(x)\ldots + A' f(x)\,\psi(x)\ldots + A'' f(x)\,\varphi(x)$$

but A, A', A'', are the first differential co-efficients of $f(x)$, $\varphi(x)$, $\psi(x)$ respectively, hence it appears, that,

To obtain the first differential co-efficient of the product of any number of simple functions, we must multiply the first differential co-efficient of each function by all the other functions, and add the whole of these products together.

Ex. 1. $y = x^m \sin. x$

The first differential co-efficients of x^m and $\sin. x$, are mx^{m-1} and $\cos. x$ respectively.

$$\therefore\ \frac{dy}{dx} = mx^{m-1}\sin. x + x^m \cos. x.$$

2. $y = \sin. x \cos. x$

$$\frac{dy}{dx} = \cos.^2 x - \sin.^2 x = 2\cos.^2 x - 1.$$

3. $y = a^x \log. x.$

$$\frac{dy}{dx} = p\ \ a^x \log. x + a^x \frac{1}{p} \cdot \frac{1}{x} = \frac{a^x}{p}\left(\frac{1+p^2\,x\log. x}{x}\right)$$

4. $y = x^m u^n z^h$ (u and z being functions of x)

$$\frac{dy}{dx} = mu^n z^h x^{m-1} + nx^m z^h u^{n-1} + hx^m u^n z^{h-1}$$

5. $y = x^r \sin. x\, a^x$

$$\frac{dy}{dx} = r\sin. x\, a^x x^{r-1} + x^r \cos. x\, a^x + p\ \ a^x x^r \sin. x$$

$$= a^x \sin. x\, x^{r-1}\,(r + x\cot. x + px)$$

3. *To find the first differential co-efficient of a fractional function.*

Let $y = \dfrac{f(x)}{\varphi(x)}$

Let x become $x + h$ and y become y'

$$y' = \frac{f(x+h)}{\varphi(x+h)}$$

$$= \frac{f(x) + Ah + Bh^2 + \ldots}{\varphi(x) + A'h + B'h^2 + \ldots}$$
$$= (f(x) + Ah + Bh^2 + \ldots)(\varphi(x) + A'h + B'h^2 + \ldots)^{-1}$$
$$= (f(x) + Ah + Bh^2 + \ldots)\left(\frac{1}{\varphi(x)} - \frac{A'h}{\{\varphi(x)\}^2} + \text{terms in } h^2 \ldots\right)$$
$$= \frac{f(x)}{\varphi(x)} + \left\{\frac{A}{\varphi(x)} - \frac{A'f(x)}{\{\varphi(x)\}^2}\right\} h + Ph^2 + Qh^3 + \ldots$$
$$\therefore \frac{dy}{dx} = \frac{A}{\varphi(x)} - \frac{A'f(x)}{\{\varphi(x)\}^2}$$
$$= \frac{A\,\varphi(x) - A'f(x)}{\{\varphi(x)\}^2}$$

But A is the first differential co-efficient of $f(x)$, and A$'$ is the first differential co-efficient of $\varphi(x)$, hence it appears, that, in order to obtain the first differential co-efficient of a fractional function, we must

Multiply the first differential co-efficient of the numerator by the denominator, subtract from this the product of the first differential co-efficient of the denominator multiplied by the numerator, and divide the whole by the square of the denominator.

Ex. 1. Let $y = \frac{u^3}{z^2}$ where u and z are functions of x

$$\frac{dy}{dx} = \frac{3u^2 z^2 - 2u^3 z}{z^4} = \frac{3u^2 z - 2u^3}{z^3}$$

2. $y = \frac{u}{z}$ where u and z are functions of x

$$\frac{dy}{dx} = \frac{z - u}{z^2}$$

3. $y = \tan. x$

$$= \frac{\sin. x}{\cos. x}$$
$$\frac{dy}{dx} = \frac{\cos.^2 x + \sin.^2 x}{\cos.^2 x} = \frac{1}{\cos.^2 x} = \sec.^2 x$$

4. $y = \cot. x$

$$= \frac{\cos. x}{\sin. x}$$
$$= -\frac{\sin.^2 x + \cos.^2 x}{\sin.^2 x} = -\frac{1}{\sin.^2 x} = -\operatorname{cosec.}^2 x$$

5. $y = \sec. x$

$$= \frac{1}{\cos. x}$$
$$\frac{dy}{dx} = \frac{\sin. x}{\cos.^2 x} = \frac{\sin. x}{1 - \sin.^2 x} = \tan. x \sec. x$$

$y = \operatorname{cosec.} x$

$$= \frac{1}{\sin. x}$$
$$\frac{dy}{dx} = -\frac{\cos. x}{\sin.^2 x} = -\frac{\cos. x}{1 - \cos.^2 x} = -\cot. x \operatorname{cosec.} x$$

4. Let $y = f(z)$ where $z = \phi(x)$, it is required to find the first differential co-efficient of y considered as a function of x

When x becomes $x + h$ let z become $z + k$

$$\therefore \quad y' = f(z + k)$$
$$= f(z) + Ak + Bk^2 + \ldots \ldots \ldots \ldots \ldots (1)$$

But

$$z + k = \phi(x + h)$$
$$= \phi(x) + A'h + B'h^2 + \ldots$$
$$\therefore \quad k = A'h + B'h^2 + \ldots \qquad \because z = \phi(x)$$

Substitute this value of k in equation (1)

$$y' = f(z) + A(A'h + B'h^2 + \ldots) + B(A'h + B'h^2 + \ldots)^2 + \ldots$$
$$= f(z) + AA'h + Ph^2 + Qh^3 + \ldots$$
$$\therefore \quad \frac{dy}{dx} = AA'$$

But A is the first differential co-efficient of y considered as a function of z, and A′ is the first differential of z considered as a function of x, hence we have the first differential co-efficient of y considered as a function of x, or

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

This theorem will be found of great use in differentiating many complicated functions, thus,

Ex. 1. Let $y = (ax^3 + bx^2 + cx + d)^6$

Put $z = ax^3 + bx^2 + cx + d$

$$\therefore \quad y = z^6$$

And by the theorem just established

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

But since $y = z^6$

$$\therefore \quad \frac{dy}{dz} = 6z^5$$

And since $z = ax^3 + bx^2 + cx + d$

$$\therefore \quad \frac{dz}{dx} = 3ax^2 + 2bx + c$$

$$\therefore \quad \frac{dy}{dz} \cdot \frac{dz}{dx} = 6z^5(3ax^2 + 2bx + c)$$

$\therefore$ Substituting for z its value we have

$$\frac{dy}{dx} = 6(ax^3 + bx^2 + cx + d)^5 \cdot (3ax^2 + 2bx + c)$$

Ex. 2. Let $y = \sin.^n x$

Put $z = \sin. x \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$

$$\therefore \quad y = z^n$$

$$\therefore \quad \frac{dy}{dz} = nz^{n-1}$$

And from (1) $\frac{dz}{dx} = \cos x \therefore \frac{dy}{dz} \cdot \frac{dz}{dx} = nz^{n-1} \cos x$;

$$\therefore \frac{dy}{dx} = nz^{n-1} \cos x = n \sin^{n-1} x \cos x.$$

Ex. 3. Let $y = (a^x)^{\frac{r}{q}} = a^{\frac{rx}{q}}$.

Put $a^x = z$, then $y = z^{\frac{r}{q}}$, and, therefore, we have

$$\frac{dy}{dz} = \frac{r}{q} z^{\frac{r}{q}-1} \text{ and } \frac{dz}{dx} = p \cdot a^x = \log. a \cdot a^x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{r}{q} z^{\frac{r}{q}-1} \log. a \cdot a^x = \frac{r}{q} (a^x)^{\frac{r}{q}-1} \log. a \cdot a^x$$

$$= \frac{r \log. a}{q} \cdot a^{\frac{rx}{q}}.$$

Ex. 4. Let $y = (\log. x)^m$.

Put $z = \log. x$; then $y = z^m$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = mz^{m-1} \cdot \frac{1}{\log. a} \cdot \frac{1}{x} = \frac{m (\log. x)^{m-1}}{\log. a \cdot x}.$$

The differential coefficients of many complicated functions may be found by first taking the Napierian logarithms of the functions.

Ex. 5. Let $y = x (a+x) (b+2x)$.

Taking the logs., and putting $z = \log. y$, we have

$$z = \log. y$$

$$z = \log. x + \log. (a+x) + \log. (b+2x)$$

$$\therefore \frac{dz}{dy} = \frac{1}{y}, \text{ or } \frac{dy}{dz} = y, \text{ and } \frac{dz}{dx} = \frac{1}{x} + \frac{1}{a+x} + \frac{2}{b+2x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = y\left(\frac{1}{x} + \frac{1}{a+x} + \frac{2}{b+2x}\right)$$

$$= x (a+x) (b+2x) \left(\frac{1}{x} + \frac{1}{a+x} + \frac{2}{b+2x}\right)$$

$$= (a+x) (b+2x) + x (b+2x) + 2x (a+x)$$

$$= ab + (4a+2b) x + 6x^2.$$

5. It frequently happens that in the equation

$$y = f(x)$$

$f(x)$ is of such a complicated form that, in order to find its first differential co-efficient, it is necessary to simplify it by the substitution of two new variables, u, z, each of which is, of course, a function of x. Hence arises the following problem :

If $y = \mathrm{F}(u, z,)$ where $u = \phi_{\prime}(x)$, $z = \psi(x)$ .... (1)

required the first differential co-efficient of y, considered as a function of x.

In order to discover this, it is manifest that we must substitute $x + h$ for x in each of the two functions u, and z, and find the co-efficient of the simple power of h in the developement of the compound function $F(u, z)$.

When x becomes $x + h$ let u become $u + k$, and let z become $z + l$.

Then $\quad u'$ or $u + k = \phi(x + h)$
$$= \phi(x) + A'h + B'h^2 + \ldots$$

But $u = \phi(x) \quad \therefore \quad k = A'h + B'h^2 + \ldots\ldots\ldots \quad (2)$

Again, $\quad z'$ or $z + l = \psi(x + h)$
$$= \psi(x) + A''h + B''h^2 + \ldots$$

But $z = \psi(x) \quad \therefore \quad l = A''h + B''h^2 + \ldots\ldots\ldots \quad (3)$

It now remains for us to substitute $u + k$, $z + l$, for u and z in $F(u, z)$, but it is manifest that if we make these two substitutions in succession, we shall obtain the same result as if we make them both at once, since u and z are considered altogether independent of each other in these substitutions.

Let us then, in the first instance, suppose that u becomes $u + k$, and that z remains constant in the equation $y = F(u, z)$.

$$\therefore \quad y' \text{ or } F(u + k, z) = F(u, z) + A_1 k + B_1 k^2 + \ldots\ldots \quad (4)$$

Let us now suppose that z becomes $z + l$, and that u remains constant, then

$F(u, z)$ becomes $F(u, z + l) = F(u, z) + A_2 l + B_2 l^2 + \ldots\ldots \quad (5)$

Since $A_1 k$ involves z, it now becomes a function of $z + l$, and being expanded as a function of $(A_1 k + l)$

$A_1 k$ becomes $= A_1 k +$ terms in kl, hl^2, $k^2 l \ldots$

$B_1 k^2 \ldots \quad = B_1 k^2 +$ terms in $k^2 l, \ldots$

Substitute then these values of $F(u, z)$, $A_1 k$, $B_1 k^2$, in (4), and we have

y' or $F(u + k, z + l) = F(u, z) + A_1 k + A_2 l +$ terms in $hl, k^2, l^2, \ldots$

Substitute for k and l their values from (2) and (3)

$$\therefore \; y' = F(u, z) + A_1(A'h + B'h^2 + \ldots) + A_2(A''h + B''h^2 + \ldots) + \ldots$$

Arranging according to powers of h

$$= F(u, z) + (A_1 A' + A_2 A'')h + Ph^2 + Qh^3 + \ldots$$

Hence by definition

$$\frac{dy}{dx} = A_1 A' + A_2 A''$$

But it is evident from (2), (3), (4), (5) that

$$A' = \frac{du}{dx}, \; A'' = \frac{dz}{dx}, \; A_1 = \frac{dy}{du}, \; A_2 = \frac{dy}{dz}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} + \frac{dy}{dz} \cdot \frac{dz}{dx}$$

In like manner it might be proved that if

$y = f(t, u, z)$ where $t = F(x)$, $u = \phi(x)$, $z = \psi(x)$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} + \frac{dy}{du} \cdot \frac{du}{dx} + \frac{dy}{dz} \cdot \frac{dz}{dx}$$

and so for any number of functions.

Hence we deduce the following general conclusion:

The first differential co-efficient of a function composed of different particular functions, will be sum of the first differential co-efficients of each of these functions considered separately and independent of each other, according to the rule established in the last article.

This principle, combined with the preceding one, will enable us to determine the first differential co-efficients of all functions of one variable, however complicated in form.

Ex. 1. Let $y = (ax^3 + bx^2 + cx + d)^m \ (ex^4 + nx^5 + rx^7)^n$.

$$\text{Let } u = ax^3 + bx^2 + cx + d$$
$$z = ex^4 + nx^5 + rx^7$$
$$\therefore \quad y = u^m z^n$$
$$\frac{du}{dx} = 3ax^2 + 2bx + c$$
$$\frac{dz}{dx} = 4ex^3 + 5nx^4 + 7rx^6$$
$$\frac{dy}{du} = mu^{m-1} z^n$$
$$\frac{dy}{dz} = nz^{n-1} u^m$$
$$\therefore \quad \text{But } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} + \frac{dy}{dz} \cdot \frac{dz}{dx}$$
$$= mu^{m-1} z^n (3ax^2 + 2bx + c) + nz^{n-1} u^m (4ex^3 + 5nx^4 + 7rx^6)$$

Substituting for u and z their values

$$= m(ax^3 + bx^2 + cx + d)^{m-1} (ex^4 + nx^5 + rx^7)^n (3ax^2 + 2bx + c)$$
$$+ n(ex^4 + nx^5 + rx^7)^{n-1} (ax^3 + bx^2 + cx + d)^m (4ex^3 + 5nx^4 + 7rx^6$$

Ex. 2. Let $y = \sin.^m x \cos.^n x$

$$\text{Let } u = \sin. x \quad \therefore \quad \frac{du}{dx} = \cos. x$$
$$z = \cos. x \quad \therefore \quad \frac{dz}{dx} = -\sin. x$$
$$\therefore \quad y = u^m z^n$$
$$\frac{dy}{du} = mu^{m-1} z^n$$
$$\frac{dy}{dz} = nz^{n-1} u^m$$
$$\text{But} \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} + \frac{dy}{dz} \cdot \frac{dz}{dx}$$
$$= mu^{m-1} z^n \cos. x - nz^{n-1} u^m \sin. x$$
$$= m \sin.^{m-1} x \cos.^n x \cos. x - n \cos.^{n-1} x \sin.^m x \sin. x$$
$$= m \sin.^{m-1} x \cos.^{n+1} x - n \cos.^{n-1} x \sin.^{m+1} x$$
$$= \sin.^{m-1} x \cos.^{n-1} x (m \cos.^2 x - n \sin.^2 x)$$

Ex. 3. Let $y = \sqrt[4]{\left\{a - \frac{b}{\sqrt{x}} + \sqrt[3]{c^2 - x^2}\right\}}$

$$\text{Let } u = a - \frac{b}{\sqrt{x}}$$

$$z = \sqrt[3]{c^2 - x^2}$$

$$y = (u + z)^{\frac{1}{4}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} + \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$- \frac{\dfrac{2x}{\sqrt[3]{(c^2 - x^2)^2}}}{4\sqrt[4]{a - \dfrac{b}{\sqrt{x}} + \sqrt[3]{(c^2 - x^2)}}}$$

By dint of practice, however, the student will be able to obtain the first differential co-efficients of all functions without actually performing the process of substitution.

On finding the differential co-efficients of equations.

We have hitherto supposed the function x to be given under the form

$$y = f(x)$$

but it frequently happens that y is given only by an equation between x and y of the form

$$F(x, y) = 0$$

The resolution indeed of this equation for y would give us y under the form

$$y = f(x)$$

but this solution is seldom possible, and wholly unnecessary for our present purpose.

If the equation, then be of the form

$$F(x, y) = 0$$

and if we suppose $y = f(x)$ to be the value of y, which would be obtained from the solution of the equation, it is manifest that if we substitute this value for y in the proposed equation, we shall arrive at an identical equation

$$u \text{ or } F\{x, f(x)\} = 0$$

whatever may be the value of x, and hence, if we substitute $x + h$ for x, the equation will still be identically $= 0$, whatever may be the value of h.

Substituting $\therefore$ $x + h$ for h

$$u' = u + Ah + Bh^2 + \ldots$$

whatever be the value of h, hence necessarily each individual term must be

$$= 0, \text{ and } \therefore u = 0, Ah = 0, Bh^2 = 0 \ldots.$$

But since

$$u = F(x, y)$$

We have by article (5) chap. III.

$$\frac{du}{dx} = \frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\dfrac{du}{dx}}{\dfrac{du}{dy}}$$

Ex. 1. Take the equation

$$x^2 + y^2 - 2rx = r^2$$

or $$u = x^2 + y^2 - 2rx - r^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\therefore \quad \frac{du}{dx} = 2x - 2r$$

$$\frac{du}{dy} = 2y$$

But $$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}}$$

$$= \frac{r - x}{y}$$

In equation (1) the differential co-efficient $\frac{du}{dx}$ was found upon the supposition that x was variable and y constant, and then the differential co-efficient $\frac{du}{dy}$ was taken upon the supposition that y was variable and x constant, hence $\frac{du}{dx}, \frac{du}{dy}$ are called *Partial Differential Co-efficients.*

Ex. 2. Let the proposed equation be

$$x^4 + 2ax^2y = ay^3$$

or $$u = x^4 + 2ax^2y - ay^3 = 0$$

$$\frac{du}{dx} = 4x^3 + 4axy$$

$$\frac{du}{dy} = 2ax^2 - 3ay^2$$

But $$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}}$$

$$= -\frac{4x^3 + 4axy}{2ax^2 - 3ay^2}$$

CHAPTER IV.

ON FINDING THE SUCCESSIVE DIFFERENTIAL CO-EFFICIENTS OF FUNCTIONS OF ONE VARIABLE.

The first differential co-efficient of any function is itself a new function o the variable, and consequently its differential co-efficient may be found according to principles already explained. This differential co-efficient of a differential co-efficient is called *the second differential co-efficient* of the original function, and if the first differential co-efficient be expressed by the symbol $\frac{dy}{dx}$ the second differential co-efficient is represented by $\frac{d^2y}{dx^2}$

In like manner this second differential co-efficient is itself a new function of the variable and its differential co-efficient may be found, this is called *the third differential co-efficient* of the original function, and is represented by the symbol $\frac{d^3y}{dx^3}$

Proceeding in the same manner, the differential co-efficient of $\frac{d^3y}{dx^3}$ is called *the fourth differential co-efficient* of the original function, and is written $\frac{d^4y}{dx^4}$, so also we shall have $\frac{d^5y}{dx^5}$, $\frac{d^6y}{dx^6}$, and so on to any extent.

Thus if

Ex. 1. $\quad y = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + gx + m$

The first differential co-efficient is

$$\frac{dy}{dx} = 6ax^5 + 5bx^4 + 4cx^3 + 3dx^2 + 2ex + g \quad\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In order to find the second differential co-efficient of y, we must take the first differential co-efficient of this new function (1), which will be

$$5 \,.\, 6 \,.\, ax^4 + 4 \,.\, 5\, bx^3 + 3 \,.\, 4\, cx^2 + 2 \,.\, 3\, dx + 1 \,.\, 2 \,.\, e$$

$$\therefore \quad \frac{d^2y}{dx^2} = 5 \,.\, 6 \,.\, ax^4 + 4 \,.\, 5\, bx^3 + 3 \,.\, 4\, cx^2 + 2 \,.\, 3 \,.\, dx + 1 \,.\, 2 \,.\, e \ldots (2)$$

Taking the first differential co-efficient of this new function (2), we shall have

$$\frac{d^3y}{dx^3} = 4 \,.\, 5 \,.\, 6 \,.\, ax^3 + 3 \,.\, 4 \,.\, 5\, bx^2 + 2 \,.\, 3 \,.\, 4 \,.\, cx + 1 \,.\, 2 \,.\, 3\, d$$

In like manner

$$\frac{d^4y}{dx^4} = 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, ax^2 + \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5\, bx + 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, c$$

$$\frac{d^5y}{dx^5} = 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, ax + 1 \,.\, 2 \,.\, 3 \;\; 4 \,.\, 5\, b$$

$$\frac{d^6y}{dx^6} = 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, a$$

$$\frac{d^7y}{dx^7} = 0$$

Ex. 2. Let $y = x^m$

$$\frac{dy}{dx} = m\, x^{m-1}$$

$$\frac{d^2y}{dx^2} = m\,(m-1)\, x^{m-2}$$

$$\frac{d^3y}{dx^3} = m\,(m-1)\,(m-2)\, x^{m-3}$$

$$\frac{d^4y}{dx^4} = m\,(m-1)\,(m-2)\,(m-3)\, x^{m-4}$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots$$

$$\frac{d^py}{dx^p} = m\,(m-1)\,(m-2)\,(m-3)\,\ldots\ldots\,(m-p+1)\, x^{m-p}$$

Ex. 3. $y = a^x$

$$\frac{dy}{dx} = p \cdot a^x, \text{ where } p = \log._e a$$

$$\frac{d^2y}{dx^2} = p^2 a^x$$

$$\frac{d^3y}{dx^3} = p^3 \cdot a^x$$

$$\ldots = \ldots\ldots$$

$$\frac{d^ny}{dx^n} = p^n a^x$$

Ex. 4. $y = \log. x$

$$\frac{dy}{dx} = \frac{1}{p} \cdot \frac{1}{x} = \frac{1}{p} x^{-1}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{p} x^{-2}$$

$$\frac{d^3y}{dx^3} = \frac{1 \cdot 2}{p} x^{-3}$$

$$\ldots = \ldots\ldots\ldots$$

$$\frac{d^ny}{dx^n} = \pm \frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1)}{p} x^{-n}$$

where the sign will be $+$ or $-$ according as n is odd or even.

Ex. 5. $y = \sin. x$

$$\frac{dy}{dx} = \cos. x$$

$$\frac{d^2y}{dx^2} = - \sin. x$$

$$\frac{d^3y}{dx^3} = - \cos. x$$

$$\frac{d^4y}{dx^4} = \sin. x$$

$$\frac{d^5y}{dx^5} = \cos. x$$

If n be an odd number

$$\frac{d^ny}{dx^n} = \pm \cos. x, + \text{ when } \frac{n-1}{2} \text{ is even}, - \text{ when } \frac{n-1}{2} \text{ is odd.}$$

If n be an even number

$$\frac{d^ny}{dx^n} = \pm \sin. x, + \text{ when } \frac{n}{2} \text{ is even}, - \text{ when } \frac{n}{2} \text{ is odd.}$$

In like manner, if

$$y = \cos. x$$

If n be an odd number

$$\frac{d^ny}{dx^n} = \pm \sin. x, + \text{ when } \frac{n-1}{2} \text{ is odd}, - \text{ when } \frac{n-1}{2} \text{ is even}$$

If n be an even number

$$\frac{d^ny}{dx^n} = \pm \cos. x, + \text{ when } \frac{n}{2} \text{ is even}, - \text{ when } \frac{n}{2} \text{ is odd.}$$

CHAPTER V.

ON INVERSE FUNCTIONS.

In the preceding trigonometrical expressions, the sines, cosines, &c., have been considered as functions of the arcs; but we shall now treat of the *inverse functions*, and consider the arcs as functions of the sine, cosine, &c., and investigate their differential coefficients.

A peculiar notation has been adopted to distinguish inverse functions.

The arc whose sine is x, is represented by the symbol .... $\sin^{-1}x$;
the arc whose cosine is x .... $\cos^{-1}x$;
the arc whose tangent is x .... $\tan^{-1}x$;
the number whose log is x .... $\log^{-1}x$

Ex. 1. Let $y = \sin^{-1}x$.

Here the *direct function* is $x = \sin y$; and, therefore,

$$\frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad (1.)$$

Ex. 2. Let $y = \cos^{-1}x$.

Then $x = \cos y$, and $\frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} \qquad (2.)$$

Ex. 3. Let $y = \tan^{-1}x$.

Then $x = \tan y$, and $\frac{dx}{dy} = \sec^2 y = 1+\tan^2 y = 1+x^2$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2} \qquad (3.)$$

Ex. 4. Let $y = \cot^{-1}x$.

Then $x = \cot y \therefore \frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1+\cot^2 y) = -(1+x^2)$

$$\therefore \frac{dy}{dx} = -\frac{1}{1+x^2} \qquad (4.)$$

Ex. 5. Let $y = \sec^{-1}x$.

Then $x = \sec y$, and $\frac{dx}{dy} = \tan y \sec y = \sec y\sqrt{\sec^2 y-1} = x\sqrt{x^2-1}$

$$\therefore \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \qquad (5.)$$

Ex. 6 Let $y = \operatorname{cosec}^{-1}x$.

Then $x = \operatorname{cosec} y \therefore \frac{dx}{dy} = -\cot y \operatorname{cosec} y = -x\sqrt{x^2-1}$

$$\therefore \frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}} \qquad (6.)$$

Ex. 7. Let $y = \text{vers}^{-1}x$.

Then $x = \text{vers}\, y \therefore \dfrac{dx}{dy} = \sin y = \sqrt{1-\cos^2 y} = \sqrt{1-(1-\text{vers}\, y)^2}$

$= \sqrt{2\,\text{vers}\, y - \text{vers}^2 y} \quad = \sqrt{2x-x^2}$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{2x-x^2}} \quad\ldots\ldots\ldots\ldots \quad (7.)$$

Ex. 8. Let $y = \text{chd}^{-1}x$.

Then $x = \text{chd}\, y = 2\sin\frac{1}{2}y \therefore \dfrac{dx}{dy} = \cos\frac{1}{2}y = \sqrt{1-\sin^2\frac{1}{2}y}$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2\frac{1}{2}y}} = \frac{1}{\sqrt{1-\frac{1}{4}x^2}} = \frac{2}{\sqrt{4-x^2}} \quad\ldots\ldots\ldots \quad (8.)$$

In the preceding expressions the radius of the arc is unity; but they may be readily adapted to radius r, by considering that $\frac{y}{x}$ and $\frac{dy}{dx}$ are numbers; therefore the numerator and denominator of each differential coefficient must be of the same dimensions. Hence, to radius r the formulas now investigated are as under.

$y = \sin^{-1}x \therefore \dfrac{dy}{dx} = \dfrac{r}{\sqrt{r^2-x^2}}$	$y = \cot^{-1}x \therefore \dfrac{dy}{dx} = -\dfrac{r^2}{r^2+x^2}$
$y = \cos^{-1}x \quad \dfrac{dy}{dx} = -\dfrac{r}{\sqrt{r^2-x^2}}$	$y = \sec^{-1}x \quad \dfrac{dy}{dx} = \dfrac{r^2}{x\sqrt{x^2-r^2}}$
$y = \text{vers}^{-1}x \quad \dfrac{dy}{dx} = \dfrac{r}{\sqrt{2rx-x^2}}$	$y = \text{cosec}^{-1}x \quad \dfrac{dy}{dx} = -\dfrac{r^2}{x\sqrt{x^2-r^2}}$
$y = \tan^{-1}x \quad \dfrac{dy}{dx} = \dfrac{r^2}{r^2+x^2}$	$y = \text{chd}^{-1}x \quad \dfrac{dy}{dx} = \dfrac{2r}{\sqrt{4r^2-x^2}}$

We may now investigate the differential coefficients of a few of the more complicated inverse functions, as in the following examples:

Ex. 1. Let $y = \text{cosec}^{-1}\dfrac{\sqrt{1+x^2}}{x}$.

Put $\dfrac{\sqrt{1+x^2}}{x} = z$; then $y = \text{cosec}^{-1}z$, or $z = \text{cosec}\, y$.

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = -\frac{1}{z\sqrt{z^2-1}}\cdot -\frac{1}{x^2\sqrt{1+x^2}} = \frac{1}{1+x^2}.$$

Ex. 2. Let $y = \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{1+2x}{\sqrt{3}}\right)$.

Put $z = \dfrac{1+2x}{\sqrt{3}}$; then $y = \dfrac{2}{\sqrt{3}}\tan^{-1}z$.

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{2}{\sqrt{3}}\cdot\frac{1}{1+z^2}\cdot\frac{2}{\sqrt{3}} = \frac{4}{3}\cdot\frac{1}{1+z^2} = \frac{1}{1+x+x^2}.$$

Examples for practice on the above rules.

1. $y = ax^n - bx^{n-1} + cx^{n-2} + e$

$\frac{dy}{dx} = nax^{n-1} - (n-1)\,bx^{n-2} + (n-2)\,cx^{n-3}$

2. $y = (a + bx)^n$

$\frac{dy}{dx} = nb\,(a + bx)^{n-1}$

3. $y = (a + bx)^2\,(a' + b'x)^3$

$\frac{dy}{dx} = 2b\,(a + bx)\,(a' + b'x)^3 + 3b'\,(a + bx)^2\,(a' + b'x)^2$

4. $y = \frac{x}{1 + x}$

$\frac{dy}{dx} = \frac{1}{(1 + x)^2}$

5. $y = \frac{x^n}{(1 + x)^n}$

$\frac{dy}{dx} = \frac{nx^{n-1}}{(1 + x)^{n+1}}$

6. $y = \frac{a + x}{b + x}$

$\frac{dy}{dx} = -\frac{a - b}{(b + x)^2}$

7. $y = \sqrt{\left(\frac{a + x}{a - x}\right)\left(\frac{b + x}{b - x}\right)}$

$\frac{dy}{dx} = \frac{(a + b)\,(ab - x^2)}{(a - x)^{\frac{3}{2}}\,(b - x)^{\frac{3}{2}}\,\sqrt{(a + x)\,(b + x)}}$

8. $y = \frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}$

$\frac{dy}{dx} = -\frac{2x}{\sqrt{1 - x^4}\,(1 - \sqrt{1 - x^4})}$

9. $y = \log.\ \frac{1 + x}{1 - x}$*

$\frac{dy}{dx} = \frac{2}{1 - x^2}$

10. $y = \log.\ \frac{x + \sqrt{1 - x^2}}{x - \sqrt{1 - x^2}}$

$\frac{dy}{dx} = \frac{-2}{(2x^2\ \text{[illegible]})\sqrt{1\ \text{[illegible]}}}$

* We suppose the logarithms to be taken in the system where $p = 1$.

$$y = -\frac{a\sqrt{a^2+x^2}}{x^2} + \log.\frac{a+\sqrt{a^2+x^2}}{x}$$

$$\frac{dy}{dx} = \frac{2a^3}{x^3\sqrt{a^2+x^2}}$$

12 $y = \log.^2 x$*

$$\frac{dy}{dx} = \frac{1}{x\log. x}.$$

13. $y = \log.^n x.$

$$\frac{dy}{dx} = \frac{1}{x\log. x\log.^2 x\log.^3 x \ldots \log.^{n-1} x}$$

14. $y = \frac{e^x}{1+x}$

$$\frac{dy}{dx} = \frac{x.\,e^x}{(1+x)^2}$$

15. $y = \log.\frac{e^x-1}{e^x+1}$

$$\frac{dy}{dx} = \frac{2e^x}{e^{2x}-1}$$

16. $y = \sqrt{x+\sqrt{x+\sqrt{x+\&c.}}}$

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{4x+1}}$$

17. $y = \sqrt{\operatorname{cosec.} x.}$

$$\frac{dy}{dx} = -\frac{\cos. x}{2\sin.^{\frac{3}{2}} x}$$

18. $y = \log.\left(\frac{1+\sqrt{-1}\tan. x}{1-\sqrt{-1}\tan. x}\right)^n$

$$\frac{dy}{dx} = 2n\sqrt{-1}$$

19. $y = \sin.^{-1}\frac{1-x^2}{1+x^2}$

$$\frac{dy}{dx} = -\frac{2}{1+x^2}$$

20. $y = \cot.^{-1}\sqrt{\frac{1-x}{x}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x-x^2}}$$

* This signifies log. log. x, or the logarithm of the logarithm of x. So $\log.^3 x$ is log. log. log. x, or the logarithm of the logarithm of the logarithm of x. Any generally $\log.^n x =$ log. log. log. . . . n terms, x

CHAPTER VI.

ON THE GENERAL FORM OF THE DEVELOPEMENT OF $f(x+h)$.

If $f(x)$ be any function of x, and if we substitute $x+h$ for x, where h is any indeterminate quantity; and if we develope $f(x+h)$ in a series according to powers of h; then, so long as no particular value is assigned to x.

1. The series can contain no negative or fractional powers of h.

2. The series will be of the form

$$f(x+h) = fx + \mathrm{P} \, . \, h + \mathrm{Q} \, . \, h^2 + \mathrm{R} \, . \, h^3 + \ldots$$

where P, Q, R, &c. are functions of x only.

Let $$f(x+h) = \mathrm{A}h^{\alpha} + \mathrm{B}h^{\beta} + \mathrm{C}h^{\gamma} + \ldots$$

1. The exponents $\alpha, \beta, \gamma, \ldots$ must be positive.

For, if any term such as $\mathrm{T}h^{-\mu}$ could enter the expansion, then on the supposition that $h = 0$.

$$f(x) = \infty$$

$$\frac{1}{f(x)} = 0$$

Hence x has some determinate value which renders $\dfrac{1}{f(x)} = 0$ which is contrary to the hypothesis.

The exponents must be integral.

For, if any term, such as $\mathrm{T}h^{\frac{\lambda}{\mu}}$ could enter the expansion, since $h^{\frac{\lambda}{\mu}}$ is susceptible of μ values, we must have μ values of $(x+h)$; but while x remains indeterminate, $f(x)$ must contain the same radicals as $f(x+h)$; and $\therefore f(x)$ must have μ values: substituting $\therefore$ the μ values of $f(x)$ successively in the values of $f(x+h)$, we shall have in the whole μ^2 values of $f(x+h)$. Thus $f(x+h)$ when developed will have μ^2 values, and when not developed it can only have the same number of values as $f(x)$, *i. e.* μ values which is impossible, except for particular values of x.

Next, to ascertain the form of the developement of $f(x+h)$

If we wish to ascertain what part of this function is independent of h, we have only to make $h = 0$, which reduces it to $f(x)$, so that $f(x+h) = f(x)$ + a quantity which disappears when $h = 0$, and which must $\therefore$ be multiplied by some positive power of h, and since no fractional power of h can enter, this quantity must be of the form $\mathrm{A}h$, A being a function of h and x, which does not become infinite when $h = 0$, thus we have

$$f(x+h) = f(x) + \mathrm{A} \, . \, h$$

$$\therefore \quad f(x+h) - f(x) = \mathrm{A} \, . \, h, \text{ and } \therefore \text{ divisible by } h$$

$$\therefore \quad \mathrm{A} = \frac{f(x+h) - f(x)}{h}$$

But since A is a new function of x and h, we may in like manner separate the part which is independent of h.

Let P be what A becomes when $h = 0$, then P is a function of x alone, and reasoning as above we prove

$$A = P + Bh$$

Bh being the part of A which $= 0$ when $h = 0$

$$\therefore \quad A - P = Bh$$

$$B = \frac{A - P}{h}$$

and proceeding as before to separate the parts of this new function of x, we find

$$B = Q + Ch$$

Then we have

$$f(x + h) = f(x) + Ah$$

$$A = P + Bh$$

$$\therefore \quad f(x + h) = f(x) + P \cdot h + Bh^2$$

$$B = Q + Ch$$

$$\therefore \quad f(x + h) = f(x) + P \cdot h + Q \cdot h^2 + Ch^3$$

when P, Q are functions of x alone, and proceeding in this manner, we get $f(x + h)$ developed in the required form.

PROP.

If the variable of a function be supposed to consist of two parts, the differential co-efficient will be the same to whichever part the variation be ascribed.

Let $\quad y = f(z)$ where $z = x + h$

Then it is required to prove that the differential co-efficient will be the same when we consider x variable and h constant, as it will be when we consider h variable and x constant.

1. If we consider x variable, then by art. 4. cap. III.

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \quad \text{.................................} \quad (1)$$

2. If we consider h variable and x constant, then

$$\frac{dy}{dh} = \frac{dy}{dz} \cdot \frac{dz}{dh} \quad \text{.................................} \quad (2)$$

But since

$$z = x + h$$

Upon the first supposition

$$\frac{dz}{dx} = 1$$

Upon the second supposition

$$\frac{dz}{dh} = 1$$

Hence, comparing (1) (2)

$$\frac{dy}{dx} = \frac{dy}{dh}$$

Having fully proved that if $y = f(x)$, and that if x become $x + h$, so long as x remains indeterminate we shall have

$$f(x+h) = f(x) + \mathrm{P}h + \mathrm{Q}h^2 + \mathrm{R}h^3 + \ldots \ldots\ldots \quad (1)$$

where P, Q, R, . . . are functions of x and do not involve h

Let us take the first differential co-efficient of $f(x+h)$ upon the supposition that x is constant and h variable

$$\frac{d\,f(x+h)}{dh} = \mathrm{P} + 2\mathrm{Q}h + 3\mathrm{R}h^2 + \ldots\ldots\ldots \quad (2)$$

Again, take the first differential co-eflicient of $f(x+h)$ upon the supposition that x is variable and h constant

$$\frac{d\,f(x+h)}{dx} = \frac{d\,f(x)}{dx} + \frac{d\mathrm{P}}{dx}\cdot h + \frac{d\mathrm{Q}}{dx}h^2 + \frac{d\mathrm{R}}{dx}h^3 + \ldots\ldots \quad (3)$$

But by the last article

$$\frac{d\,f(x+h)}{dh} = \frac{d\,f(x+h)}{dx}$$

comparing therefore the homologous terms in the identical series (2) and (3)

$$\mathrm{P} = \frac{d\,f(x)}{dx} \text{ or } \frac{dy}{dx}$$

$$\mathrm{Q} = \tfrac{1}{2}\frac{d\mathrm{P}}{dx} = \tfrac{1}{2}\frac{d^2y}{dx^2}$$

$$\mathrm{R} = \tfrac{1}{3}\frac{d\mathrm{Q}}{dx} = \frac{1}{1\,.\,2\,.\,3}\frac{d^3y}{dx^3}$$

Substituting these values of P, Q, R, ... in (1)

$$f(x+h) = f(x) + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \ldots$$

$$\text{or} \quad y' = y + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \ldots$$

which is the series known by the name of TAYLOR'S THEOREM, perhaps the most important in the whole range of pure mathematics.

Sometimes for the sake of brevity the differential co-efficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, are represented by $p, q, r, \ldots$ respectively, in which case the series may be written

$$y' \text{ or } f(x+h) = y + p\,h + q\cdot\frac{h^2}{1\,.\,2} + r\cdot\frac{h^3}{1\,.\,2\,.\,3} + \ldots$$

According to the notation of Lagrange, the first differential co-efficient of $f(x)$, or as he designates it, *the first derived function* of $f(x)$ is represented by $f'(x)$, the second differential co-efficient by $f''(x)$, the third by $f'''(x)$, and so on, in his works, therefore, the above series appears under the form

$$f(x+h) = f(x) + f'(x)\frac{h}{1} + f''(x)\frac{h^2}{1\,.\,2} + f'''(x)\frac{h^3}{1\,.\,2\,.\,3} + \ldots$$

If h be negative, substituting $-h$ for $+h$, the series will become

$$f(x-h) = y - \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} - \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \ldots$$

in which the terms are alternately positive and negative.

TAYLOR'S THEOREM AND MACLAURIN'S THEOREM.

Let y be a function of x, which it is possible to develop in a series of positive ascending powers of that variable, and let us suppose that

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots\dots\dots \quad (1)$$

and when x becomes $x + h$, let y becoms y'; then we have

$$\begin{aligned} y' &= A + B(x+h) + C(x+h)^2 + D(x+h)^3 + \dots \\ &= \left.\begin{aligned} &A + Bx + Cx^2 + Dx^3 + \dots\dots \\ &+ Bh + 2Cxh + 3Dx^2h + \dots\dots \\ &+ Ch^2 + 3Dxh^2 + \dots\dots \\ &+ Dh^3 + \dots\dots \end{aligned}\right\} \dots\dots\dots\dots \quad (2) \end{aligned}$$

But by eq. (1) we have the first, second, &c. differential co-efficients, as follows:—

$$\left.\begin{aligned} \frac{dy}{dx} &= B + 2Cx + 3Dx^2 + 4Ex^3 + \dots\dots \\ \frac{d^2y}{dx^2} &= 1{\cdot}2C + 2{\cdot}3Dx + 3{\cdot}4{\cdot}Ex^2 + \dots\dots \\ \frac{d^3y}{dx^3} &= 1{\cdot}2{\cdot}3D + 2{\cdot}3{\cdot}4Ex + \dots\dots \\ \&c. & \qquad \&c. \end{aligned}\right\} \dots\dots\dots\dots \quad (3)$$

hence by multiplying these differential co-efficients respectively by h, $\frac{h^2}{1{\cdot}2}$, $\frac{h^3}{1{\cdot}2{\cdot}3}$, &c. and substituting the results in equation (2) we have finally

$$y' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1{\cdot}2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1{\cdot}2{\cdot}3} + \frac{d^4y}{dx^4}\cdot\frac{h^4}{1{\cdot}2{\cdot}3{\cdot}4} + \dots\dots\dots \quad (A)$$

Again, since the co-efficients A, B, C, D, &c. in (1) do not involve x, they will remain unchanged whatever value be assigned to x. Let then the particular values of y, and its successive differential co-efficients, be expressed by means of brackets, and when $x = 0$, we shall have by (1) and (3)

$$\begin{aligned} (y) &= A \\ \left(\frac{dy}{dx}\right) &= B \\ \left(\frac{d^2y}{dx^2}\right) &= 1{\cdot}2C \quad \therefore\ C = \frac{1}{2{\cdot}3}\left(\frac{d^2y}{dx^2}\right), \&c. \\ \left(\frac{d^3y}{dx^3}\right) &= 2{\cdot}3D \quad \therefore\ D = \frac{1}{2.3}\left(\frac{d^3y}{dx^3}\right), \&c. \end{aligned}$$

Hence by substitution in (1) we have

$$y = (y) + \left(\frac{dy}{dx}\right)x + \frac{1}{1{\cdot}2}\left(\frac{d^2y}{dx^2}\right)\cdot x^2 + \frac{1}{1{\cdot}2{\cdot}3}\cdot\left(\frac{d^3y}{dx^3}\right)x^3 + \dots\dots\dots \quad (B)$$

The former of these equations (A) is Taylor's theorem, and the latter (B) is Maclaurin's theorem; and the demonstrations we have given of these most important theorems will be readily comprehended by the student. We regret that room will not permit us to exemplify the latter of these theorems.

Cases in which Taylor's Theorem fails.

In the preceding demonstration of Taylor's theorem, we have supposed with regard to $f(x)$ that x remains indeterminate, and $\therefore$ that $f(x)$ has as many values as $f(x+h)$. But when we assign particular values to x, the above reasonings will not always hold good, and we shall not in all cases obtain the true expansion of $f(x+h)$ . . . .

1. Let $f(x) = \infty$ when $x = a$, then the expansion of $f(x+h)$ must contain negative powers of h.

For a will be determined from the equation

$$f(x) = \infty \qquad \text{or} \qquad \frac{1}{f(x)} = 0$$

$$\therefore \qquad \frac{1}{f(x)} = \frac{(x-a)^n}{\phi(x)}.$$

n being any positive whole number whatever, and $\phi(x)$ some function of x which does not become 0 or ∞ when $x = a$

$$\therefore \qquad fx = \frac{\phi(x)}{(x-a)^n}.$$

Then, putting $(a+h)$ for x

$$f(a+h) = \frac{\phi(a+h)}{h^n}$$

$$= \frac{1}{h^n}\left\{\phi(a) + \phi'(a).h + \phi''.(a)\frac{h^2}{1.2} + \dots\right\}$$

which contains negative powers of h.

2. Let $f(x)$ contain a radical which disappears when $x = a$.

In this case, either the radical itself must vanish when $x = a$, or its co-efficient must vanish.

If the radical itself vanish in $f(x)$ when $x = a$, it must be of the form $(x-a)^{\frac{m}{n}}$, m and n being whole numbers; hence $f(x+h)$ will contain the corresponding radical $(x-a+h)^{\frac{m}{n}}$, which, on making $x = a$ becomes $h^{\frac{m}{n}}$, so that the developement of $f(a+h)$ according to powers of h may contain the radical $h^{\frac{m}{n}}$ and its powers.

If the co-efficients of the radical vanish when $x = a$, then this co-efficient must be of the form $(x-a)^n$, n being a whole number, in this case the radical will disappear in the differential co-efficients $f'(a)$, $f''(a)$ . . . . $f^{n-1}(a)$, but will be found in those of higher orders. In general the following proposition will hold good:

When we assign a particular value to x in the developement of $f(x+h)$, if a term appear containing a fractional power of h which lies between h^n and h^{n+1}, then Taylor's theorem will hold good for the first n terms only.

Let

$$f(a+h) = A + Bh + Ch^2 + Dh^3 + \dots + Mh^n + Nh^{n+\frac{1}{2}} + Lh^{n+1} + \dots$$

Take the differential co-efficients regarding h as the variable, and let us denote the successive co-efficients by $f'(a+h)$, $f''(a+h)$, &c.

$$f'(a+h) = B + 2Ch + 3Dh^2 + \ldots + M'h^{n-1} + N'h^{n+\frac{1}{2}-1} + L'h^n + \ldots$$

$$f''(a+h) = 1.2.C + 2.3.Dh + \ldots + M''h^{n-2} + N''h^{n+\frac{1}{2}-2} + L''h^{n-1} + \ldots$$

$$\ldots = \ldots \ldots \ldots \ldots \ldots \ldots$$

$$f^n(a+h) = \qquad\qquad + M_, + N_,h^{\frac{1}{2}} + Lh + \ldots$$

Making $h = 0$, we find

$$f(a) = A, \quad f'(a) = B, \quad f''(a) = 1.2.C\ldots.$$

The co-efficients A, B, C, ... are the values which $f(x)$ and its differential co-efficients assume when $x = a$, precisely as in the series of Taylor. But at each differentiation the first term disappears because it is constant. When we arrive at the n^{th} differential co-efficient, on the supposition that $h = 0$,

$$f^n(a) = M,$$

but for the $(n+1)^{\text{th}}$ co-efficient

$$f^{n+1}(a+h) = N_{,,}h^{\frac{1}{2}-1}$$

and, since $\frac{1}{2}$ is $\angle$ 1

$$f^{n+1}(a+h) = \frac{N_{,,}}{h^{1-\frac{1}{2}}}$$

and therefore the supposition $h = 0$, gives

$$f^{n+1}(a) = \infty$$

and all the succeeding differential co-efficients will, in like manner, be infinite.

It only now remains for us to show how we can obtain the developement of $f(x+h)$ when Taylor's theorem fails.

If, then, we wish to obtain the developement of $f(x+h)$ when $x = a$, we must calculate the terms of the series

$$f(x) + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \ldots.$$

but if, in effecting this calculation, we find that one of the differential co-efficients becomes infinite upon the supposition that $x = a$, we must employ the following process.

Substitute $(x+h)$ for x in $f(x)$; then the term which contains $x - a$ in the denominator, will now contain $x - a + h$, and will no longer become infinite when $x = a$, but will become a term involving a fractional power of h.

For example, let

$$f(x) = 2ax - x^2 + a\sqrt{x^2 - a^2}$$

$$\therefore \qquad \frac{dy}{dx} = 2(a-x) + \frac{ax}{\sqrt{x^2 - a^2}}$$

Substitute this value and the values of $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c. in Taylor's theorem, we shall then find

$$f(x+h) = 2ax - x^2 + a\sqrt{a^2 - x^2} + \left\{2(a-x) + \frac{ax}{\sqrt{x^2 - a^2}}\right\}h + \ldots$$

But, when $x = a$ the term multiplied by h becomes infinite, hence Taylor's theorem fails and the developement is no longer possible.

But in the above case, since

$$f(x) = 2ax - x^2 + a\sqrt{x^2 - a^2}$$

According to the rule just given, substitute $x + h$ for x, then $f(x + h) = 2ax + 2ah - x^2 - 2xh - h^2 + a\sqrt{x^2 + 2xh + h^2 - a^2}$ which, upon the supposition that $x = a$ becomes

$$f(a + h) = a^2 - h^2 + a\sqrt{2ah + h^2}$$
$$= a^2 - h^2 + ah^{\frac{1}{2}}\sqrt{2a + h}$$

Expanding $\sqrt{2a + h}$ by the binomial, and representing the co-efficients by A, B, C.

$$\sqrt{2a + h} = (2a + h)^{\frac{1}{2}} = \mathrm{A} + \mathrm{B}h + \mathrm{C}h^2 + \mathrm{D}h^3 + \ldots$$

Substituting

$$f(a + h) = a^2 - h^2 + a\mathrm{A}h^{\frac{1}{2}} + a\mathrm{B}h^{\frac{3}{2}} + a\mathrm{C}h^{\frac{5}{2}} + \ldots$$

a series which gives the true developement of $f(a + h)$ but which does not proceed by integral powers of h.

CHAPTER VII.

APPLICATIONS OF THE DIFFERENTIAL CALCULUS.

ON THE THEORY OF VANISHING FRACTIONS.

When a fraction $\frac{\mathrm{P}}{\mathrm{Q}}$ both of whose terms are functions of x becomes $\frac{0}{0}$ when a particular value is assigned to the variable as $x = a$, it shows that $(x - a)$ is a common factor both of numerator and denominator, and in order to find the real value of the fraction, we must make this factor disappear from one or both terms.

$$\text{Let } \frac{f(x)}{\varphi(x)} = \frac{0}{0} \text{ when } x = a$$

$$\therefore \quad f(x) = \mathrm{F}(x)(x - a)^m$$
$$\varphi(x) = \psi(x)(x - a)^n$$

Let x become $a + h$

$$\therefore \quad \frac{f(a + h)}{\varphi(a + h)} = \frac{\mathrm{F}(a + h)\,h^m}{\psi(a + h)\,h^n}$$

$$\text{Hence if } m > n, \frac{f(a)}{\varphi(a)} = 0$$

$$\ldots\ldots m = n \quad \ldots = \frac{\mathrm{F}(a)}{\psi(a)}$$

$$\ldots\ldots m < n \ldots = \infty$$

Now, when Taylor's theorem can be applied to expand $f(a + h)$ and $\varphi(a + h)$, we have

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + f'(a)\cdot\frac{h}{1} + f''\cdot(a)\frac{h^2}{1\cdot 2} + \ldots}{\phi(a) + \phi'(a)\cdot\frac{h}{1} + \phi''(a)\frac{h^2}{1\cdot 2} + \ldots}$$

$$= \frac{f'(a) + f''(a)\frac{h}{1\cdot 2} + \ldots}{\phi'(a) + \phi''(a)\frac{h}{1\cdot 2} + \ldots} \text{ (since } f(a) = 0, \text{ and } \phi(a) = 0)$$

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}$$

If $f'(a) = 0$, $\phi'(a) = 0$, then, $\frac{f'(a)}{\phi'(a)} = \frac{f''(a)}{\phi''(a)}$ and so on

Hence the rule to find the value of a vanishing fraction.

Differentiate both terms of the fraction the same number of times, until one or other ceases to become 0, on the supposition that $x = a$. Then substitute a for x in both terms of the fraction, and the result will be the value required.

Ex. 1. The fraction $\frac{ax^2 + ac^2 - 2acx}{bx^2 - 2bcx + bc^2}$ becomes $\frac{0}{0}$ when $x = c$.

The first differential co-efficient

$\frac{dy}{dx} = \frac{ax - ac}{bx - bc}$ is likewise $\frac{0}{0}$ when $x = c$.

But the second differential co-efficient

$\frac{d^2y}{dx^2} = \frac{a}{b}$ the true value of the fraction.

It is necessary to take the second differential co-efficient, because the common factor of the two terms of the original fraction is $(x - c)^2$.

Ex. 2. $\frac{x^3 - ax^2 - a^2x + a^3}{x^2 - a^2}$ becomes $\frac{0}{0}$ when $x = a$.

$\frac{dy}{dx} = \frac{3x^2 - 2ax - a^2}{2x}$ which $= \frac{0}{2a}$ when $x = a$

therefore the true value of the fraction is 0, the factor of the numerator is $(x - a)^2$, that of the denominator is $(x - a)$.

Ex. 3. $\frac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4} = \frac{0}{0}$ when $x = a$

$\frac{dy}{dx} = \frac{a - 2x}{-2a^3 + 6ax^2 - 4x^3} = -\frac{a}{0} = \infty$ the true value of the fraction.

Ex. 4. When $x = 0$

$\frac{a^x - b^x}{x} = \frac{0}{0} = \log.\ \frac{a}{b}$

Ex. 5. When $x = a$

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = \frac{0}{0} = \frac{16a}{9}$$

Ex. 6. When $x = 1$

$$\frac{1 - x + \log.\ x}{1 - \sqrt{2x - x^2}} = \frac{0}{0} = -1$$

Ex. 7. When $x = 1$

$$\frac{x^x - x}{1 - x + \log. x} = \frac{0}{0} = -2$$

If Taylor's theorem fails to give the expansions of $f(a+h)$, $\varphi(a+h)$, which will happen wherever $f(x)$, or $\varphi(x)$ contains a radical which vanishes for $x = a$, we must obtain the expansion by some other method.

Substitute $\therefore$ $(a+h)$ for x in both terms of the fraction, and developing by binomial theorem, we shall have

$$\frac{f(a+h)}{\varphi(a+h)} = \frac{Ah^{\alpha} + Bh^{\beta} + Ch^{\gamma} + \dots}{A'h^{\alpha'} + B'h^{\beta'} + C'h^{\gamma'} + \dots}$$

$$= \frac{Ah^{\alpha - \alpha'} + Bh^{\beta - \alpha'} + \dots}{A' + B'h^{\beta' - \alpha'} + \dots}$$

Dividing both the numerator and denominator by the lowest power of h. Now make $h = 0$

$$\text{if} \quad \alpha = \alpha' \quad \frac{f(a)}{\varphi(a)} = \frac{A}{A'}$$

$$\text{..} \quad \alpha > \alpha' \quad \frac{f(a)}{\varphi(a)} = 0$$

$$\text{..} \quad \alpha < \alpha' \quad \frac{f(a)}{\varphi(a)} = \infty$$

Ex. 1. When $x = a$

$$\frac{(x^2 - a^2)^{\frac{3}{2}}}{(x - a)^{\frac{3}{2}}} = \frac{0}{0}$$

It is useless to take the differential co-efficients of the terms in this case, because they become infinite. Making $x = a + h$, we find for $h = 0$

$$\frac{(2ah + h^2)^{\frac{3}{2}}}{h^{\frac{3}{2}}} = \frac{(2a+h)^{\frac{3}{2}}}{1} = (2a)^{\frac{3}{2}}$$

Ex. 2. When $x = a$

$$\frac{\sqrt{x} - \sqrt{a} + \sqrt{x - a}}{\sqrt{x^2 - a^2}} = \frac{0}{0}$$

Make $x = a + h$, we have

$$\frac{(a+h)^{\frac{1}{2}} - a^{\frac{1}{2}} + h^{\frac{1}{2}}}{(2ah + h^2)^{\frac{1}{2}}} = \frac{h^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}h + \dots}{h^{\frac{1}{2}}(2a+h)^{\frac{1}{2}}}$$ developing by binomial theorem

$$= \frac{1}{\sqrt{2a}} \text{ when } h = 0$$

Ex. 3. When $x = c$

$$\frac{(x - c)\sqrt{x - b} + \sqrt{x - c}}{\sqrt{2c} - \sqrt{x + c} + \sqrt{x - c}} = 0$$

We may here employ Taylor's theorem to determine those terms of the series for which it holds good, we shall thus obtain upon substituting $c + h$ for x

$$\frac{\sqrt{h} + h\sqrt{c - b} + \dots}{\sqrt{h} - \frac{1}{2}h(2c)^{-\frac{1}{2}}}$$

dividing by $\sqrt{h}$ and then making $h = 0$ we find 1 for the true value of the fraction.

Ex. 4. When $x = a$

$$\frac{(x^2 - a^2)^{\frac{1}{2}} + x - a}{(1 + x - a)^3 - 1} = \frac{0}{0}$$

$$= \frac{h + (2ah)^{\frac{1}{2}} + \dots}{3h + 3h^2 + \dots} \text{ substituting } a+h \text{ for } x$$

$$= \tfrac{1}{3} \text{ making } h = 0$$

When $x = a$ gives a product $f(x) \times \phi(x)$ of the form $0 \times \infty$,

Then
$$f(x) \times \phi(x) = 0 \times \infty$$

$$= 0 \times \frac{1}{0}$$

$$\therefore \quad \frac{f(x)}{\dfrac{1}{\phi(x)}} = \frac{0}{0} \text{ which may be treated by the rule.}$$

Thus when $x = 1$

$$(1 - x) \tan. \frac{\pi x}{2} = 0 \times \infty$$

$$\therefore \quad \frac{1 - x}{\cot. \dfrac{\pi x}{2}} = \frac{0}{0}$$

$$= \frac{2}{\pi}$$

Let
$$\frac{f(x)}{\phi(x)} = \frac{\infty}{\infty} = \frac{\dfrac{1}{0}}{\dfrac{1}{0}}$$

$$\therefore \quad \frac{1}{\phi(x)} + \frac{1}{f(x)} = \frac{0}{0} \quad \text{which may be treated by the rule.}$$

Thus if $f(x) = \tan. \frac{\pi x}{2a}$, and $\phi(x) = \frac{x^2}{a(x^2 - a^2)}$, the fraction $\frac{f(x)}{\phi(x)}$ becomes $\frac{\infty}{\infty}$ on the supposition that $x = a$, but by the above process we shall have

$$\frac{a(x^2 - a^2)}{x^2 \cot. \dfrac{\pi x}{2a}} = \frac{0}{0} = \frac{2a^2}{-\frac{1}{2}\pi a} = -\frac{4a}{\pi}$$

Lastly, let

$$f(x) - \phi(x) = \infty - \infty = \frac{1}{0} - \frac{1}{0}$$

$$\frac{\dfrac{1}{\phi(x)} - \dfrac{1}{f(x)}}{\dfrac{1}{f(x) \times \phi(x)}} = \frac{0 - 0}{0 \times 0} = \frac{0}{0}$$

Thus, if $x = \frac{\pi}{2}$ or 90°.

$$x \tan. x - \tfrac{1}{2}\pi \sec. x = \infty - \infty$$

Whence
$$\frac{x \sin. x - \frac{1}{2}\pi}{\cos. x} = \frac{0}{0}$$

$$\frac{dy}{dx} = \frac{x \cos. x + \sin x}{-\sin. x} = -1.$$

CHAPTER VIII.

ON THE THEORY OF MAXIMA AND MINIMA.

When the variable upon which any proposed function depends passes successively through all degrees of magnitude, the different values of the function may form first an increasing and then a decreasing series, or *vice versa*, and may go on increasing or decreasing repeatedly, and *vice versa.*

That value at which an increase of the function ends, and a decrease begins, is called a *maximum*, and that at which a diminution ends, and an increase begins, is called a *minimum.*

The essential characteristic of a *maximum* consists in its being greater than each of the values of the function which immediately precede and follow it; and that of a *minimum* in being less than both these values.

Let y be any function of x in which this variable has attained a value which constitutes it either a *maximum* or a *minimum.* Then if x be increased and diminished by an indefinitely small quantity h, the developements of $(x + h)$ and $(x - h)$ will exhibit the values of x, immediately adjacent on each side to that value which renders y a *maximum* or *minimum.* Hence it follows from our definition, that the values of y corresponding to $(x + h)$ and $(x - h)$ will in the one case be both less than the *maximum*, and in the other both greater than the *minimum.*

Let
$$y = f(x)$$
$$y' = f(x+h) = y + ph + q \cdot \frac{h^2}{1 . 2} + \ldots .$$
$$y_1 = f(x-h) = y - ph + q \cdot \frac{h^2}{1 . 2} - \ldots .$$
$$\therefore \quad y' - y = ph + q \cdot \frac{h^2}{1 . 2} + r \cdot \frac{h^3}{1 . 2 . 3} + \ldots\ldots\ldots\ldots (1)$$
$$y_1 - y = -ph + q \cdot \frac{h^2}{1 . 2} - r \cdot \frac{h^3}{1 . 2 . 3} + \ldots\ldots\ldots (2)$$

Now, in order that y may be a *maximum* or *minimum*, the values of y' and y_1 which immediately precede and follow it must be both less or both greater than y.

$\therefore$ When y is a *maximum* or *minimum*, $(y' - y)$ and $(y_1 - y)$ must both have the same sign.

But when h is assumed infinitely small, the whole of the expansions (1) and (2) will have the same signs as their first terms.

Hence, $(y_1 - y)$ and $(y' - y)$ cannot have the same sign, unless p vanishes, $\therefore$ in order that y may be a *maximum* or a *minimum*, the condition requisite is that,

$$p \text{ or } \frac{dy}{dx} = 0$$

If the same value of x which renders $\frac{dy}{dx} = 0$ renders $q = 0$ also, then

$$y' - y = r \cdot \frac{h^3}{1 . 2 . 3} + \ldots$$

And in order that y may be a *maximum* or *minimum*, we must have $\frac{d^3y}{dx^3} = 0$, and generally y cannot be a *maximum* or *minimum* unless the first differential co-efficient, which does not vanish for a particular value of x be of an even order.

Upon inspecting the series (1) and (2) it will be seen that,

When $\left.\begin{matrix} y' - y \\ y_1 - y \end{matrix}\right\}$ are both negative, then since in this case y is greater than y', and y_1, y must be a *maximum;* and since the whole expansions are in this case negative, $\frac{d^2y}{dx^2}$ will have a negative sign. The reverse takes place when y is a *minimum,* and in this case $\frac{d^2y}{dx^2}$ is positive.

Cor.—If the equation $\frac{dy}{dx} = 0$ has $(m - 1)$ equal roots each $= \alpha$, then $\frac{d^2y}{dx^2}$ has $(m - 2)$ of these roots, $\frac{d^3y}{dx^3}$ has $(m - 3)$ of them, and so on; till we come to $\frac{d^my}{dx^m}$ which is the first differential co-efficient which does not contain the root, and in this case, the values y', y_1, corresponding to $(x + h)$, and $(x - h)$, are

$$y' = y + \frac{d^my}{dx^m} \cdot \frac{h^m}{1.2...m}$$

$$y_1 = y \pm \frac{d^my}{dx^m} \cdot \frac{h^m}{1.2...m}$$

The sign of the second term in this last expansion being $+$ or $-$ according as m is even or odd. Hence $(y' - y)$, and $(y_1 - y)$ cannot have the same sign if m be odd, and $\therefore$ in this case y is neither a *maximum nor minimum.* But if m be even, then $(y' - y)$ and $(y_1 - y)$ will have the same sign, and y is a *maximum* or a *minimum* according as $\frac{d^my}{dx^m}$ is positive or negative.

Ex. 1. Let $y = \sqrt{2mx}$

Required to determine the value of x which will render y a maximum or minimum.

$$\frac{dy}{dx} = \frac{m}{\sqrt{2mx}}$$

Since we should obtain no result by equating this quantity to 0, it appears that y is not susceptible of a maximum or minimum value.

Ex. 2. $\quad y = b - (x - a)^2$

$$\therefore \quad \frac{dy}{dx} = -2(x - a) = 0 \quad \therefore \quad x = a$$

$$\frac{d^2y}{dx^2} = -2$$

Hence it appears that y is a *maximum* when $x = a$.

Ex. 3. $\quad y = b + (x - a)^2$

$$\frac{dy}{dx} = 2(x - a) = 0 \quad \therefore \quad x = a$$

$$\frac{d^2y}{dx^2} = +2$$

Hence, in this case, y is a *minimum* when $x = a$.

Ex 4. $$y = \frac{x}{1 + x^2}$$

$$\frac{dy}{dx} = \frac{1 - x^2}{(1 + x^2)^2} = 0 \quad \therefore \quad x = \pm 1$$

The equation $\frac{dy}{dx} = 0$ gives in this case $x = \pm 1$ and $\therefore y = \pm \frac{1}{2}$ and $\frac{d^2y}{dx^2} = \mp \frac{1}{2}$.

Hence the value $x = + 1$ gives $+ \frac{1}{2}$ as the *maximum* value of y.
. . . . $x = - 1$ gives $- \frac{1}{2}$ as the *minimum* value of y.

Ex. 5. $$y^2 - 2mxy + x^2 - a^2 = 0$$

$$\frac{dy}{dx} = \frac{my - x}{y - mx}$$

Putting $\frac{dy}{dx} = 0$ we have $my = x$, and eliminating x and y by the original equation, we have

$$x = \frac{\pm ma}{\sqrt{1 - m^2}}, \; y = \frac{\pm a}{\sqrt{1 - m^2}}, \; \frac{d^2y}{dx^2} = \frac{\mp 1}{a \sqrt{1 - m^2}}$$

Hence $$y = \frac{+ a}{\sqrt{1 - m^2}} \text{ a } maximum.$$

$$y = \frac{- a}{\sqrt{1 - m^2}} \text{ a } minimum.$$

Ex. 6. *To divide a given number* a *into two parts, so that the product of the* m^{th} *power of the one multiplied by the* n^{th} *power of the other, shall be the greatest possible.*

Let x be one of the parts, and let y be the product of the two parts; then it is required to find the value of x which will render the quantity

$$y = x^m (a - x)^n \text{ a } maximum.$$

We have $$\frac{dy}{dx} = x^{m-1} (a - x)^{n-1} \{ma - x (m + n)\}$$

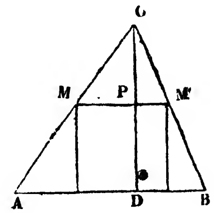
$$\frac{d^2y}{dx^2} = x^{m-2} (a - x)^{n-2} \{(m + n - 1) (m + n) x^2 - . .$$

Putting $\frac{dy}{dx} = 0$ we have $x = 0$, $x = a$, $x = \frac{ma}{m + n}$, this last root gives the *maximum* which is $m^m n^n \left(\frac{a}{m + n}\right)^{m+n}$.

The two other roots correspond to the *minima* when m and n are equal.

A great number of interesting geometrical problems may be solved by the application of these principles. The following are a few examples.

Ex. 1. *To inscribe the greatest rectangle in a triangle.*



Let CP $= x$, AB $= c$, CD $= p$

$\therefore$ PD $= (p - x)$

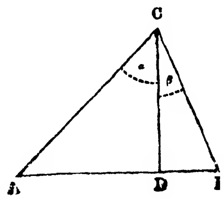
$$\text{CD} : \text{CP} :: \text{AB} : \text{MM}' = \frac{\text{AB} \cdot \text{CP}}{\text{CD}} = \frac{cx}{p}$$

$$y = \text{MM}' \times \text{PD} = \frac{cx}{p} \times (p - x) = \frac{cpx - cx^2}{p}$$

$$\frac{dy}{dx} = \frac{cp - 2cx}{p} = 0$$

$$\therefore x = \frac{p}{2}$$

Ex. 2. *Given the base and perpendicular of a triangle, to describe it so that the vertical angle may be a maximum.*



$$\text{AB} = c$$
$$\text{DC} = p$$
$$\text{AD} = x$$
$$\therefore \text{DB} = c - x$$
$$\frac{\text{AD}}{\text{DC}} = \tan.\ \alpha = \frac{x}{p}$$
$$\frac{\text{DB}}{\text{DC}} = \tan.\ \beta = \frac{c - x}{p}$$

$$\tan.\ \text{angle C} = \tan.\ (\alpha + \beta) = \frac{\tan.\ \alpha + \tan.\ \beta}{1 - \tan.\ \alpha \tan.\ \beta}$$

$$= \frac{\frac{x}{p} + \frac{c - x}{p}}{1 - \frac{x (c - x)}{p^2}}$$

$$= \frac{cp^2}{p^2 - cx + x^2} \text{ a } maximum$$

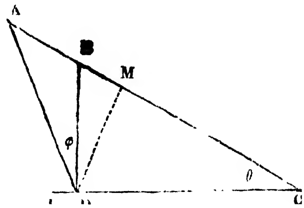
$$\therefore y = x^2 - cx + p^2 = minimum.$$

$$\therefore \frac{dy}{dx} = 2x - c = 0$$

$$\therefore x = \frac{c}{2}$$

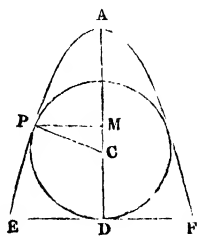
$\therefore$ the triangle is isosceles

Ex. 3. *To find the point* D *in the straight line* CE, *from which* AB *subtends the greatest angle.*



$$AC = a$$
$$CB = b$$
$$CD = x$$
$$\tan. ADB = \tan. (ADM - BDM)$$
$$= \frac{\frac{AM}{MD} - \frac{BM}{MD}}{1 + \frac{AM \cdot MB}{MD^2}}$$
$$= \frac{(AM - BM)\, MD}{MD^2 + AM \cdot BM}$$
$$MD = x \sin. \theta$$
$$AM = a - x \cos. \theta$$
$$BM = b - x \cos. \theta$$
$$\therefore \tan. \phi = \frac{(a-b)\, x \sin. \theta}{x^2 \sin.^2\theta + (a - x\cos. \theta)(b - x \cos. \theta)} \text{ a maximum.}$$
$$\therefore y = \frac{x^2 \sin.^2 \theta + (a - x\cos. \theta)(b - x\cos. \theta)}{(a-b)\, x \sin. \theta} \text{ a minimum.}$$
$$\frac{dy}{dx} = 1 - \frac{ab}{x^2} = 0$$
$$x = \pm \sqrt{ab}$$

Ex. 4. *To find the least parabola which shall circumscribe a given circle.*



Since the parabola and the circle touch at P

$\therefore$ CP is a normal to the parabola and CM is the subnormal $= \frac{1}{2}$ latus rectum.

Let CM $= z$

$\therefore$ equation to the parabola is

$$y^2 = 2z \cdot x \quad \ldots\ldots\ldots\ldots\ldots (1)$$
$$PM^2 = r^2 - z^2$$
$$= 2z \cdot AM$$
$$\therefore \quad AM = \frac{r^2 - z^2}{2z}$$
$$AD = AM + MC + CD$$
$$= \frac{r^2 - z^2}{2z} + z + r$$
$$= \frac{(r+z)^2}{2z}$$

Now area EAF* $= \frac{4}{3}$ AD . DE and DE $= \sqrt{2z \cdot AD}$

* This will be proved afterwards by the Integral Calculus.

$$\therefore \text{ area EAF} = \frac{4}{3}.\text{AD}.\sqrt{2z.\text{AD}} = \frac{2}{3}.\frac{(r+z)^3}{z}$$

$$\therefore u = \frac{(r+z)^3}{z} = \text{maximum}$$

$$\text{hence } \frac{du}{dz} = \frac{(r+z)^2}{z^2}(3z - r - z) = 0$$

$$\therefore 3z - r - z = 0, \text{ and } z = \frac{r}{2} = \text{semi-parameter.}$$

$$\text{AD} = \frac{(r+z)^2}{2z} = \frac{9r}{4}, \text{ and DE} = \sqrt{r.\text{AD}} = \pm\frac{3}{2}r.$$

Ex. 5. *To divide an angle* 2ϕ *into two parts, such that the* m*th power of the sine of one part by the* n*th power of the sine of the other part may be the greatest possible.*

Let $\phi + \theta$ be the one part, then $\phi - \theta$ is the other; hence

$$\sin^m(\phi+\theta)\sin^n(\phi-\theta) = \text{maximum}$$

$$\therefore y = m \log.\sin(\phi+\theta) + n\log.\sin(\phi-\theta) = \text{maximum}$$

$$\therefore \frac{dy}{d\theta} = \frac{m\cos(\phi+\theta)}{\sin(\phi+\theta)} - \frac{n\cos(\phi-\theta)}{\sin(\phi-\theta)} = m\cot(\phi+\theta) - n\cot(\phi-\theta) = 0$$

$$\therefore m\cot(\phi+\theta) = n\cot(\phi-\theta), \text{ or } m\tan(\phi-\theta) = n\tan(\phi+\theta)$$

$$\therefore m:n::\tan(\phi+\theta):\tan(\phi-\theta)$$

$$\therefore m+n:m-n::\tan(\phi+\theta)+\tan(\phi-\theta):\tan(\phi+\theta)-\tan(\phi-\theta)$$

$$::\sin 2\phi \qquad :\sin 2\theta$$

$$\therefore \sin 2\theta = \frac{m-n}{m+n}\sin 2\phi; \text{ hence the two parts are known.}$$

Ex. 6. *The four edges of a rectangular piece of lead,* a *inches in length, and* b *inches in breadth, are to be turned up perpendicularly, so as to form a vessel that shall hold the greatest quantity of water; how much of the edge must be turned up?*

Let x = breadth of edge turned up; then

$$x(a-2x)(b-2x) = \text{maximum}$$

$$\therefore y = abx - 2(a+b)x^2 + 4x^3 = \text{maximum}$$

$$\therefore \frac{dy}{dx} = ab - 4(a+b)x + 12x^2 = 0$$

$$\therefore \frac{d^2y}{dx^2} = -4(a+b) + 24x$$

Hence $12x^2 - 4(a+b)x + ab = 0$, and therefore

$$x = \frac{a+b\pm\sqrt{a^2-ab+b^2}}{6}$$

$$\therefore \frac{d^2y}{dx^2} = \pm 4\sqrt{a^2-ab+b^2}$$

and $x = \frac{1}{6}\{a+b-\sqrt{a^2-ab+b^2}\}$ gives the maximum vessel.

Examples in Maxima and Minima.

(1.) Of all triangles on the same base, having the same given perimeter, to find that whose surface is the greatest.

(2.) Given the hypothenuse of a right-angled triangle, to determine the other sides, when the surface is the greatest possible.

(3.) The whole surface of a cylinder being given $= a^2$, to find its base and altitude, when the volume of the cylinder is a maximum.

(4.) The volume of a cylinder $= b^3$; find its base and altitude, when its whole surface is a minimum.

(5.) Of all the squares inscribed in a given square, find that which is the least.

(6.) Cut the greatest parabola from a given cone.

(7.) Inscribe the greatest rectangle in a given ellipse.

(8.) Find the longest straight pole that can be put up a chimney, when the height from the floor to the mantel $= a$, and the depth from front to back $= b$.

(9.) AB is the diameter of a given semicircle; it is required to draw a chord PQ parallel to AB; so that if AQ and BP be joined intersecting in R, the triangle PQR may be a maximum.

(10.) Inscribe the greatest cone in a sphere whose radius is a.

ON THE METHOD OF LEAST SQUARES.

In astronomical and physical researches, it is frequently required to determine the values of several quantities from a number of simple equations, and when the number of these equations is greater than the number of unknown quantities, they may be combined in a variety of ways, and each mode of combination will produce a different value of the unknown quantities. Hence it is a question of the highest importance to determine in what manner these equations are to be combined so as to give the values of the unknowns affected with the smallest probable errors, or in what way the values of these unknowns are to be found, so that each of the given equations may be satisfied with the greatest accuracy. Thus, for example, if from observation we have the four equations

$$3 - x + y - 2z = 0 \ldots\ldots\ldots\ldots\ldots (1.)$$
$$5 - 3x - 2y + 5z = 0 \ldots\ldots\ldots\ldots\ldots (2.)$$
$$21 - 4x - y - 4z = 0 \ldots\ldots\ldots\ldots\ldots (3.)$$
$$14 + x - 3y - 3z = 0 \ldots\ldots\ldots\ldots\ldots (4.)$$

and it is required to find the values of x, y, z, we may pursue various methods and obtain various results for these unknowns. If the coefficient of x be made the greatest possible, while those of y and z are the smallest possible, we shall evidently have the most accurate value of x; because the value of x depends on those of y and z, and when their coefficients are the smallest possible, the terms in which y and z appear will then have the smallest influence on the value of x. In order, therefore, to obtain the most accurate value of x, we must change the signs of all the terms in eq. (4), and then by addition we get

$$15 - 9x + y + 2z = 0 \ldots\ldots\ldots\ldots\ldots (5.)$$

To make the coefficient of y a maximum, change the signs in eq. (1) and add, then we have

$$37 - 5x - 7y = 0 \ldots\ldots\ldots\ldots\ldots (6.)$$

Similarly for z, change the signs of eq. (2) and add, then we have

$$33 - x - y - 14z = 0 \ldots\ldots\ldots\ldots (7.)$$

Hence from (5), (6), (7), we have by the usual mode of elimination

$$x = 2{\cdot}4853, \quad y = 3{\cdot}5104, \quad z = 1{\cdot}9289.$$

This method is practised in astronomy; but in point of accuracy it yields to the *method of least squares*, invented by Gauss, and to which modern astronomy owes much of its precision.

Suppose then that $e, e', e'', \ldots.$ are the errors of a series of observations, and that we have the equations

$$\begin{aligned} e &= h + ax + by + cz + \ldots\ldots \\ e' &= h' + a'x + b'y + c'z + \ldots\ldots \\ e'' &= h'' + a''x + b''y + c''z + \ldots\ldots \\ e''' &= h''' + a'''x + b'''y + c'''z + \ldots\ldots \text{ \&c.} \end{aligned}$$

to determine the values of $x, y, z, \ldots$ so that the errors e, e', e'', e''', in reference to the whole of the observations shall be the least possible. Now if we were to take simply the sum of these errors, and put the differential coefficient of each of the variables equal to zero, we could not obtain an equation for the determination of the unknown quantity; but if we square each of the equations, we should have

$$\begin{aligned} e^2+e'^2+e''^2+e'''^2+\ldots = x^2(a^2+a'^2+\ldots)+2x\{ah+a'h'+\ldots+a(by+cz+\ldots) \\ +a'(b'y+c'z+\ldots)+a''(b''y+c''z+\ldots)\}+h^2+h'^2+\ \ldots \end{aligned}$$

where the terms involving y and z are not written down, being exactly of the same form as the terms involving x. Let then, for the sake of brevity,

$$u = e^2 + e'^2 + e''^2 + e'''^2 + \ldots. = Ax^2 + 2Bx + C + \ldots..$$

and, therefore, putting the first differential coefficient of this equation equal to zero, we have, considering x alone as the variable,

$$\frac{du}{dx} = 2Ax + 2B = 0 \quad \therefore \ Ax + B = 0$$

hence $x\ (a^2+a'^2+\ldots)+ah+a'h'+\ldots a(by+cz+\ldots)+a'(b'y+c'z+\ldots)+\text{ \&c.}=0$

or $a\ (h+ax+by+cz)+a'(h'+a'x+b'y+c'z)+ \ldots\ldots\ldots\ldots\ldots =0$

and therefore to form an equation that gives a minimum for any one of the unknown quantities, as x, we must multiply each equation of condition by the coefficient of the unknown quantity in that equation, taken with its proper sign, and equate the sum to zero. Proceed in the same manner for $y, z, \ldots$ and we shall have as many equations of the first degree as there are unknown quantities, which may then be obtained by the usual mode of elimination. To apply this method to the preceding example, we have these equations:—

$$\begin{array}{llrl} (1.) & \times -1 \text{ gives} & -3 + & x - y + 2z = 0 \\ (2.) & \times -3 \ \ldots. & -15 + & 9x + 6y - 15z = 0 \\ (3.) & \times -4 \ \ldots. & -84 + & 16x + 4y + 16z = 0 \\ (4.) & \times \ \ 1 \ \ldots. & 14 + & x - 3y - 3z = 0 \end{array}$$

and putting the sum of these equations $= 0$, we obtain

$$27x + 6y - 88 = 0 \ \ldots\ldots\ldots \quad (8.)$$

Proceeding in a similar manner for y and z, we derive the equations

$$6x + 15y + z = 70 \ \ldots\ldots\ldots\ldots \quad (9.)$$

$$y + 54z = 107 \ \ldots\ldots\ldots\ldots \quad (10.)$$

and from these three equations, (8), (9), (10), we have

$$x = 2{\cdot}4702, \quad y = 3{\cdot}5507, \quad z = 1{\cdot}9157.$$

The preceding example is from Gauss (*Theoria Motus,*) where he has proved that the *Method of Minimum Squares* gives the most probable values of the unknown quantities. For a more detailed account of this method, the student may consult Galloway's "Treatise on Probability" (1839.) We shall add only one example by way of exercise.

Ex. Suppose that by observation the four following equations have been formed, viz.:—

$$x + \cdot 96y = -11''.2 \ldots\ldots\ldots\ldots\ldots (1.)$$
$$x - \cdot 93y = -12''.0 \ldots\ldots\ldots\ldots\ldots (2.)$$
$$x + \cdot 62y = -14''.8 \ldots\ldots\ldots\ldots\ldots (3.)$$
$$x - \cdot 85y = +15''.0 \ldots\ldots\ldots\ldots\ldots (4.)$$

it is required to find the most probable values of x and y, by the method of least squares. Ans. $x = -6''.14$, and $y = -7''.86$

CHAPTER IX.

TO CHANGE THE INDEPENDENT VARIABLE.

If we reduce an equation between x and y to the form

$$y = f(x)$$

x is called the *independent variable* and y the *dependent variable.*

Let it be required to change the differential co-efficients found on the supposition that $y = f(x)$ into others where $x = \phi(y)$; that is, where y is the independent variable.

Let h and k be the contemporaneous increments of x and y.

Then if $p, q, r \ldots$ represent, $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$

And $p', q', r' \ldots\ldots\ldots \frac{dx}{dy}, \frac{d^2x}{dy^2}, \frac{d^3x}{dy^3}, \ldots$

We have by Taylor's theorem

$$f(y + k) - f(x) \text{ or } k = f(x + h) - fx$$
$$= ph + q \cdot \frac{h^2}{1 \cdot 2} + r \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots\ldots (1)$$

And similarly $$h = p'k + q' \cdot \frac{k^2}{1 \cdot 2} + r' \frac{k^3}{1 \cdot 2 \cdot 3} + \ldots\ldots (2)$$

Substituting the value of k found in (1) in this last equation

$$h = p'\left(ph + q \cdot \frac{h^2}{1 \cdot 2} + \ldots\right) + \frac{q'}{1 \cdot 2}\left(ph + q \cdot \frac{h^2}{1 \cdot 2} + \ldots\right)^2 + \ldots$$
$$= p'ph + \left(\frac{p'q}{1 \cdot 2} + \frac{p^2 q'}{1 \cdot 2}\right) h^2 + \ldots$$

And comparing co-efficients on both sides

$$pp' = 1 \quad \therefore p' = \frac{1}{p}$$

$$\frac{p'q + p^2 q'}{1 \cdot 2} = 0 \quad \therefore \; q' = -\frac{p'q}{p^2} = -\frac{q}{p^3}$$

&c. &c. &c.

Let us now take a more general case.

Let $\quad y = f(x)$ .... (1)

And $\quad x = \varphi(t)$ .... (2)

$\therefore \quad y = \psi(t)$ .... (3)

The form of the function $\psi(t)$ being unknown.

It is required to change a differential expression found on the supposition that y is a function of x, into another in which both x and y are considered functions of a third variable t.

Let the contemporaneous increments of x, y, t, be h k, l.

By (2) $h = p \cdot l + q \cdot \frac{l^2}{1 \cdot 2} + r \cdot \frac{l^3}{1 \cdot 2 \cdot 3} + \ldots$ where $p = \frac{dx}{dt}$, $q = \ldots$

(3) $k = p' \cdot l + q' \cdot \frac{l^2}{1 \cdot 2} + r' \cdot \frac{l^3}{1 \cdot 2 \cdot 3} + \ldots\ldots\ldots$ $p' = \frac{dy}{dt}$, $q' = \ldots$

(1) $k = p'' \cdot h + q'' \cdot \frac{h^2}{1 \cdot 2} + r'' \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots\ldots\ldots$ $p'' = \frac{dy}{dx}$, $q'' = \ldots$

Substitute for h in this last equation its value from 1st series.

$$\therefore \quad k = p''\left(p\, l + q \cdot \frac{l^2}{1 \cdot 2} + \ldots\right)$$
$$+ \frac{q''}{1 \cdot 2}\left(p^2 l^2 + 2pq \cdot \frac{l^3}{1 \cdot 2} + \ldots\right)$$

Compare this with value of k (3) and equating similar powers of l.

$p''p = p'$

$$p'' = \frac{p'}{p} \text{ or } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

$q''p^2 + p''q = q'$

$$\therefore \quad q'' = \frac{q' - p''q}{p^2}$$

$$\text{or } \frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2} - \frac{dy}{dx} \cdot \frac{d^2x}{dt^2}}{\frac{dx^2}{dt^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}$$

If
$$y = f(x)$$
$$x = \varphi(t)$$

Then by (A), we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

By (B) $$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} \cdot \frac{dx^2}{dt^2} + \frac{dy}{dx} \cdot \frac{d^2x}{dt^2}$$

If $$y = f(x)$$
$$x = \varphi(y)$$

And we wish to change some differential expression, found on the supposition that $y = f(x)$, into others where $x = \varphi(y)$, $t = y$ in the equations (A) and (B), whence

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$\frac{d^2x}{dy^2} = \frac{-\frac{d^2y}{dx^2} \cdot \frac{dx^2}{dy^2} + \frac{d^2y}{dy^2}}{\frac{dy}{dx}}$$

But since y is the independent variable $\frac{d^2y}{dy^2} = 0$

$$\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$

CHAPTER X.

ON THE APPLICATION OF THE DIFFERENTIAL CALCULUS TO THE THEORY OF CURVES.

To draw a tangent to any curve at a given point.

Let P P′ P″ be any curve whose equation is $f(x, y) = 0$

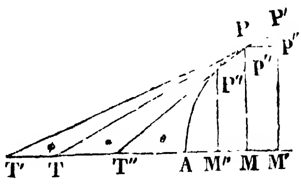
... P be any point whose co-ordinates are x, y.

... $AM = x$, $MM' = MM'' = h$.

... P T be a tangent at P.

... P P″ T″ a secant through P P″.

... P′ P T′ . . . . P P′.



Then the angles θ, α, φ are in the order of their magnitude. The form of equation to tangent will be

$$Y - y = \tan.\ \alpha\,(X - x)$$

where X and Y are the variable co-ordinates of the straight line, and x and y those of the point of contact.

Now, $$\tan.\ \theta = \frac{Pp''}{P''p''} = \frac{f(x) - f(x - h)}{h}$$

$$\tan.\ \alpha = \tan.\ \alpha$$

$$\tan.\ \varphi = \frac{P'p}{Pv} = \frac{f(x + h) - f(x)}{h}$$

$$\tan.\ \theta = \frac{dy}{dx} - \frac{d^2y}{dx^2} \cdot \frac{h}{1.2} \ldots \ldots\ldots\ldots \quad (1)$$

$$\tan.\ \alpha = \tan.\ \alpha \ldots\ldots\ldots\ldots \quad (2)$$

$$\tan.\ \phi = \frac{dy}{dx} + \frac{d^2y}{dx^2} \cdot \frac{h}{1.2} + \ldots\ldots\ldots \quad (3)$$

Which three series must be in the order of magnitude, whatever be the value of h, $\therefore$ their first terms must be in the order of magnitude.

$$\therefore \qquad \tan.\ \alpha = \frac{dy}{dx}$$

Hence equation to tangent is

$$Y - y = \frac{dy}{dx}(X - x)$$

Hence it appears

1°. Since $\tan.\ \alpha = \frac{dy}{dx}$

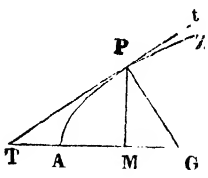
$$\cos.\ \alpha = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$\sin.\ \alpha = \frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

2°. The normal PG makes with the axis of x's, the angle PGT, but

$$\tan.\ \text{PGT} = -\cot.\ \text{PTG}$$

$$= -\frac{1}{\frac{dy}{dx}}$$



Hence the equation to the normal is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x)$$

or $\frac{dy}{dx}(Y - y) + X - x = 0$

3°. Making $Y = 0$ in the equations to the tangent and normal we find the values of AT and AM, the abscissas of the points in which the tangent and normal cut the axis of x's, hence we find

$$x - X \text{ or subtangent MT} = \frac{y}{\frac{dy}{dx}}$$

$$\text{subnormal MG} = y\frac{dy}{dx},$$

4°. Since PMT and PMG are right angles

Tangent $\quad PT = \sqrt{PM^2 + MT^2}$

$$= \sqrt{y^2 + \frac{y^2}{\left(\frac{dy}{dx}\right)^2}}$$

$$= \frac{y}{\frac{dy}{dx}} \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Normal $\quad PG = \sqrt{PM^2 + MG^2}$

$$= \sqrt{y^2 + y^2 \left(\frac{dy}{dx}\right)^2}$$

$$= y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Ex. 1. *To find the subtangent in the parabola.*

The equation to the parabola is

$$y^2 = 4mx$$

$$\therefore \quad \frac{dy}{dx} = \frac{2m}{y}$$

But $\quad MT = \frac{y}{\frac{dy}{dx}}$

In this case $\quad = \frac{y^2}{2m}$

$$= 2x$$

x and y are the co-ordinates of *any* point in the curve; if x' y' be the co-ordinates of the point of contact in the present instance; we have of course,

$$MT = 2x' \quad \text{a known quantity.}$$

Ex. 2. *To find the subnormal in the ellipse.*

The equation to the ellipse referred to its centre as origin is

$$a^2y^2 + b^2x^2 = a^2b^2$$

$$\therefore \quad \frac{dy}{dx} + \frac{b^2x}{a^2y} = 0 \qquad \therefore \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

Substituting this value of $\frac{dy}{dx}$ in the equation

$$MG = y\frac{dy}{dx}$$

We have subnormal of ellipse $= -\frac{b^2}{a^2}x'$, x' and y' being the co-ordinates of the point of contact.

Ex. 3. *To find the subtangent in the Cissoid*

The equation to the cissoid, is*

$$y^2 = \frac{x^3}{a - x}$$

$$\therefore \text{ the subtangent } = \frac{2x'(a - x')}{3a - 2x'}$$

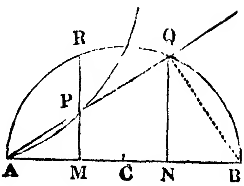
Ex. 4. *To find the subtangent in the conchoid.*

The equation to the conchoid, is†

$$x^2 y^2 = (a + x)^2 (b^2 - x^2)$$

$$\text{Hence the subtangent } = -\frac{x(a + x)(b^2 - x^2)}{x^3 + ab^2}$$

* *To find the equation to the Cissoid.*



AQB is a semicircle, C its centre.
Take any equal parts CM, CN, and draw the ordinates MR, NQ
Join AQ cutting MR in P.
The locus of the point P is the curve called the *cissoid.*
Let A be the origin to co-ordinates.
$\therefore$ AB $= a$, AM $= x$, MP $= y$.

Then PM^2 : AM^2 :: QN^2 : AN^2

$y^2 : x^2$ :: AN . NB : AN^2

:: NB : AN

:: x : $(a - x)$

$\therefore$ $y^2 = \frac{x^3}{a - x}$ which is the equation required.

To find the Polar equation.

Let A be the pole; join QB, let AP $= r$, angle QAB $= \theta$, and AB $= 2a$.

Then AQ $= 2a \cos. \theta$ . . . . (1)

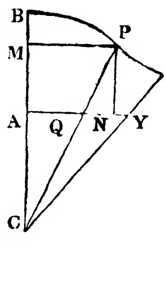
And AQ $= \frac{\text{QN}}{\sin. \theta} = \frac{\text{BN}}{\tan. \theta \sin. \theta} = \frac{\text{AM}}{\tan. \theta \sin. \theta} = \frac{r \cos. \theta}{\tan. \theta \sin. \theta}$ . . . . . . (2)

Equating (1) and (2)

$$\frac{r \cos. \theta}{\tan. \theta \sin. \theta} = 2a \cos. \theta$$

$$\therefore \quad r = \frac{2a \sin.^2 \theta}{\cos. \theta} \text{ the equation required.}$$

† *To find the equation to the Conchoid.*



C is a given point, and AY a straight line given in position.

Draw CB at right angles to AY, and draw CP, CP$_1$ . . . making QP, Q$_1$ P$_1$ . . . . always equal to AB.

The locus of the point P is the curve called the *conchoid.*

Let A be the origin of co-ordinates.

$\therefore$ CA $= a$, AB $= b$, AM $= x$, MP $= y$.

Then CM^2 : NP^2 :: PM^2 : NQ^2

$(a + x)^2$: x^2 :: y^2 : $b^2 - x^2$

$\therefore$ $x^2 y^2 = (a + x)^2 (b^2 - x^2)$ which is the equation required.

To find the Polar equation.

Let C be the pole, CP $= r$, angle BCP $= \theta$

AC $=$ CQ cos. θ

$= (r - b)$ cos. θ the equation required.

Ex. 5. *The tangent to a cycloid at any point is parallel to the corresponding chord of the generating circle.*

The general expression for the normal, is

$$N = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

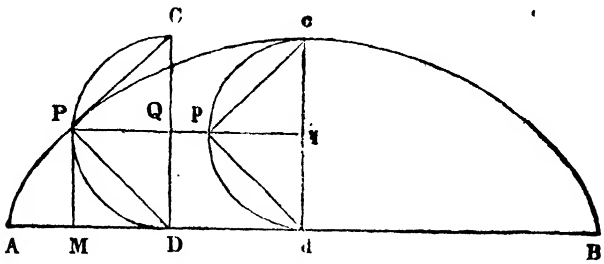
The equation to the cycloid referred to its extremity as origin, is*

$$x = \text{versin.}^{-1}\, y - \sqrt{2ay - y^2}$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{2a - y}{y}} \qquad \left(\frac{dy}{dx}\right)^2 = \frac{2a - y}{y}$$

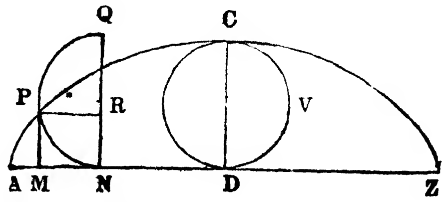
$$\therefore\ y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{2ay} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let P be a point in a cycloid, and CP the position of the generating circle. Draw PQ pq parallel to AB, join P, C; P, D; p, c; p, d;
Then by the property of the circle CPD is a right angle.



* *To find the equation to the Cycloid.*

Let AZ be a straight line.
CVD a circle.
If the circle CVD be supposed to roll along the straight line AZ, the curve traced out by any fixed point P in the circumference of the circle is called a *cycloid.*
Let ACZ be a cycloid.
AZ is called the base of the cycloid.
Let C be the position of the fixed point when the circle begins to roll along AZ.

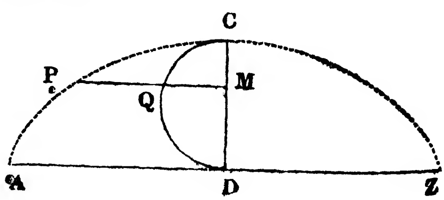


Then C is called the vertex of the cycloid, and CD the axis.

1. Let the origin be at A, the extremity of the curve.
Let QPN be any position of the rolling circle, and P the fixed point.
$\therefore$ AM $= x$, MP $= y$, QN or CD $= 2r$, arc NP $= \theta$

Then $x =$ AM
$=$ AN $-$ NM
$= \theta - \sin.\ \theta$
$= \text{versin.}^{-1}\, y - \sqrt{2ry - y^2}$

2. Let the origin be at the vertex
CM $= x$, MP $= y$
$y =$ MQ $+$ QP
$=$ MQ $+$ arc QC
$= \sin.$ QC $+ \text{versin.}^{-1}$ MC
$= \text{versin.}^{-1}\, x + \sqrt{2rx - x^2}$



$\because$ QD : PD :: PD : DC

$\therefore$ $PD^2 = DC . QD$

$= 2a . y$

$\therefore$ $PD = \sqrt{2ay}$, and is therefore by (1) a normal to the curve at P; and since CPD is a right angle, CP is perpendicular to PD, and is $\therefore$ a tangent at P.

And since the triangles CPQ, *cpq*, are equal and similar, $\therefore$ CP is parallel to *cp*.

Hence the construction to draw a tangent geometrically is obvious.

Asymptotes.

Let the equations to two plane curves which have infinite branches be

$$f(x, y) = 0 \qquad \phi(x, y') = 0$$

y and y' being the values of y in the two curves corresponding to the same values of x. The distance between the two curves measured in a direction parallel to the axis of y's is $(y - y')$. Then, if as x increases without limit either positively or negatively, the distance $(y - y')$ diminishes without limit, but vanishes only when x becomes infinite, the infinite branch of the one curve is said to be an *asymptote* to the other.

In order that this may be the case, it is necessary that the quantity $(y - y')$ when developed, should contain *negative* powers only of x; if it contained a positive power, then $(y - y')$ would be rendered infinite by x becoming infinite, and if it contained any term independent of x, it would be finite when x was infinite.

Hence the developement of $(y - y')$ must have the form

$$y - y' = ax^{-\alpha} + bx^{-\beta} + \ldots$$

The exponents being supposed to descend.

It follows, $\therefore$ that if the developement of y by descending powers of x contain any positive powers of x, or a term independent of x, all these terms must be common to the developement of y', in order that they may disappear by subtraction. Hence, if the developement of y be

$$y = ax^{\alpha} + bx^{\beta} + \ldots r + a'x^{-\alpha'} + b'x^{-\beta'} + \ldots$$

The developement of y' must be

$$y' = ax^{\alpha} + bx^{\beta} + \ldots r + \ldots$$

The terms which succeed r, or which involve the negative powers of x being unrestricted.

Since the terms of the developement which succeed r are arbitrary, it follows that there may be an infinite number of asymptotes to the same curve, and that each of these will be asymptotes to each other. The most simple asymptote of which the curve admits, at least that whose developement is most simple, is the curve represented by the equation

$$y' = ax^{\alpha} + bx^{\beta} + \ldots r^{1}$$

The curve represented by equation

$$y'' = ax^{\alpha} + bx^{\beta} + \ldots + r + a'x^{-\alpha'}$$

is also an asymptote, and approaches closer to the curve than the former, since

by increasing x it is manifest that y'' approaches nearer to an equality with y than y' does. In like manner the curve represented by

$$y''' = ax^{\alpha} + bx^{\beta} + \ldots + r + a'x^{-\alpha'} + b'x^{-\beta}$$

has an asymptotism of a still higher order with the given curve.

Thus it appears that there are orders of asymptotism in some degree analogous to the orders of contact, as explained in the following article. Curves which admit of asymptotes are sometimes divided into hyperbolic and parabolic. Hyperbolic are those which admit of a rectilinear asymptote. Parabolic those which do not.

All hyperbolic curves $\therefore$ must be involved in the class

$$y = Ax + B + ax^{-\alpha} + bx^{-\beta} + \ldots$$

The equation to the asymptote being

$$y' = Ax + B$$

Whatever has been said with regard to the difference $y - y'$, is equally applicable to the difference $(x - x')$ for the same ordinate y.

Examples.

1. Let $$y = \pm \frac{b}{a}(x^2 - a^2)^{\frac{1}{2}}$$
$$= \pm \frac{b}{a} \cdot x \mp \frac{ba}{2} x^{-1} \pm \ldots.$$

Hence the curve has two rectilinear asymptotes represented by the equations

$$y' = \pm \frac{b}{a} x$$

2. Let $$xy = c^2$$
$$\therefore \quad y = c^2 x^{-1}$$
$$x = c^2 y^{-1}$$

Hence the asymptotes are the axes of co-ordinates themselves.

3. $$y^2(x^2 - a^2) = b^4$$
$$y = \pm b^2 x^{-1} + \ldots.$$
$$x = \pm a \pm \tfrac{1}{2} \cdot \frac{b^4}{a} y^{-2}$$

Hence the axis of x is an asymptote, and there are two other asymptotes parallel to the axis of y represented by.

$$x = \pm a$$

There are also two hyperbolas, $xy = \pm b^2$ which are asymptotes.

4. $$y^2 x - px^2 - a^3 = 0$$
$$y^2 = px + a^3 x^{-1}$$

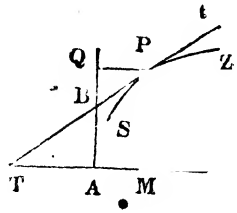
Hence the asymptote to this curve is the common parabola.

Rectilinear Asymptotes

Besides the general method already given for determining the asymptotes to a curve, there is another method of determining whether the curve admits

of a rectilinear asymptote, founded on the consideration that, *a tangent to a curve, when the point of contact is removed to an infinite distance, becomes an asymptote.*

Let Tt be a tangent to the curve SZ at any point P, whose co-ordinates are $(x' y')$.



Let AB the distance of the point B where Tt cuts $Ay = Y$.

Let AT the distance of the point T where Tt cuts $Ax = X$.

$$-X = AT = MT - AM$$

$$\therefore X = -\left(\frac{y'}{\frac{dy'}{dx'}} - x'\right) \quad \ldots\ldots (1)$$

$$Y = AB = AQ - BQ = AQ - PQ \tan. PTA$$

$$= y' - \frac{x'dy'}{dx'} \quad \ldots\ldots (2)$$

We must obtain the values of X and Y from the equation to the proposed curve in terms of x, and we must suppose x increased without limit, then

1. If the limits of X and Y are finite, they determine a rectilinear asymptote.

2. If X have a limit, and Y be infinite, then there is an asymptote parallel to the axis Ay, at the distance X from origin.

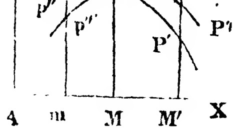
3. If Y have a limit, and X none, there is an asymptote parallel to the axis Ax, at the distance Y from origin.

4. If both X and Y are infinite, the curve does not admit of a rectilinear asymptote, and the same if the values be impossible.

5. If $X = 0$, $Y = 0$, the asymptote passes through the origin, and its direction is found by determining the values of $\frac{dy}{dx}$ when x becomes infinite.

On the Principles of Contact.

Let PP', PP'', PP''' be three curves related to the same axes and passing through the point P whose co-ordinates are x', y'.



Let their equations be generally

$$y = \varphi(x), \qquad y = f(x), \qquad y = F(x).$$

Then for the point P, $\varphi(x'), = f(x'), = F(x')$.

Let x' become $x' + h$, taking $MM' = h$, and draw the ordinate $M'P'P''P'''$.

$$\text{Then. } M'P' = \varphi(x' + h) = \varphi(x') + \frac{d\varphi(x')}{dx'} \cdot \frac{h}{1} + \frac{d^2\varphi(x')}{dx'^2} \cdot \frac{h^2}{1.2} + \ldots$$

$$M'P'' = f(x' + h) = f(x') + \frac{df(x')}{dx'} \cdot \frac{h}{1} + \frac{d^2f(x')}{dx'^2} \cdot \frac{h^2}{1.2} + \ldots$$

$$M'P''' = F(x' + h) = F(x') + \frac{dF(x')}{dx'} \cdot \frac{h}{1} + \frac{d^2F(x')}{dx'^2} \cdot \frac{h^2}{1.2} + \ldots$$

Now, if all the terms in each of these developements are equal to the corresponding terms in the others, then the curves will be identical; if the first terms only be equal, the curves will have but one common point; if besides the first term we have also, $\frac{d\phi\,(x')}{dx'} = \frac{df\,(x')}{dx'} = \frac{d\mathrm{F}\,(x')}{dx'}$, the curves will approach more nearly than before, and the greater number of terms which are equal in the developements, the greater will be the intimacy of the curves.

If, in the above expansions, $\phi\,(x') = f\,(x') = \mathrm{F}\,(x')$ and also $\frac{d\phi\,(x')}{dx'} = \frac{df\,(x')}{dx'}$ then the two curves, PP′ and PP″ will have a common rectilineal tangent at P, and the curve PP‴ which does not fulfil this condition, must be altogether on the same side of the two curves which have the common tangent, and cannot pass between them. In general it may be proved that if two curves have n differential co-efficients equal, no curve which has $(n - 1)$ or a smaller number of its co-efficients equal to those of the other two can pass between the proposed curves, but must lie altogether without them.

Generally when two curves have the two first terms of their developements equal, they are said to osculate and to have a contact of the first order, when the second differential co-efficients are equal in each, they have a contact of the second order, and when n differential co-efficients are equal in each, they are said to have a contact of the nth order.

If it be required to find the highest order of contact which any curve $y = \phi\,(x)$ can have with any other curve, $y = f\,(x)$, we can determine this by the number of constants contained in the equation $y = \phi\,(x)$, for if there be n constants in this equation, if both equations be differentiated $(n - 1)$ times, the values of the constants in $y = \phi\,(x)$ will be determined by elimination between the equations, $y = y'$, $\frac{dy}{dx} = \frac{dy'}{dx'}$, $\frac{d^2y}{dx^2} = \frac{d^2y'}{dx'^2} \ldots \frac{d^{n-1}y}{dx^{n-1}} = \frac{d^{n-1}y'}{dx'^{n-1}}$ and these values substituted in the original equation $y = \phi\,(x)$ will give the equation to the osculating curve required. The n constants satisfy these conditions, and the contact will be of the $(n - 1)^{\text{th}}$ order, but it cannot be of a higher order, since n constants can fulfil only n equations.

We may further observe, that if in the figure we take M m = M M′, but measured in the opposite direction, and $\therefore = -h$, and if we develope, $\phi\,(x' - h)$, $f\,(x' - h)$, $\mathrm{F}\,(x' - h)$ as before, and the three curves be supposed to have a contact of the first order, then the relative magnitude of M′ P′, M′ P″, M′ P‴, will depend upon the sign of $\frac{d^2\,\phi\,(x')}{dx'^2}$, $\frac{d^2\,f\,(x')}{dx'^2}$, $\frac{d^2\,\mathrm{F}\,(x')}{dx'^2}$, since h may be assumed so small that these terms shall be greater than the sum of those which follow: now, in this case, in both developements, the signs of these terms are positive, $\therefore$ the relative magnitude of M′ P′, M′ P″, M′ P‴, are the same for $+h$ and $-h$.

But if the curves be supposed to have a contact of the second order, then the relative value of these ordinates will depend upon $\frac{d^3\phi(x')}{dx'^3}$ . . . and since the signs of these terms is different in the two developements, the order of the magnitude of M′ P′, M′ P″, M′ P‴ will be inverted, and $\therefore$ the curves will intersect in P, and so on for the succeeding orders of contact.

From this it follows that contact of an odd order is contact only, but that contact of an even order is both contact and intersection.

Let $y_1 = f(x)$

$y_2 = \varphi(x)$

be the equations of two curves which intersect, then at the point of intersection we shall have the ordinates in each curve equal for the same value of abscissa. Let us now consider the course of the curves beyond this point, and for this purpose, substitute $x + h$ for x.

$$\therefore \quad y_1 = f(x) + p \cdot \frac{h}{1} + q \cdot \frac{h^2}{1 \cdot 2} + r \cdot \frac{h^3}{1 \cdot 2 \cdot 3}$$

$$y_2 = \varphi(x) + \mathrm{P} \cdot \frac{h}{1} + \mathrm{Q} \cdot \frac{h^2}{1 \cdot 2} + \mathrm{R} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} \dots$$

Let δ = distance between the two curves, then since $f(x) = \varphi(x)$

$$\delta = (p - \mathrm{P})\frac{h}{1} + (q - \mathrm{Q})\frac{h^2}{1 \cdot 2} + (r - \mathrm{R})\frac{h^3}{1 \cdot 2 \cdot 3} + \dots$$

Now if the value of x which renders $y_1 = y_2$, render also $p = \mathrm{P}$

$$\delta = (q - \mathrm{Q})\frac{h^2}{1 \cdot 2} + (r - \mathrm{R})\frac{h^3}{1 \cdot 2 \cdot 3} + \dots$$

and our two curves approach more closely to each other than any other curve which, passing through the same point does not fulfil this condition also. For let the equation to this curve be

$$y_3 = \psi(x)$$

Then if Δ be the distance between the points of this curve and of the first, whose abscissa is $(x + h)$

$$\Delta = (p - p'')\frac{h}{1} + (q - q'')\frac{h^2}{1 \cdot 2} + \dots$$

Now the values of δ and Δ are of the form

$$\delta = \mathrm{B}h^2 + \mathrm{C}h^3 + \dots \qquad \Delta = ah + bh^2 + \dots$$

$$\therefore \quad \Delta - \delta = ah + (b - \mathrm{B})h^2 + (c - \mathrm{C})h^3 + \dots$$

And, since we may assume h so small that the sign of the first term shall be the sign of the whole series, it follows that $\Delta > \delta$.

Osculating Circles.

The general equation to the circle is

$$(y - y')^2 + (x - x')^2 = \mathrm{R}^2$$

in which x', y', are the co-ordinates of the centre, and R is the radius.

Since the equation contains three arbitrary constants, the circle will have a contact of the second order with any proposed curve, whose equation is

$$f(x, y) = 0$$

At the point of contact the co-ordinates of the circle and the proposed curve will be identical, or $x_1\ y_1$ is the same for each.

Also, since the osculation is of the second order, the first and second differential co-efficients, obtained by differentiating the equations to the curve, and the circle will be identical.

If $\therefore$ we eliminate x', y', from the above equation to the curve, we shall obtain a general value of R, which in that case will be the radius of the osculating circle to the proposed curve at any point (x, y).

The radius of the osculating circle at any point of a proposed curve is called the "Radius of Curvature."

To find the radius of curvature in any curve.

The general equation to a circle is

$$R^2 = (y - y')^2 + (x - x')^2 \text{ .... } (1)$$

Differentiating and putting $\frac{dy}{dx} = p$, $\frac{d^2y}{dx^2} = q$

$$0 = p\,(y - y') + (x - x') \text{ .... } (2)$$

Again $$0 = (y - y')\,q + p^2 + 1 \text{ .... } (3)$$

From (3), $y - y' = -\dfrac{1 + p^2}{q}$

$\therefore$ From (2), $x - x' = -\,p\,(y - y')$

$$= p\,\frac{1 + p^2}{q}$$

Substituting these values of $(x - x')$ and $(y - y')$ in (1), we have

$$R^2 = \frac{(1 + p^2)^2}{q^2} + \frac{p^2\,(1 + p^2)^2}{q^2}$$

$$= \frac{(1 + p^2)\,(1 + p^2)^2}{q^2}$$

$$= \frac{(1 + p^2)^3}{q^2}$$

$$\therefore \quad R = \pm\,\frac{(1 + p^2)^{\frac{3}{2}}}{q} \text{ .... } (A)$$

We must take the negative sign when the curve is concave towards the axis of abscissas, and the positive sign when it is convex. For it will be proved that the sign of $\frac{d^2y}{dx^2}$ will depend on these circumstances.

To apply this to any curve we have only to find the value of p^2 and q from equation to curve, and these when substituted in A will give the value of R required.

Ex. 1. To find the value of R *in the ellipse.*

The equation to the ellipse is

$$a^2\,y^2 + b^2\,x^2 = a^2\,b^2$$

$$\frac{dy}{dx} = -\,\frac{b}{a}\cdot\frac{x}{\sqrt{a^2 - x^2}}$$

$$\frac{d^2y}{dx^2} = -\frac{ab}{(a^2 - x^2)^{\frac{3}{2}}}$$

$$R = a^2 b^2 \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{\frac{3}{2}}$$

Ex. 2. *To find the value of* R *in the Cissoid.*

The equation to the cissoid is

$$y^2 = \frac{x^3}{a - x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{3a}{2} - x\right)x^{\frac{1}{2}}}{(a - x)^{\frac{3}{2}}}$$

$$\frac{d^2y}{dx^2} = \frac{3a^2}{4x^{\frac{1}{2}}(a - x)^{\frac{5}{2}}}$$

$$\therefore \quad R = \frac{ax^{\frac{1}{2}}(4a - 3x)^{\frac{3}{2}}}{6(a - x)^2}$$

Ex. 3. *To find the value of* R *in the Cycloid.*

The equation to the cycloid, the extremity of the curve being the origin, is

$$x = \text{versin.}^{-1} y + \sqrt{2ry - y^2}$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{2r - y}{y}}$$

$$\frac{d^2y}{dx^2} = -\frac{r}{y^2}$$

$$\therefore \quad R = 2\sqrt{2ry}$$

Now, generally in any curve

$$\text{Normal} = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\therefore \quad \text{Normal in cycloid} = y\sqrt{\frac{2r}{y}}$$

$$= \sqrt{2ry}$$

Hence it appears, that

The radius of curvature at any point of the cycloid is equal to twice the normal.

Evolutes.

The evolute of a curve is the locus of the centres of the radii of curvature.

The general equation to a circle is

$$(y - y')^2 + (x - x')^2 - R^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

in which, x', y', are the co-ordinates of the centre, and R the radius.

Now, if a series of circles be described osculating a curve whose equation is $f(x, y) = 0$ in every point, it is manifest that as we pass from one point to another, R will continually vary, and $(x' y')$ will be the variable co-ordinates of any point in the evolute. To obtain $\therefore$ the equation to the evolute of any curve, we must eliminate (x, y) the co-ordinates of any point in the curve and the osculating circle at that point, and in this manner obtain a relation between (x', y') alone.

Now, if we suppose the circle (1) to be an osculating circle to a curve $f(x, y) = 0$ at a point whose co-ordinates are x and y, we shall have for this point the x, y of the curve and the circle identical, and also the first and second differential co-efficients derived from the equation to curve identical with the same functions derived from the equation to the circle. Substituting $\therefore$ y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, obtained from the equation to curve, for the same quantities obtained from the equation to the circle, we shall obtain two equations containing only x, x', y', and eliminating x between these two equations, we shall arrive at a relation between $x' y'$, which will be the equation to the evolute of the proposed curve.

The general equation to the circle is

$$(y - y')^2 + (x - x')^2 - R^2 = 0 \text{ .... } (1)$$

Differentiating $(y - y')\,p + (x - x') = 0$ .... (2)

Again, $(y - y')\,q + p^2 + 1 = 0$ .... (3)

From (3) $$y - y' = -\frac{1 + p^2}{q}$$

$$\therefore \quad y' = y + \frac{1 + p^2}{q} \text{ .... } (A)$$

By (2) $$x - x' = -p\,(y - y')$$

$$= p \cdot \frac{1 + p^2}{q}$$

$$\therefore \quad x' = x - p \cdot \frac{1 + p^2}{q} \text{ .... } (B)$$

The quantities y', x', are the variable co-ordinates of any point in the evolute, substituting $\therefore$ in (A) and (B), the values of y, p, q, given by equation to curve $f(x, y) = 0$, the two equations (A) and (B) will involve x, y', x', alone, and $\therefore$ eliminating x between these equations, we arrive at the sought for relation between $(x' y')$.

To find the Evolute the common parabola.

If x', y', be the co-ordinates to any point in evolute we have shown that

$$y' = y + \frac{1 + p^2}{q} \text{ .... } (A)$$

$$x' = x - p\,\frac{1 + p^2}{q} \text{ .... } (B)$$

Let the equation to the curve be

$$y^2 = 2mx \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$2y \cdot \frac{dy}{dx} = 2m$$

$$\therefore \quad p = \frac{m}{y} = \frac{m^{\frac{1}{2}}}{\sqrt{2x}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$1 + p^2 = 1 + \frac{m^2}{y^2} = \frac{2mx + m^2}{2mx} = \frac{2x + m}{2x} \quad \ldots\ldots\ldots\ldots (3)$$

$$q, \text{ or } \frac{dp}{dx} = -\frac{m^2}{y^3}$$

$$\therefore \quad q = -\frac{m^2}{(2mx)^{\frac{3}{2}}} = -\frac{m^{\frac{1}{2}}}{(2x)^{\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots (4)$$

$$y' = y + \frac{1 + p^2}{q}$$

$$= (2mx)^{\frac{1}{2}} - \frac{2x + m}{2x} \cdot \frac{(2x)^{\frac{3}{2}}}{m^{\frac{1}{2}}}$$

$$= \frac{m\,(2x)^{\frac{1}{2}} - (2x)^{\frac{3}{2}} - m\,(2x)^{\frac{1}{2}}}{m^{\frac{1}{2}}}$$

$$= -\frac{(2x)^{\frac{3}{2}}}{m^{\frac{1}{2}}} \quad \therefore \quad x = +\frac{m^{\frac{1}{3}} y'^{\frac{2}{3}}}{2} \quad \ldots\ldots\ldots\ldots (C)$$

and $x' = x - p \cdot \dfrac{1 + p^2}{q}$

$$= x + \frac{m^{\frac{1}{2}}}{(2x)^{\frac{1}{2}}} \cdot \frac{2x + m}{2x} \cdot \frac{(2x)^{\frac{3}{2}}}{m^{\frac{1}{2}}}$$

$$= 3x + m \quad \therefore \quad x = \frac{x' - m}{3} \quad \ldots\ldots\ldots\ldots (D)$$

Equating the values of x found in (C) and (D), we have

$$\frac{m^{\frac{1}{3}} y'^{\frac{2}{3}}}{2} = \frac{x' - m}{3}$$

$$y'^2 = \frac{8\,(x' - m)^3}{27m}$$

which is the equation to the evolute.

At the points of greatest and least curvature the contact of the circle of curvature is of the third order.

The points of greatest and least curvature will manifestly be the points at which the radius of curvature is a *minimum* and a *maximum*, and to find these points we must differentiate the radius of curvature, and put the result $= 0$. Now

$$\frac{(1 + p^2)^3}{q^2} = R^2$$

$$3pq^2 - (1 + p^2)\, r = 0$$

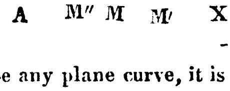
$$\therefore \quad r \text{ or } \frac{d^3y}{dx^3} = \frac{3pq^2}{(1+p^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

Now take the equation to the circle and differentiate three times

$$(y-y')^2 + (x-x')^2 - R^2 = 0$$
$$(y-y')\,p + (x-x') = 0$$
$$(y-y')\,q + p^2 + 1 = 0 \therefore y - y' = -\frac{1+p^2}{q}$$
$$(y-y')\,r + 3pq = 0$$
$$\therefore \quad \frac{d^3y}{dx^3} \text{ or } r = -\frac{3pq}{y-y'}$$
$$= \frac{3pq}{\dfrac{1+p^2}{q}}$$
$$= \frac{3pq^2}{1+p^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}$$

Comparing (A) and (B) they will be found identical, *i. e.* the third differential co-efficient derived from curve, is, at the points of greatest and least curvature, identical with the third differential co-efficient derived from equation to circle. Hence at these points the circle has a contact of the third order.

A Curve is convex or concave towards the axis of the abscissas, according as $\dfrac{d^2y}{dx^2}$ *is positive or negative.*



If RZ be any plane curve, it is manifest that if a tangent be drawn at any point P, and if two points P′, P″, be taken very near to P on opposite sides of it, then if the curve be convex both the ordinates M′P′, M″P″ to the curve, will be greater than the corresponding ordinates to the tangent M′Q′, M″Q″, and if the curve be concave they will both be less than these.

$AM = x$, $MM' = MM'' = h$, $MP = f(x)$, $M'P' = f(x+h)$, $M''P'' = f(x-h)$

$$\therefore \quad M'P' = y + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \cdots$$
$$M''P'' = y - \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\,.\,2} - \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\,.\,2\,.\,3} + \cdots$$

$$\left.\begin{aligned} M'Q' &= y + \frac{dy}{dx} \cdot \frac{h}{1} \\ M''Q'' &= y - \frac{dy}{dx} \cdot \frac{h}{1} \end{aligned}\right\}$$ ∵ the tangent has an osculation of first order.

$$\therefore \; M'P' - M'Q' = \frac{d^2y}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \ldots .$$

$$M''P'' - M''Q'' = \frac{d^2y}{dx^2} \cdot \frac{h^2}{1 \cdot 2} - \ldots .$$

Now, these quantities will be positive or negative according as the curve is convex or concave towards the axis of abscissas.

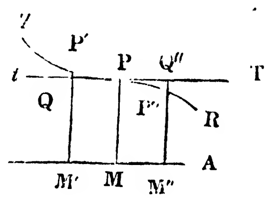
∴ The curves will be convex or concave to the axes, according as $\frac{d^2y}{dx^2}$ is positive or negative, since by assuming h sufficiently small the sign of the whole series may be made to depend on the signs of these terms.

At a point of contrary flexure, $\frac{d^2y}{dx^2} = 0$ or $= \alpha$

Or generally, there cannot be a point of contrary flexure unless the first differential, which does not vanish for a particular value of the abscissa be of an odd order.

At a point of contrary flexure a curve from being convex to the axis of abscissas, becomes concave, or *vice versa*.

The contiguous ordinates of a convex curve are both greater, and of a concave curve both less than the corresponding ordinates to the tangent, but at a point of contrary flexure the ordinates to two points in the curve being near the point of inflexion on each side of it, must be one greater and the other less than the corresponding ordinates to the tangent.



Let RZ be a plane curve, P a point of inflexion, Tt a tangent at P

AM $= x$, MM' $=$ MM'' $= h$, MP $= f(x)$, M' P' $= f(x + h)$, M'' P'' $= f(x - h)$

$$\therefore \quad M'P' = y + \frac{dy}{dx} \cdot \frac{h}{1} + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \frac{d^3y}{dx^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots$$

$$M''P'' = y - \frac{dy}{dx} \cdot \frac{h}{1} + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1 \cdot 2} - \frac{d^3y}{dx^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots$$

$$M'Q' = y + \frac{dy}{dx}$$

$$M''Q'' = y - \frac{dy}{dx}$$

$$\therefore M P' - M' Q' = \frac{d^2y}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \frac{d^3y}{dx^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots$$

$$M''P'' - M''Q'' = \frac{d^2y}{dx^2} \cdot \frac{h^2}{1 \cdot 2} - \frac{d^3y}{dx^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \ldots$$

But at a point of contrary flexure these differences must have a contrary sign, which cannot be unless $\frac{d^2y}{dx^2} = 0$ or ∞, and if the same value of x which makes $\frac{d^2y}{dx^2}$ vanish, makes $\frac{d^3y}{dx^3}$ vanish also, then in order that there may be a point of inflexion, $\frac{d^4y}{dx^4}$ must vanish also, and so on.

At a multiple point in a curve $\frac{dy}{dx} = \frac{0}{0}$

A multiple point is a point in which two or more branches of the curve intersect or touch each other, and is called a double, a triple, &c. point, according to the number of these branches.

Let (A) $f(x, y) = 0$, be the equation to curve divested of radicals

Let (B) $Mp + N = 0$, the first derived equation, then

(1). If the branches of the curve cut one another at the point, there will be several tangents at that point; and $\therefore$ for the value of x and y which belongs to this point, $\frac{dy}{dx}$ will have as many values as there are branches. Suppose that there are only two branches, and let the two values of p corresponding to these be α, β. Then by equation (B), we must have

$$M\alpha + N = 0$$
$$M\beta + N = 0$$
$$\text{and} \quad \therefore \quad M(\alpha - \beta) = 0$$

$\therefore$ Since α and β are supposed to be unequal, we must have $M = 0$, and $\therefore N = 0$, hence by equation (B)

$$p = -\frac{N}{M}$$
$$= -\frac{0}{0}$$

(2). If the branches of the curve touch with a contact of the first order, there will be only one value of $\frac{dy}{dx}$, but there will be several of q or $\frac{d^2y}{dx^2}$, and in general if the contact be of the n^{th} order, the first n differential co-efficients will have but one value, but the $(n+1)^{th}$ will have several, we shall in that case have

$$M \cdot \frac{d^{n+1}y}{dx^n} + L = 0$$

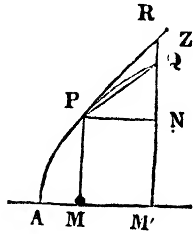
where M is the same as in equation (B), and L is a *rational* function of x, y, and the first n differential co-efficients. Hence it may be shown as before, that $M = 0$, $N = 0$, and $\therefore$

$$\frac{d^{n+1}y}{dx^n} = \frac{0}{0}$$

The converse, however, does not hold, for it does not follow that these values of x which render $\frac{dy}{dx} = \frac{0}{0}$ necessarily belong to a multiple point.

Points of the second species where branches of the curve touch are sometimes by way of distinction called osculating points.

To find the first differential co-efficient of the Arc of a Curve considered as a function of the abscissa.



Let equation to curve AZ be

$$y = f(x)$$

Let arc AP $= s = \varphi(x)$

AM $= x$ MP $= y$, MM' $= h$.

Then it is manifest that the increment PQ of the arc must always be $>$ chord PQ and $<$ (PR + RQ), whatever be the values of h.

$$\text{Now chord PQ} = \sqrt{h^2 + \{f(x+h) - fx\}^2}$$

$$= h\sqrt{1 + p^2 + Qh + } \ldots\ldots\ldots\ldots (1)$$

$$\text{Arc PQ} = \varphi(x+h) - \varphi(x)$$

$$= \frac{ds}{dx} \cdot h + \frac{d^2s}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \ldots\ldots\ldots\ldots (2)$$

$$\text{PR + RQ} = \sqrt{h^2 + \left\{p\frac{h}{1} + q\frac{h^2}{1 \cdot 2} + \ldots\right\}^2}$$

$$= h\sqrt{1 + p^2 + Q'h + } \ldots\ldots\ldots\ldots (3)$$

Now, series (1), (2), (3) are in the order of their magnitude, whatever be the values of h; $\therefore$ their first terms are in the order of magnitude, and these are

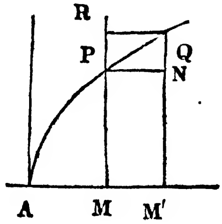
$$h\sqrt{1 + p^2}$$

$$h \cdot \frac{ds}{dx}$$

$$h\sqrt{1 + p^2}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1 + p^2}$$

To find the first differential co-efficient of the Area of a Curve considered as a function of the abscissa.



Equation to curve

$$y = f(x)$$

Area APM $=$ A $= \varphi(x)$

AM $= x$, MP $= y$, MM' $= h$.

Now, it is manifest that MPQ the increment of the area is always $>$ parallelogram MPNM' and $<$ parallelogram MRQM' whatever be the value of h.

$$\text{Now, parallelogram PM}' = y \cdot h \ldots\ldots\ldots\ldots (1)$$

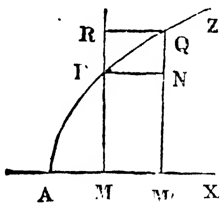
$$\text{Area MPQM}' = \frac{dA}{dx} \cdot \frac{h}{1} + \frac{d^2A}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \ldots\ldots\ldots\ldots (2)$$

$$\text{Parallelogram RM}' = h\left(y + p\frac{h}{1} + q \cdot \frac{h^2}{1 \cdot 2} \cdots\right) \ldots\ldots (3)$$

And series (1), (2), (3), are in the order of their magnitude, whatever be the value of h, and $\therefore$ their first terms are so; hence

$$\frac{dA}{dx} = y$$

To find the first differential co-efficient of the surface of a solid of Revolution, considered as a function of the abscissa of the generating curve.



Let the surface be generated by the revolution of curve AZ whose equation, is

$$y = f(x)$$

round AX as an axis.

Let surface generated by arc AP ($= s$) be S, and let PQ the increment of arc s be k, $S = \phi(x)$

$$AM = x,\ MP = y,\ MM' = h.$$

Then it is manifest that the increment of the surface generated by PQ, is always less than the surface generated by PQ stretched out perpendicular to M'Q from Q, and always greater than the surface generated by PQ stretched out perpendicular to MP from P.

i. e. The surface generated by PQ $>$ surface of cylinder rad. $=$ MP, height $= k$
. . . . $<$ . . . . rad. $=$ M'Q, height $= k$

Now, surface of 1st cylinder $= 2\pi y \,.\, k$

$$= 2\pi y \,.\left(\frac{ds}{dx}\cdot\frac{h}{1} + \frac{d^2s}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \ldots\right) \ldots\ldots (1)$$

$$\text{Surface generated by PQ} = \frac{dS}{dx}\cdot\frac{h}{1} + \frac{d^2S}{dx^2}\cdot\frac{h^2}{1\,.\,2} + \ldots \ldots\ldots\ldots (2)$$

$$\text{Surface of 2nd cylinder} = 2\pi\left(y + p\,.\,\frac{h}{1} + q\,.\,\frac{h^2}{1\,.\,2} + \ldots\right)\left(\frac{ds}{dx}\cdot\frac{h}{1} + \ldots\right) \ldots (3)$$

And series (1), (2), (3), are in the order of magnitude, whatever be the value of h, and $\therefore$ their first terms are in order of magnitude; hence

$$\frac{dS}{dx} = 2\pi y\,\frac{ds}{dx}$$
$$= 2\pi y\,\sqrt{1 + p^2}$$

To find the first differential co-efficient of the volume of a solid of Revolution, considered as a function of the abscissa of the generating curve.

Let the solid be generated by the revolution of a curve whose equation is $y = f(x)$ round the axis of x, and its volume $= V = \phi(x)$.

Then every section of the solid made by a plane perpendicular to the axis of x will be a circle.

Let the area of circular plane whose abscissas is x $= A$.
. . . . $x + h = A'$.

Then the increment of solid is manifestly always $>$ than solid generated by plane A moving parallel to itself through h, and $<$ than the solid generated by A' moving parallel to itself through h.

(1). Now, first solid or $Ah = \pi y^2 h$

(2). Increment of volume $= \frac{dV}{dx} \cdot \frac{h}{1} + \frac{d^2V}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \ldots$

(3). Second solid or $A'h = \pi h \left(y + p \cdot \frac{h}{1} + q \cdot \frac{h^2}{1 \cdot 2} \ldots\right)^2$

And these three series are in the order of their magnitude, whatever be the value of h, and $\therefore$ their first terms are so; hence we have

$$\frac{dV}{dx} = \pi y^2$$

METHOD OF LIMITS.

PROPOSITION.—*If there be an equation of the form*

$$A + x = B + y$$

where A *and* B *are constant quantities, and x and y are susceptible of all degrees of magnitude, then* $A = B$, *and* $x = y$.

For if A be not equal to B, let their difference be represented by P

$$A - B = \pm P$$

whence

$$y - x = \pm P$$

that is, the variables y and x have a constant difference P, and therefore cannot be made less than P, which is contrary to the hypothesis.

This principle is the foundation of the *method of limits*, which is used extensively in the investigations of the higher geometry, and has been employed by many writers to establish the doctrines of the differential calculus.

DEFINITION.—*When a variable quantity by being continually increased or continually diminished, approaches towards a certain fixed quantity, and approaches nearer to this quantity than any assignable difference, but never actually reaches or becomes equal to it, then that fixed quantity is called the* LIMIT *of the variable quantity.*

Thus a circle is the limit of the area of the inscribed and circumscribed polygons. For by continually increasing the number of sides in the polygon, its area will approach nearer to the area of the circle than by any assignable difference, but the sides of the polygon being straight lines, can never actually coincide with the curved perimeter of the polygon, so that the figures should be equal, and, therefore, by the above definition, the circle is the limit of the inscribed and circumscribed polygons.

In like manner if we can make a variable magnitude $A - \alpha$ approach another magnitude A which is fixed, so as to render their difference α less than any assignable magnitude, but without their ever becoming actually equal, then the fixed magnitude A is the limit of the variable magnitude $A - \alpha$.

Let us now consider the differential calculus with reference to these principles. Let y be a function of x, such that

$$y = x^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Let x become $x + h$, and let the corresponding change in the value of y be denoted by y'

$$\therefore \quad y' = (x + h)^3$$
$$= x^3 + 3x^2h + 3xh^2 + h^3$$

Subtract from this equation (1), then

$$y' - y = 3x^2h + 3xh^2 + h^3$$

Divide both sides of the equation by h

$$\frac{y' - y}{h} = 3x^2 + 3xh + h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Here $y' - y$ represents the increment which y receives when x becomes $x + h$, and h is the increment of x.

Hence the expression $\frac{y' - y}{h}$ is the ratio of the contemporaneous increments of y and x: it is manifest from considering the second member of equation (2) that this ratio will diminish as h diminishes, and when h becomes $= 0$ the ratio becomes $= 3x^2$.

This term $3x^2$ is therefore the limit to which the ratio $\frac{y' - y}{h}$ tends, as we diminish h, and the quantity to which it becomes equal when $h = 0$.

But $3x^2$ is the first differential co-efficient of x^3 or y, hence we perceive that in this case the first differential co-efficient of y may be considered as *the limit of the ratio of the increment of the function to the increment of the variable.*

It must be remarked, that upon the supposition $h = 0$, the increment of y becomes also nothing, and consequently the expression $\frac{y' - y}{h}$ is reduced to the form $\frac{0}{0}$ and equation (2) assumes the form

$$\frac{0}{0} = 3x^2$$

This equation involves no absurdity, because Algebra teaches us that $\frac{0}{0}$ may represent every description of quantity.

Generally, let

$$y = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let x become $x + h$ and y become y'

Let $f(x + h)$ be developed in a series ascending regularly by powers of h, so that

$$y' = f(x + h)$$
$$= f(x) + Ah + Bh^2 + Ch^3 + \ldots\ldots\ldots\ldots (2)$$

Subtracting (1) from (2)

$$y' - y = Ah + Bh^2 + Ch^3 + \ldots$$

Divide both sides of the equation by h

$$\frac{y' - y}{h} = A + Bh + Ch^2 + \ldots$$

which represents the ratio of the increment of the function to the increment of the variable; it is manifest that this ratio will diminish as h diminishes.

To find the limit of this ratio make $h = 0$

$$\therefore \quad \frac{y' - y}{h} = \mathrm{A}, \text{ or } \frac{0}{0} = \mathrm{A}$$

but according to the principles which we have already explained at the commencement of the treatise, A is the first differential co-efficient of y or $f'(x)$, hence it appears that

If y be a function of x, the first differential co-efficient of y may be considered as the limit of the ratio of the increment of the function to the increment of the variable.

In all extensive treatises upon the differential calculus, the manner in which the differential co-efficients of all algebraic, transcendental, and circular functions may be obtained by the doctrine of limits, and the different rules established, will be found fully detailed. What has been said above will suffice to give the student a general idea of the nature of this method.

The infinitesimal method.

The ideas which we entertain with regard to an infinite magnitude may be reduced to the following proposition: A quantity is not infinite when it is susceptible of increase, consequently, if we have a quantity $x + a$, and if we suppose x to become infinite, we must suppress a, otherwise we should suppose that x was increased by a, which is contrary to our definition.

This may perhaps be made more evident in the following manner. Let

$$\frac{1}{a} + \frac{1}{x} = \mathrm{M} \qquad (1)$$

Reducing $\quad x + a = \mathrm{M}ax \qquad (2)$

Now, if we suppose that x becomes infinite, the term $\frac{1}{x}$ in equation (1) must disappear or be $= 0$, hence the equation assumes the form

$$\mathrm{M} = \frac{1}{a}$$

Substituting this value in (2)

$$x + a = x$$

which shows that when x is infinite $x + a$ is equivalent to x.

The quantity a, in comparison with which x is infinite is called an *infinitesimal*, or infinitely small quantity, in reference to x.

Since we are now considering the relative values only of quantities, the preceding demonstration will hold good even when x has a finite value, provided only that a be infinitely small in comparison with x. The theory of fractions will enable us to make this truth more manifest. For if we compare the finite quantity b with the fraction $\frac{b}{z}$, it is clear that in proportion as the denominator z becomes greater the fraction itself will become less, so that when z becomes infinite the fraction will become 0, and as such, must be suppressed with reference to b, and thus b will be infinite relatively to $\frac{b}{z}$.

Although two quantities be infinitely small, it does not follow that their ratio will be nothing, for

$$\frac{a}{\alpha} : \frac{b}{\alpha} :: a : b$$

Hence if we represent two infinitely small quantities by dy and dx, their ratio $\frac{dy}{dx}$ may represent any quantity whatever, a result the same as that which we obtained by the consideration of limits.

When a quantity x is infinitely small relatively to a finite magnitude a, the square of x or x^2 is infinitely small relatively to x. For the proportion

$$1 : x :: x : x^2$$

shows that x^2 is involved in x as often as x is involved in unity, that is to say, an infinite number of times. We may demonstrate in the same manner by the proportion

$$x : x^2 :: x^2 : x^3$$

that if x^2 is infinitely small relatively to x, the term x^3 must be infinitely small relatively to x^2. According to this view, infinitesimals are divided into different orders, thus, in the preceding examples, x is an infinitesimal of the first order, x^2 is an infinitesimal of the second order, x^3 is an infinitesimal of the third order, and so on.

We may remark, that if x is infinitely small relatively to a, it will likewise be infinitely small when multiplied by a finite quantity b. In fact, since x may be considered as a fraction whose denominator is infinity, we may represent x by $\frac{p}{\alpha}$, but whether we have $\frac{p}{\alpha}$ or $\frac{bp}{\alpha}$ these quantities will equally be nothing relatively to a.

Since an infinitesimal of the first order must be disregarded when connected with a finite quantity, which it cannot increase, so, in like manner, an infinitesimal of the second order must be disregarded when connected with an infinitesimal of the first order, and so on.

The product of two infinitesimals x and y, of the first order, is an infinitesimal of the second order; for from the product xy we derive the proportion

$$1 : y :: x : xy$$

which shows, that since y is infinitely small relatively to unity, xy will be infinitely small relatively to x; that is to say, xy will be an infinitesimal of the second order.

In like manner, we might prove that the product of three infinitesimals of the first order, is an infinitesimal of the third order.

The differential calculus may be deduced from the theory of infinitesimals. This method of considering the subject is less philosophical than either of the preceding but the results are precisely the same, and as the principles employed will greatly abbreviate many of the processes of the integral calculus, we shall briefly explain their application.

Let y be a function of x, such that

$$y = ax \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Let x be increased by an infinitely small quantity which we shall represent by dx, and let the corresponding infinitely small increment of y be represented

by dy. Hence when x becomes $x + dx$, y will become $y + dy$, and we shall have from the above equation

$$y + dy = a(x + dx)$$
$$= ax + adx$$
$$\therefore \text{ By (1)} \quad dy = adx \quad \text{.......... (2)}$$

The quantity dy is called *the differential of* y, and the quantity dx is called *the differential of* x.

If we divide both sides of equation (2) by dx, we shall have

$$\frac{dy}{dx} = a$$

But we know that a is the differential co-efficient of ax; hence it appears, in the present case, that the first differential co-efficient is the same thing as the ratio of the infinitely small increment of y to the infinitely small increment of x, and that the differential of y is equal to its first differential co-efficient multiplied by the differential of x.

Again, let it be required to find the differential of a function of x, such as ax^3.

Let $$y = ax^3 \quad \text{.......... (1)}$$

Let x become $x + dx$, and let the corresponding change in y be represented by $y + dy$.

$$\therefore \quad y + dy = a(x + dx)^3$$
$$= ax^3 + 3ax^2\,dx + 3ax\,(dx)^2 + (dx)^3$$
$$\therefore \quad dy = 3ax^2\,dx + 3ax\,(dx)^2 + a\,(dx)^3 \quad \text{.......... (2)}$$

But $a\,(dx)^3$ being an infinitesimal of the third order, cannot augment $3a\,(dx)^2$, and may therefore be rejected, and in like manner $3a\,(dx)^2$ being an infinitesimal of the second order, cannot augment $3ax^2\,dx$, and may therefore be rejected, so that equation (2) is reduced to

$$dy = 3ax^2\,dx$$

dividing both sides of the equation by dx we have

$$\frac{dy}{dx} = 3ax^2$$

but $3ax^2$ is the differential co-efficient of ax^3, so that in this case also the differential co-efficient is the same as the ratio of the infinitely small increments of y and x, and the differential of y is equal to its first differential co-efficient multiplied by the differential of x.

Generally, let

$$y = f(x)$$

let x become $x + dx$ and y become $y + dy$

$$y + dy = f(x + dx)$$
$$= f(x) + \mathrm{A}dx + \mathrm{B}\,(dx)^2 + \mathrm{C}\,(dx)^3 + \ldots$$
$$\therefore \quad dy = \mathrm{A}dx + \mathrm{B}\,(dx)^2 + \mathrm{C}\,(dx)^3 + \ldots$$

But $(dx)^2$, $(dx)^3$, . . . being infinitesimals of the second, third, . . . orders, cannot augment $\mathrm{A}dx$, and may therefore be rejected, hence the above equation becomes

$$dy = \mathrm{A}dx$$

Whence $$\frac{dy}{dx} = A$$

but according to the principles which we have already explained, A is the first differential co-efficient of y or $f(x)$, hence it appears that

If y be a function of x, the first differential co-efficient of y may be considered as the ratio of the differentials, or infinitely small increments of y and x; and the differential of y is always equal to the first differential co-efficient of y, multiplied by the differential of x.

In order to find the differential of the product of two variables u and z, each of which is a function of x, we shall suppose that when x becomes $x + dx$, u becomes $u + du$, and z becomes $z + dz$.

$$\therefore \text{ Let } \quad y = f(x)$$
$$= uz$$
$$y + dy = (u + du)(z + dz)$$
$$= uz + udz + zdu + du \,.\, dz$$
$$\therefore \quad dy = udz + zdu + du \,.\, dz$$

But $du\, dz$ being an infinitesimal of the second order, may be neglected.

$$\therefore \quad dy = udz + zdu$$

which agrees with the result already found by a different process.

To find the differential of sin. x according to this method.

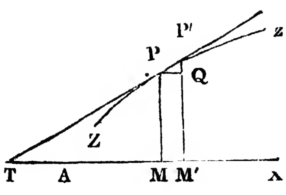
$$y = \sin. x$$
$$y + dy = \sin. (x + dx)$$
$$= \sin. x \cos. dx + \sin. dx \cos. x$$

but the arc dx being infinitely small

$$\sin. dx = dx, \cos. dx = 1$$
$$\therefore \quad y + dy = \sin. x + \cos. xdx$$
$$\therefore \quad dy = \cos. xdx$$
$$\text{or} \quad \frac{dy}{dx} = \cos. x, \text{ as before.}$$

Let us now show how we may resolve the problem of tangents by the method of infinitesimals.

Let Zz be a curve, PM, P′M′ two ordinates infinitely near to each other.



Let PQ be drawn parallel to AX, and let PT be a tangent to the curve at P.

The tangent PT may be considered as a production of P′P the element of the curve Zz, because this element or infinitely small portion of the curve may be considered as a straight line.

Let AM $= x$, PM $= y$

P′Q which is the infinitely small increment of y will be represented by dy

PQ = MM′ . . . . x . . . . dx

The infinitely small triangle P′ Q P being similar to the triangle P M T, we have,

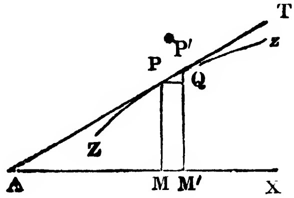
$$P'Q : PQ :: PM : MT$$

or, $$dy : dx :: y : MT$$

$$\therefore \quad MT = y\,\frac{dx}{dy}$$

The subtangent MT being thus known, we can immediately determine the normal and tangent, and the equation to these lines.

To find the differential of an arc of a curve, we may consider the infinitely small arc PP′ included between the ordinates PM, P′M′, as a straight line, and calling the whole arc of the curve s, the infinitely small portion PP′ will be represented by ds.



The right-angled triangle PP′Q gives

$$PP'^2 = PQ^2 + P'Q^2$$

or $$ds^2 = dx^2 + dy^2$$

$$\therefore \quad ds = \sqrt{dx^2 + dy^2}$$

or $$\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} \text{ as before.}$$

To find the differential of the area comprised between two ordinates PM, P′ M′ of a curve which are infinitely near to each other, neglecting the area PP′Q, if we call the whole area A, the area of the rectangle PM′ may be taken for dA

$$\therefore \quad dA = PM \times PQ$$
$$= ydx$$

Polar Curves.

In applying the differential calculus to the theory of curves, we have hitherto considered only such as are referred to rectangular co-ordinates. The various propositions which we have demonstrated may, however, be applied to polar curves also, either directly by Taylor's theorem, or, by adapting the expressions already deduced, by aid of the chapters on the transformation of co-ordinates and the change of the independent variable.

The principles of the infinitesimal calculus may also be employed with much elegance in these investigations. Thus, for example,

To find the angle under the radius vector and a tangent at any point of a polar curve.

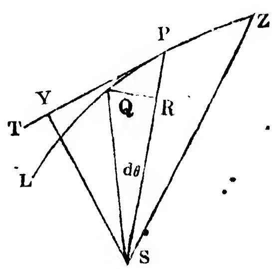
Let LZ be a curve referred to Polar Co-ordinates.

Let S be the pole and SZ the straight line from which the angles are measured.

Take any point P, and draw a tangent PT.

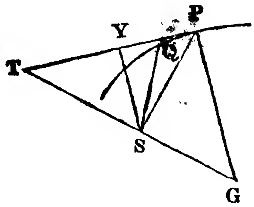
Let SP $= r$, angle ZSP $= \theta$.

Take a point Q infinitely near to P, then the arc PQ may be considered ultimately as coinciding with tangent and angle PSQ $= d\theta$, draw QR perpendicular to SP, and SY perpendicular on tangent.



$$\text{tangent SPY} = \frac{SY}{PY}$$
$$= \frac{QR}{PR} \text{ by similar triangles.}$$
$$= \frac{rd\theta}{dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Through S draw TSG perpendicular to SP, meeting the tangent at P in T, and the normal at P in G.



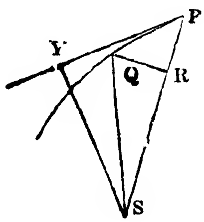
Then ST is the Polar Subtangent.

.... . SG .... Subnormal.

$$ST = SP \text{ tan. SPY.}$$
$$= \frac{r^2 d\theta}{dr} \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$SG : SP :: PY : SY$$
$$SG = \frac{PY}{SY} \cdot SP$$
$$= \text{cotan. SPY . SP}$$
$$= \frac{1}{\frac{d\theta}{dr}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

The following expressions are much employed in the investigations of physical astronomy.



1\. Let
$$ds = \text{arc } PQ$$
$$PQ^2 = QR^2 + PR^2$$
or
$$ds^2 = r^2 d\theta^2 + dr^2$$
$$\therefore \frac{ds}{d\theta} = \sqrt{r^2 + \frac{dr^2}{d\theta^2}}$$

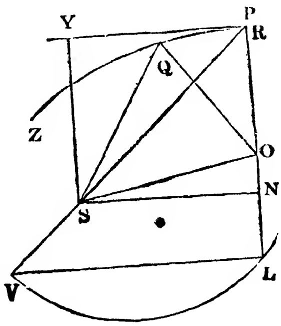
2\.
$$PR : QR :: PY : SY$$
or
$$dr : rd\theta :: \sqrt{r^2 - p^2} : p$$
$$\therefore \frac{dr}{d\theta} = \frac{r\sqrt{r^2 - p^2}}{p}$$

3\. Let area of sector
$$PSQ = dA$$
$$dA = \tfrac{1}{2}\, SP.QR$$
$$= \tfrac{1}{2}\, r^2 d\theta$$
$$\therefore \frac{dA}{d\theta} = \frac{r^2}{2}$$

$$SY : SP :: QR : PQ$$
$$p : r :: rd\theta : \sqrt{dr^2 + r^2 d\theta^2}$$
$$p = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}$$

$$PQ : PR :: SP : PY$$
$$ds : dr :: r : \sqrt{r^2 - p^2}$$
$$\therefore \frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}$$

To find the radius and chord of curvature in polar curves.



Let ZR be a polar curve, S the pole, P any point, Q a point infinitely near to P.

Draw normals at P and Q intersecting in O, O is the centre of curvature.

Produce PO and draw SN perpendicular on it.

$$SY = p, \quad SP = r, \quad OP = \varrho.$$

Now while the arc of curve receives the increment PQ, and SP varies from SP to SQ; the point O remains fixed, and $\therefore$ OP and SO remain constant. But

$$SO^2 = SP^2 + PO^2 - 2PO \cdot PN$$
$$= r^2 + \varrho^2 - 2p\varrho$$

$\therefore$ differentiating $o = rdr - dp \cdot \varrho$

$$\varrho = r \cdot \frac{dr}{dp} \quad \text{.........} \quad (1)$$

To find the chord of curvature through S, produce PS and PO to meet the circle of curvature in V and L.

Then since the angle at V is a right angle being in a semicircle, the triangles PVL, PSN are similar.

$$\therefore \; PV : PL :: PN : PS$$

$$PV = \frac{2\varrho \cdot p}{r}$$
$$= \frac{2 \cdot \frac{rdr}{dp} \cdot p}{r}$$
$$= \frac{2pdr}{dp} \quad \text{.........} \quad (2)$$

We shall conclude by showing how the first of the above propositions may be established by the transformation of co-ordinates.

To find the angle under the radius vector and tangent, in a spiral curve.

Let RZ be a spiral curve whose pole is S and equation

$$r = f\ (\theta)$$

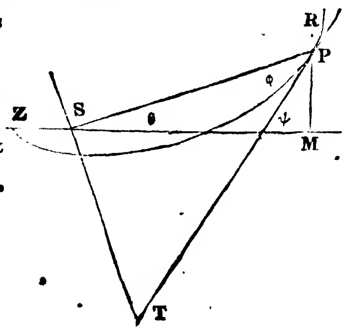
PT is a tangent at point P.

r and θ are polar co-ordinates of the point P.

x and y ... rectangular ....

$$SM = x, \qquad MP = y.$$

$$\tan.\ \varphi = \tan.\ (\psi - \theta)$$
$$= \frac{\tan.\ \psi - \tan.\ \theta}{1 + \tan.\ \theta \tan\ \psi}$$



$$= \frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{y}{x} \cdot \frac{dy}{dx}}$$

If we wish to transform this expression into another in which r is the independent variable, we shall have

$$\tan.\ \phi = \frac{\frac{\frac{dy}{dr}}{\frac{dx}{dr}} - \frac{y}{x}}{1 + \frac{y}{x}\,\frac{\frac{dy}{dr}}{\frac{dx}{dr}}} \quad \text{.......................................} \quad (1)$$

Now $\qquad y = r \sin.\ \theta, \qquad x = r \cos.\ \theta.$

$$\therefore \qquad \frac{dy}{dr} = \sin.\ \theta + r \cos.\ \theta \cdot \frac{d\theta}{dr}, \quad \frac{dx}{dr} = \cos.\ \theta - r \sin.\ \theta \cdot \frac{d\theta}{dr}$$

Substituting these values in (1), we have

$$\tan.\ \phi = \frac{\dfrac{\sin.\ \theta + r \cos.\ \theta \frac{d\theta}{dr}}{\cos.\ \theta - r \sin.\ \theta \frac{d\theta}{dr}} - \dfrac{\sin.\ \theta}{\cos.\ \theta}}{1 + \dfrac{\sin.\ \theta}{\cos.\ \theta} \cdot \dfrac{\sin.\ \theta + r \cos.\ \theta \frac{d\theta}{dr}}{\cos.\ \theta - r \sin.\ \theta \frac{d\theta}{dr}}}$$

$$= r \frac{d\theta}{dr}$$

$$ST = r^2 \frac{d\theta}{dr}$$

INTEGRAL CALCULUS.

CHAPTER I.

The object of the Integral Calculus is to discover the primitive function from which a given differential co-efficient has been derived.

This primitive function is called the *integral* of the proposed differential co-efficient, and is obtained by the application of the different principles established in finding differential co-efficients and by various transformations. In order to avoid the embarrassment which would arise from the perpetual changes of the independent variable, which it would be necessary to effect if we restricted ourselves to the use of differential co-efficients alone, we shall generally employ differentials according to the infinitesimal method explained in the preceding chapter.

When we wish to indicate that we are to take the integral of a function we prefix the symbol $\int$. Thus, if

$$y = ax^4$$

We know that $dy = 4ax^3\, dx$

If then, the quantity $4ax^3\, dx$ be given in the course of any calculation, and we are desirous to indicate that the primitive function from which it has been derived is ax^4, we express this by writing

$$\int 4ax^3\, dx = ax^4$$

When constant quantities are combined with variable quantities by the signs $+$ or $-$ we know that they disappear in taking the differential co-efficients, and therefore they must be restored in taking the integral.

Thus, if $\qquad y = ax^3 + b$

or, $\qquad y = ax^3 - b$

or, $\qquad y = ax^3$

In each of the three cases equally

$$dy = 3ax^2\, dx$$

Hence in taking the integral of any function it is proper always to add a constant quantity, which is usually represented by the symbol C. Thus, if it be required to find the integral of a quantity such as

$$dy = 3ax^2\,dx$$
$$y = \int 3ax^2\,dx$$
$$= ax^3 + C$$

where C may be either positive, negative, or 0. We cannot determine the value of C in an abstract example, but when particular problems are submitted to our investigation, they usually contain conditions by which the value of C can be ascertained. This will be clearly seen when we treat of the applications of the integral calculus.

By reversing the principal rules established for finding the differential co-efficients, or differentials of functions, we shall obtain an equal number of rules for ascending to the integrals from the derived functions. Recurring therefore to these we shall perceive that

I. *The integral of the sum of any number of functions is equal to the sum of the integrals of the individual terms, each term retaining the sign of its co-efficient.* Thus, if

$$dy = 4ax^3\,dx + 3bx^2\,dx - 2bx\,dx + dx$$
$$y = \int 4ax^3\,dx + \int 3bx^2\,dx - \int 2\,bx\,dx + \int dx + C$$

II. Since, if

$$y = ax^m$$
$$dy = max^{m-1}\,dx$$

it is manifest that

The integral of a function raised to any power is obtained by adding unity to the exponent of the function, and dividing the function by the exponent so increased, and by the differential of the function.

Ex. 1.
$$dy = ax^n\,dx$$
$$y = \frac{ax^{n+1}}{n+1} + C$$

Ex. 2.
$$dy = \frac{a}{x^n}\,dx$$
$$= ax^{-n}\,dx$$
$$y = \frac{ax^{-(n-1)}}{-(n-1)}$$
$$= -\frac{a}{(n-1)\,x^{n-1}}$$

Ex. 3.
$$dy = (a+x)^n\,dx$$
$$y = \frac{(a+x)^{n+1}}{n+1} + C$$

$$dy = \frac{1}{(a+x)^n}\,dx$$
$$= (a+x)^{-n}\,dx$$

$$y = -\frac{1}{(n-1)(a+x)^{n-1}}$$

This rule applies to all functions of the form

$$dy = (a + bx^n)^m \, cx^{n-1} \, dx$$

for these can all be reduced to the form $az^m \, dz$. Thus,

Let $\quad a + bx^n = z$

$\therefore \quad nbx^{n-1} \, dx = dz$

$$x^{n-1} \, dx = \frac{1}{nb} dz$$

$$cx^{n-1} \, dx = \frac{c}{nb} \, dz$$

$$\therefore \quad y = \int (a + bx^n)^m \, cx^{n-1} \, dx = \int \frac{c}{nb} z^m \, dz$$

$$= \frac{c}{nb} \cdot \frac{z^{m+1}}{m+1} + C$$

$$= \frac{c}{n.b} \cdot \frac{(a + bx^n)^{m+1}}{m+1} + C$$

This formula is very extensive in its application, since we have all integrals of the form $(\phi(x))^n \, d.\phi(x)$ composed of two factors, where the one is the differential of the other which is within the bracket.

Ex. 5.

Here $\quad a = a^2, b = 1, n = 2, m = -\frac{1}{2}, c = 1$

$$y = \sqrt{a^2 + x^2} + C$$

Ex. 6. $\quad dy = \dfrac{xdx}{\sqrt{a^2 - x^2}}$

$$y = -\sqrt{a^2 - x^2} + C$$

Ex. 7. $\quad dy = \dfrac{xdx}{(a^2 + x^2)^{\frac{3}{2}}}$

$$y = -(a^2 + x^2)^{-\frac{1}{2}} + C$$

Ex. 8. $\quad dy = \dfrac{xdx}{(a^2 - x^2)^{\frac{3}{2}}}$

$$y = (a^2 - x^2)^{-\frac{1}{2}} + C$$

Ex. 9. $\quad dy = (adx - xdx)(2ax - x^2)^{\frac{1}{2}}$

Let $\quad 2ax - x^2 = z \qquad \therefore \; adx - xdx = \frac{1}{2} dz$

$$(adx - xdx)(2ax - x^2)^{\frac{1}{2}} = \frac{z^{\frac{1}{2}} \, dz}{2}$$

$$y = \tfrac{1}{3} z^{\frac{3}{2}} = \frac{(2ax - x^2)^{\frac{3}{2}}}{3} + C$$

Ex. 10. $$dy = \frac{adx + xdx}{\sqrt{2ax + x^2}}$$

Let $2ax + x^2 = z \quad \therefore \quad adx + xdx = \frac{1}{2} dz$

$$(adx + xdx)(2ax + x^2)^{-\frac{1}{2}} = \frac{z^{-\frac{1}{2}}\, dz}{2}$$

$$y = z^{\frac{1}{2}} = (2ax + x^2)^{\frac{1}{2}} + C$$

Ex. 11. $$dy = \frac{\sqrt{2ax - x^2} \,.\, dx}{x^{\frac{1}{2}}}$$

$$= \sqrt{2a - x} \,.\, dx$$

$$y = -\tfrac{2}{3}(2a - x)^{\frac{3}{2}} + C$$

Ex. 12. $$dy = (x^3 + y^2)^{\frac{3}{2}}(6x^2\, dx + 4y\, dy)$$

Let $x^3 + y^2 = z \;\therefore\; 3x^2\, dx + 2y\, dy = dz \;\therefore\; 6x^2\, dx + 4y\, dy = 2dz$

$$(x^3 + y^2)^{\frac{3}{2}}(6x^2\, dx + 4y\, dy) = 2z^{\frac{3}{2}}\, dz$$

$$y = \frac{4}{5} z^{\frac{5}{2}} = \frac{4}{5}(x^3 + y^2)^{\frac{5}{2}} + C$$

III. The above rule fails when $n = -1$, since in that case we should find $\int z^{-1}\, dz = \infty$, but this arises from the circumstance that the integral belongs to another kind of function.

We know that $\int \frac{dz}{z} = \log.\, z + c$; and, in like manner $\int \frac{dz}{a + z} = \log.\,(a + z) + c$, hence,

The integral of every fraction whose numerator is the differential of its denominator, is the logarithm of the denominator.

IV. *The integral of every fraction whose denominator is a radical of the second degree, and whose numerator is the differential of the quantity under the radical sign, is equal to twice that radical.* Thus,

If $$dy = \frac{dx}{\sqrt{x}} \quad \therefore \quad y = 2\sqrt{x} + C$$

V. A most important process is that which is called *Integration by parts;* it depends on the following consideration, if y be a function of x

$$d(xy) = xdy + ydx$$

$$\therefore \quad xy = \int xdy + \int ydx$$

$$\int xdy = xy - \int ydx$$

hence it appears, that

Having resolved a differential into two factors, one of which can be imme-

diately integrated, we may take this integral regarding the other factor as constant; we must then differentiate the result thus obtained, upon the supposition that the factor which we considered constant is the only variable, and then subtract the integral of this differential from the first result.

Thus, in order to obtain the integral of log. $x\,dx$, let us consider this differential as composed of two factors, log. x and dx. The integral of dx is x; and therefore, considering log. x as constant, the integral of log. $x\,dx$ will be x log. x; now differentiate this result upon the supposition that log. x alone is variable, and we have $x \cdot \frac{dx}{x}$; subtract the integral of this differential from the integral first obtained, and we shall have the whole integral required.

$$\therefore \int \log. x\, dx = x \log. x - \int x \frac{dx}{x}$$
$$= x \log. x - \int dx$$
$$= x \log. x - x + C$$

Numerous examples of the application of this principle will occur in what follows.

VI. From the operations performed in the differential calculus, we know by reverting the fundamental processes, that since

$d\,x^m$	$= mx^{m-1}dx$	$\therefore \int x^{m-1}dx$	$= \frac{1}{m} \cdot x^m + C$
$d\,a^x$	$= \log. a.a^x dx$	$\int a^x\,dx$	$= \frac{1}{\log. a} \cdot a^x + C$
$d\,e^x$	$= e^x dx$	$\int e^x\,dx$	$= e^x + C$
$d \log. x$	$= \frac{1}{\log. a} \cdot \frac{dx}{x}$	$\int \frac{dx}{x}$	$= \log. a \log. x + C$
$d \log._e x$	$= \frac{dx}{x}$	$\int \frac{dx}{x}$	$= \log._e x + C$
$d \sin. x$	$= \cos. x dx$	$\int \cos. x dx$	$= \sin. x + C$
$d \cos. x$	$= - \sin. x dx$	$\int \sin. x dx$	$= - \cos. x + C$
$d \tan. x$	$= \sec.^2 x dx$	$\int \sec.^2 x dx$	$= \tan. x + C$
$d \cot. x$	$= - \operatorname{cosec}.^2 x dx$	$\int \operatorname{cosec}.^2 x dx$	$= - \cot. x + C$
$d \sec. x$	$= \tan. x \sec. x dx$	$\int \tan. x \sec. x dx$	$= \sec. x + C$
$d \operatorname{cosec}. x$	$= - \cot. x \operatorname{cosec}. x dx$	$\int \cot. x \operatorname{cosec}. x\, dx$	$= - \operatorname{cosec}. x + C$
$d \operatorname{vers}. x$	$= \sin. x dx$	$\int \sin. x dx$	$= \operatorname{vers}. x + C.$
$d \sin mx$	$= m \cos. mx dx$	$\int \cos. mx dx$	$= \frac{\sin. mx}{m} + C$
$d \cos. mx$	$= - m \sin. mx dx$	$\int \sin. mx dx$	$= - \frac{\cos. mx}{m} + C$
$d \sin.^n x$	$= n \sin.^{n-1} x \cos. x dx$	$\int \sin.^{n-1} x \cos. x dx$	$= \frac{\sin.^n x}{n} + C$

And in a similar manner, from the chapter on Inverse Functions, we have

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin.^{-1}x + C \qquad \int \frac{dx}{\sqrt{a^2-b^2x^2}} = \frac{1}{b}\sin.^{-1}\frac{b}{a}x$$

$$\int \frac{-dx}{\sqrt{1-x^2}} = \cos.^{-1}x + C \qquad \int \frac{-dx}{\sqrt{a^2-b^2x^2}} = \frac{1}{b}\cos.^{-1}\frac{b}{a}x$$

$$\int \frac{dx}{1+x^2} = \tan.^{-1}x + C \qquad \int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab}\tan.^{-1}\frac{b}{a}x$$

$$\int \frac{-dx}{1+x^2} = \cot.^{-1}x + C \qquad \int \frac{-dx}{a^2+b^2x^2} = \frac{1}{ab}\cot.^{-1}\frac{b}{a}x$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec.^{-1}x + C \qquad \int \frac{dx}{x\sqrt{b^2x^2-a^2}} = \frac{1}{a}\sec.^{-1}\frac{b}{a}x$$

$$\int \frac{-dx}{x\sqrt{x^2-1}} = \text{cosec.}^{-1}x + C \qquad \int \frac{-dx}{x\sqrt{b^2x^2-a^2}} = \frac{1}{a}\text{cosec.}^{-1}\frac{b}{a}x$$

$$\int \frac{dx}{\sqrt{2x-x^2}} = \text{vers.}^{-1}x + C \qquad \int \frac{dx}{\sqrt{a^2x-b^2x^2}} = \frac{1}{b}\text{vers.}^{-1}\frac{2b^2}{a^2}x$$

$$\int \frac{-dx}{\sqrt{2x-x^2}} = \text{covers.}^{-1}x + C \qquad \int \frac{-dx}{\sqrt{a^2x-b^2x^2}} = \frac{1}{b}\text{covers.}^{-1}\frac{2b^2}{a^2}x$$

In all these integrals the radius of the arcs is unity, and the arbitrary constant is not annexed to the integrals in the right hand column, for want of breadth of page. As it is frequently desirable to integrate differentials in which the radius is a instead of unity, we shall exhibit a few of those which most frequently occur to that radius. In the left hand column of the above differentials, write $\frac{x}{a}$ for x, and we have to radius a.

$$\int \frac{adx}{\sqrt{a^2-x^2}} = \sin.^{-1}x + C \qquad \int \frac{-adx}{\sqrt{a^2-x^2}} = \cos.^{-1}x + C$$

$$\int \frac{a^2dx}{a^2+x^2} = \tan.^{-1}x + C \qquad \int \frac{-a^2dx}{a^2+x^2} = \cot.^{-1}x + C$$

$$\int \frac{a^2dx}{x\sqrt{x^2-a^2}} = \sec.^{-1}x + C \qquad \int \frac{-a^2dx}{x\sqrt{x^2-a^2}} = \text{cosec.}^{-1}x + C$$

$$\int \frac{adx}{\sqrt{2ax-x^2}} = \text{vers.}^{-1}x + C \qquad \int \frac{-adx}{\sqrt{2ax-x^2}} = \text{covers.}^{-1}x + C$$

These are the elementary forms to which every differential whose integral is required must be decomposed; and the reduction of expressions to one or more of these fundamental formulæ is the object of almost every process in the Integral Calculus. The following are a few examples.

(1.) Let it be required to integrate $du = \frac{mdx}{a+bx^2}$

$$\text{Here } du = \frac{mdx}{a+bx^2} = m \cdot \frac{dx}{a+bx^2}$$

$$\therefore\ u = m\int \frac{dx}{a+bx^2} = \frac{m}{\sqrt{ab}}\tan.^{-1}x\sqrt{\frac{b}{a}} + C$$

(2.) Let it be required to integrate $du = \frac{mdx}{\sqrt{a^2-bx^2}}$

$$\text{Here } u = m\int \frac{dx}{\sqrt{a^2-bx^2}} = \frac{m}{\sqrt{b}} \cdot \sin.^{-1}\frac{\sqrt{b}}{a}x + C$$

(3.) $du = ax^3dx$ $\qquad \therefore v = \frac{ax^4}{4} + C$

(4.) $du = adx + \frac{bdx}{x^3} + x^{-\frac{1}{2}}dx$ $\qquad u = ax - \frac{b}{2x^2} + 2x^{\frac{1}{2}} + C$

(5.) $du = (a^2-x^2)^{\frac{3}{2}}xdx$ $\qquad u = -\frac{(a^2-x^2)^{\frac{5}{2}}}{5} + C$

(6.) $du = \frac{dx}{x\sqrt{2ax-x^2}}$ $\qquad u = -\frac{\sqrt{2a-x}}{a\sqrt{x}} + C$

(7.) $du = \sin.^{n-1}x \cos. xdx$ $\qquad u = \frac{\sin.^n x}{n} + C$

(8.) $du = (\tan.^5 x + \tan.^7 x)dx$ $\qquad u = \frac{1}{6}\tan.^6 x + C$

(9.) $du = \frac{2dx}{4+x^2}$ $\qquad u = \tan.^{-1}\frac{x}{2} + C$

(10.) $du = \frac{mdx}{\sqrt{1-m^2x^2}}$ $\qquad u = \sin.^{-1}mx + C$

(11.) $du = \sin. 2xdx$ $\qquad u = \sin.^2 x + C$

(12.) $du = \frac{3x^2+2x+1}{x^3+x^2+x+1}dx$ $\qquad u = \log.(x^3 + x^2 + x + 1) + C$

(13.) $du = \frac{3dx}{1+4x^2}$. $\qquad u = \frac{3}{2}\tan.^{-1}2x + C$

On the Integration of Binomial Differentials.

Let it be required to integrate

$$du = x^{m-1}(a+bx^n)^{\frac{p}{q}}dx.$$

This function can always be rendered rational, whenever $\frac{m}{n}$ is an integer, or when $\frac{m}{n}+\frac{p}{q}$ is an integer, or zero.

I. Let $a + bx^n = v^q \;\therefore\; bx^n = v^q - a$

$$\therefore b^{\frac{m}{n}}x^m = (v^q-a)^{\frac{m}{n}} \;\therefore\; b^{\frac{m}{n}}x^{m-1}dx = \frac{q}{n}v^{q-1}(v^q-a)^{\frac{m}{n}-1}dv$$

$$\therefore du = \frac{q}{nb^{\frac{m}{n}}}v^{p+q-1}(v^q-a)^{\frac{m}{n}-1}dv;$$

an expression which is easily integrated when $\frac{m}{n}$ is an integer.

II. Let $a + bx^n = v^q x^n \;\therefore\; x^n = a(v^q-b)^{-1}$

$$\therefore x^m = a^{\frac{m}{n}}(v^q-b)^{-\frac{m}{n}}$$

$$\therefore x^{m-1}dx = -\frac{qa^{\frac{m}{n}}}{n}\cdot\frac{v^{q-1}dv}{(v^q-b)^{\frac{m}{n}+1}}$$

and $(a+bx^n)^{\frac{p}{q}} = (v^q x^n)^{\frac{p}{q}} = v^p.a^{\frac{p}{q}}.(v^q-b)^{-\frac{p}{q}}$

$$\therefore du = -\frac{q.a^{\frac{m}{n}+\frac{p}{q}}}{n}\cdot\frac{v^{p+q-1}}{(v^q-b)^{\frac{m}{n}+\frac{p}{q}+1}}.$$

which is rational, when $\frac{m}{n}+\frac{p}{q}$ is an integer, or zero.

EXAMPLES.

(1.) Let $du = \dfrac{x^5 dx}{a^2+x^2} = x^5 (a^2+x^2)^{-1} dx.$

Here $\dfrac{m}{n} = \dfrac{6}{2} = 3$, an integer.

$$\text{Put } a^2+x^2 = v^2 \quad \therefore x^2 = v^2 - a^2$$
$$\therefore x^6 = v^6 - 3v^4 a^2 + 3v^2 a^4 - a^6$$
$$\therefore x^5 dx = (v^5 - 2a^2 v^3 + a^4 v)\, dv$$
$$\therefore \frac{x^5 dx}{a^2+x^2} = (v^3 - 2a^2 v + a^4 v^{-1})\, dv$$
$$\therefore a = \int v^3 dv - 2a^2 \int v\, dv + a^4 \int \frac{dv}{v}$$
$$= \frac{v^4}{4} - a^2 v^2 + a^4 \log. v$$
$$= \frac{v^2 (v^2 - 4a^2)}{4} + a^4 \log. v$$
$$= \frac{(a^2+x^2)(x^2-3a^2)}{4} + a^4 \log. \sqrt{a^2+x^2} + C.$$

(2.) Let $du = \dfrac{x^3 dx}{\sqrt{a^2-x^2}} = x^3 (a^2-x^2)^{-\frac{1}{2}} dx$

Here $\dfrac{m}{n} = 2$, and the differential can easily be integrated.

$$\text{Put } a^2 - x^2 = v^2 \quad \therefore x^2 = a^2 - v^2$$
$$\therefore x^4 = a^4 - 2a^2 v^2 + v^4$$
$$\therefore x^3 dx = \quad - a^2 v\, dv + v^3 dv$$
$$\therefore \frac{x^3 dx}{\sqrt{a^2-x^2}} = \quad - a^2 dv + v^2 dv$$
$$\therefore u = -a^2 v + \frac{v^3}{3} = \frac{v}{3} (-3a^2 + v^2)$$
$$= -\frac{2a^2+x^2}{3} \sqrt{a^2-x^2} + C.*$$

(3.) Let $du = \dfrac{dx}{x^4 \sqrt{a^2-x^2}} = x^{-4} (a^2-x^2)^{-\frac{1}{2}} dx$

Here $\dfrac{m}{n} = -\dfrac{3}{2}$, $\dfrac{p}{q} = -\dfrac{1}{2}$ $\therefore$ $\dfrac{m}{n} + \dfrac{p}{q} = -2$, an integer.

Put $\sqrt{a^2-x^2} = vx$ $\therefore$ $a^2 - x^2 = v^2 x^2$, and $x^{-2} = \dfrac{1+v^2}{a^2}$

$$\therefore x^{-4} = \frac{(1+v^2)^2}{a^4}, \text{ and, since } x = a(1+v^2)^{-\frac{1}{2}}$$
$$\therefore dx = -a(1+v^2)^{-\frac{3}{2}} v\, dv$$
$$\therefore du = \frac{(1+v^2)^2}{a^4} \times \frac{1}{vx} \times - \frac{av\,dv}{(1+v^2)^{\frac{3}{2}}}$$
$$= -\frac{1}{a^3}(1+v^2)\, dv, \text{ for } x = a(1+v^2)^{-\frac{1}{2}}$$
$$\therefore u = -\frac{1}{a^4}\left(v + \frac{v^3}{3}\right) = -\frac{v(v^2+3)}{3a^4}$$
$$= -\frac{(a^2+2x^2)}{3a^4 x^3} \sqrt{a^2-x^2} + C.$$

* In this manner a certain class of integrals may be very elegantly obtained.

CHAPTER II.

Rational Fractions.

It is readily proved, by the theory of equations, that every rational fraction of the form $\frac{P}{Q}$ may be decomposed into others, which must have one of the following forms:

$$\frac{A}{x-a}, \quad \frac{A}{(x-a)^n}, \quad \frac{Ax+B}{x^2+px+q}, \quad \frac{Ax+B}{(x^2+px+q)^n} \ldots\ldots\ldots\ldots\ldots (1)$$

where the quantities A, B, p, q, n, ... are constants, and the factors of the expression x^2+px+q are imaginary.

Put $x=z-\frac{p}{2}$, and, by substitution, we have

$$x^2+px+q=\left(z-\frac{p}{2}\right)^2+p\left(z-\frac{p}{2}\right)+q$$

$$=z^2+q-\frac{p^2}{4}=z^2+a^2, \text{ if } a=\sqrt{q-\frac{p^2}{4}};$$

Hence the last two of the forms in (1) are reduced to

$$\frac{Az+B'}{z^2+a^2} \text{ and } \frac{Az+B'}{(z^2+a^2)^n}.$$

I. Let it be required to integrate $du=\frac{dx}{a^2-x^2}$.

Assume $\frac{1}{a^2-x^2}=\frac{1}{(a+x)(a-x)}=\frac{A}{a+x}+\frac{B}{a-x}$; whence we have

$$1=A(a-x)+B(a+x).$$

Let $x=\quad a \therefore 1=2aB \therefore B=\frac{1}{2a}$

$x=-a \therefore 1=2aA \therefore A=\frac{1}{2a}$

hence, $du=dx\left(\frac{A}{a+x}+\frac{B}{a-x}\right)=\frac{1}{2a}\left(\frac{dx}{a+x}+\frac{dx}{a-x}\right)$

$$\therefore u=\frac{1}{2a}\int\frac{dx}{a+x}+\frac{1}{2a}\int\frac{dx}{a-x}$$

$$=\frac{1}{2a}\{\log.(a+x)-\log.(a-x)+\log. C\}=\frac{1}{2a}.\log. C\frac{a+x}{a-x}.$$

II. Let it be required to integrate $du=\frac{Adx}{(x-a)^n}$.

Here $du=A(x-a)^{-n}dx$; therefore, by integration, we have

$$u=A\int(x-a)^{-n}dx=-\frac{A}{(n-1)(x-a)^{n-1}}.$$

Ex. Let $du=\frac{x^3+x^2+2}{x^5-2x^3+x}dx$.

Here $x^5-2x^3+x=x(x^4-2x^2+1)=x(x^2-1)^2=x(x+1)^2(x-1)^2$,

and, therefore, we assume

$$\frac{x^3+x^2+2}{x^5-2x^3+x} = \frac{A}{x} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)};$$

which, reduced to a common denominator, gives the equation

$$x^3+x^2+2 = A(x^2-1)^2 + Bx(x-1)^2 + Cx(x+1)(x-1)^2 + Dx(x+1)^2 + Ex(x+1)^2(x-1).$$

$$\begin{aligned} \text{Let } x &= 0; \text{ then } 2 = A \\ x &= 1; \text{ then } 4 = 4D \therefore D = 1 \\ x &= -1; \text{ then } 2 = -4B \therefore B = -\frac{1}{2} \end{aligned}$$

Substitute these values of A, B, D, in the above equation, and we have

$$x^3+x^2+2 = 2(x^2-1)^2 - \tfrac{1}{2}x(x-1)^2 + x(x+1)^2 + Cx(x+1)(x-1)^2 + Ex(x+1)^2(x-1),$$

or, $-2x^2(x^2-1) + \frac{1}{2}x(x^2-1) = Cx(x+1)(x-1)^2 + Ex(x+1)^2(x-1)$

and, dividing both sides by $x(x+1)(x-1)$, we have

$$-2x + \frac{1}{2} = C(x-1) + E(x+1).$$

$$\begin{aligned} \text{Let } x &= 1; \text{ then } -\frac{3}{2} = 2E \quad E = -\frac{3}{4}. \\ x &= -1; \text{ then } \frac{5}{2} = -2C \quad C = -\frac{5}{4}. \end{aligned}$$

$$\text{Hence, } du = 2\frac{dx}{x} - \frac{1}{2} \cdot \frac{dx}{(x+1)^2} \quad \frac{5}{4} \cdot \frac{dx}{x+1} + \frac{dx}{(x-1)^2} - \frac{3}{4} \cdot \frac{dx}{x-1}$$

$$\therefore u = 2\log. x + \frac{1}{2} \cdot \frac{1}{x+1} - \frac{5}{4}\log.(x+1) - \frac{1}{x-1} - \frac{3}{4}\log.(x-1) + C.$$

III. Let it be required to integrate $du = \dfrac{Ax+B}{x^2+a^2}dx$.

Here $du = \dfrac{Ax\,dx}{x^2+a^2} + \dfrac{B\,dx}{x^2+a^2}$, and, therefore, we have

$$\begin{aligned} u &= A\int \frac{xdx}{x^2+a^2} + B\int \frac{dx}{x^2+a^2} \\ &= \frac{A}{2}\log.(x^2+a^2) + \frac{B}{a}\tan.^{-1}\frac{x}{a} + C. \end{aligned}$$

Ex. Let $du = \dfrac{xdx}{x^3-1}$.

Here $\dfrac{xdx}{x^3-1} = \dfrac{xdx}{(x-1)(x^2+x+1)}$, and, therefore, we must assume

$$\frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\begin{aligned} \therefore x &= A(x^2+x+1) + Bx(x-1) + C(x-1) \\ &= (A+B)x^2 + (A-B+C)x + A - C; \end{aligned}$$

hence, by the method of undetermined coefficients, we have

$$\begin{aligned} A + B &= 0 \\ A - B + C &= 1 \\ A - C &= 0 \end{aligned}$$

from which $A = C = \dfrac{1}{3}$, and $B = -A = -\dfrac{1}{3}$;

$$\therefore du = \frac{1}{3}\frac{dx}{x-1} - \frac{1}{3}\frac{(x-1)\,dx}{x^2+x+1}$$

$$\therefore u = \frac{1}{3}\int\frac{dx}{x-1} - \frac{1}{3}\int\frac{xdx}{x^2+x+1} + \frac{1}{3}\int\frac{dx}{x^2+x+1}.$$

Now, $x^2+x+1 = x^2+x+\frac{1}{4}+\frac{3}{4} = (x+\frac{1}{2})^2+\frac{3}{4}$,
and, if $x+\frac{1}{2}=y$, or $x = y-\frac{1}{2}$; then we have $xdx = ydy - \frac{1}{2}dy$.

$$\therefore\ u = \tfrac{1}{3}\int\frac{dx}{x-1} - \tfrac{1}{3}\int\frac{ydy}{y^2+\frac{3}{4}} + \tfrac{1}{2}\int\frac{dy}{y^2+\frac{3}{4}}$$

$$= \tfrac{1}{3}\ \log(x-1) - \tfrac{1}{6}\log(y^2+\tfrac{3}{4}) + \tfrac{1}{2}\ \frac{2}{\sqrt{3}}\tan^{-1}\frac{2y}{\sqrt{3}}$$

$$= \tfrac{1}{3}\ \log(x-1) - \tfrac{1}{3}\log\sqrt{x^2+x+1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$$

$$= \tfrac{1}{3}\left\{\log(x-1) - \log\sqrt{x^2+x+1} + \sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}}\right\} + \text{C}.$$

IV. Let it be required to integrate $du = \dfrac{\text{A}x+\text{B}}{(x^2+a^2)^n}\,dx$.

Here $du = \dfrac{\text{A}xdx}{(x^2+a^2)^n} + \dfrac{\text{B}dx}{(x^2+a^2)^n}$,

and $u = -\dfrac{\text{A}}{2\,(n-1)\,(x^2+a^2)^{n-1}} + \text{B}\displaystyle\int\frac{dx}{(x^2+a^2)^n}$.

To obtain the integral of the differential $\dfrac{dx}{(x^2+a^2)^n}$, we may assume

$$\int\frac{dx}{(x^2+a^2)^n} = \frac{\text{H}x}{(x^2+a^2)^{n-1}} + \int\frac{\text{K}dx}{(x^2+a^2)^{n-1}}\ \ldots\ldots\ldots\ldots\ldots\ (1)$$

Differentiate this equation, and reduce to a common denominator; then,

$$1 = \{(\text{H}+\text{K}) - 2\,(n-1)\,\text{H}\}\,x^2 + (\text{H}+\text{K})\,a^2$$

$$\therefore\ (\text{H}+\text{K})\,a^2 = 1, \text{ and } 2\,(n-1)\,\text{H} = \text{H}+\text{K};$$

whence $\text{H} = \dfrac{1}{2a^2\,(n-1)}$, and $\text{K} = \dfrac{2n-3}{2a^2\,(n-1)}$, and eq. (1) becomes

$$\int\frac{dx}{(x^2+a^2)^n} = \frac{x}{2a^2\,(n-1)\,(x^2+a^2)^{n-1}} + \frac{2n-3}{2a^2\,(n-1)}\int\frac{dx}{(x^2+a^2)^{n-1}}\ \ldots\ (\text{A})$$

a formula which, by successive operations, diminishes the exponent n, and finally reduces the differential to $\dfrac{dx}{x^2+a^2}$, whose integral is known.

Ex. Let $du = \dfrac{x^4+2x^3+3x^2+3}{(x^2+1)^3}\,dx$.

Assume $\dfrac{x^4+2x^3+3x^2+3}{(x^2+1)^3} = \dfrac{\text{A}x+\text{B}}{(x^2+1)^3} + \dfrac{\text{C}x+\text{D}}{(x^2+1)^2} + \dfrac{\text{H}}{x^2+1}$

$$\therefore\ x^4+2x^3+3x^2+3 = \text{A}x+\text{B} + (\text{C}x+\text{D})\,(x^2+1) + \text{H}\,(x^2+1)^2\ \ldots\ldots\ (1)$$

Let $x^2+1 = 0$, or $x^2 = -1$; then we have $1-2x = \text{A}x+\text{B}$;
hence eq. (1) becomes by substituting for $\text{A}x+\text{B}$ its value $-2x+1$

$$x^4+2x^3+3x^2+2x+2 = (\text{C}x+\text{D})\,(x^2+1) + \text{H}\,(x^2+1)^2$$

$$\text{or } (x^2+1)\,(x^2+2x+2) = (\text{C}x+\text{D})\,(x^2+1) + \text{H}\,(x^2+1)^2$$

$$\therefore\ x^2+2x+2 = \text{C}x+\text{D}+\text{H}\,(x^2+1)\ \ldots\ldots\ldots\ldots\ldots\ (2)$$

Let $x^2+1 = 0$; then, as before, we have $2x+1 = \text{C}x+\text{D}$;
and this, substituted in (2), gives $x^2+1 = \text{H}\,(x^2+1)$, or $\text{H} = 1$.

$$\therefore\ du = \frac{(-2x+1)\,dx}{(x^2+1)^3} + \frac{(2x+1)\,dx}{(x^2+1)^2} + \frac{dx}{x^2+1}$$

$$= \frac{-2x\,dx}{(x^2+1)^3} + \frac{2x\,dx}{(x^2+1)^2} + \frac{dx}{x^2+1} + \frac{dx}{(x^2+1)^3} + \frac{dx}{(x^2+1)^2}.$$

By formula (A) we have, making $n=3$, and $n=2$,

$$\int\frac{dx}{(x^2+1)^3}=\frac{x}{4\,(x^2+1)^2}+\frac{3}{4}\int\frac{dx}{(x^2+1)^2}$$

$$\int\frac{dx}{(x^2+1)^2}=\frac{x}{2\,(x^2+1)}+\frac{1}{2}\int\frac{dx}{x^2+1}$$

$$\therefore\ u=-\int\frac{2x\,dx}{(x^2+1)^3}+\int\frac{2x\,dx}{(x^2+1)^2}+\frac{x}{4(x^2+1)^2}+\frac{7}{8}\cdot\frac{x}{x^2+1}+\frac{15}{8}\int\frac{dx}{x^2+1}$$

$$=\frac{1}{2\,(x^2+1)^2}-\frac{1}{x^2+1}+\frac{x}{4\,(x^2+1)^2}+\frac{7}{8}\cdot\frac{x}{x^2+1}+\frac{15}{8}\tan^{-1}x$$

$$=\frac{x+2}{4\,(x^2+1)^2}+\frac{7x-8}{8\,(x^2+1)}+\frac{15}{8}\tan^{-1}x+C.$$

Examples in Rational Fractions.

(1.) $\frac{du}{dx}=\frac{x^2-x+1}{x^3+x^2+x+1}$.

(2.) $\frac{du}{dx}=\frac{1}{x^3+3x-4}$.

(3.) $\frac{du}{dx}=\frac{x^2-7x+1}{x^3-6x^2+11x-6}$.

(4.) $\frac{du}{dx}=\frac{1}{x^3+x^2+x+1}$.

(5.) $\frac{du}{dx}=\frac{3x+2}{(x-1)(x+2)(x-3)}$.

(6.) $\frac{du}{dx}=\frac{x}{x^3+5x^2+8x+4}$.

(7.) $\frac{du}{dx}=\frac{a+bx}{(x-1)(x^2+x+1)}$.

(8.) $\frac{du}{dx}=\frac{x^4+x^3+x^2+x}{x^3-7x+6}$.

(9.) $\frac{du}{dx}=\frac{x^2}{(x+a)(x+b)^2}$.

(10.) $\frac{du}{dx}=\frac{1}{x^4+4x^3+5x^2+4x+4}$.

(11.) $\frac{du}{dx}=\frac{1}{x^8+x^7-x^4-x^3}$.

(12.) $\frac{du}{dx}=\frac{x^3+1}{x^5+x^4-x-1}$.

(13.) $\frac{du}{dx}=\frac{b^3}{x^6-a^6}$.

(14.) $\frac{du}{dx}=\frac{x^3-6x^2+4x-1}{x^4-3x^3-3x^2+7x+6}$

Answers.

(1.) $\frac{3}{2}\log(x+1)-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+C.$

(2.) $\frac{1}{6}\log(x-1)-\frac{1}{12}\log(x^2+x+4)-\frac{1}{2\sqrt{15}}\tan^{-1}\frac{2x+1}{\sqrt{15}}+C.$

(3.) $-\frac{5}{2}\log(x-1)+9\log(x-2)-\frac{11}{2}\log(x-3)+C.$

(4.) $\frac{1}{2}\log(x+1)-\frac{1}{4}\log(x^2+1)+\frac{1}{2}\tan^{-1}x+C.$

(5.) $-\frac{5}{6}\log(x-1)-\frac{4}{15}\log(x+2)+\frac{11}{10}\log(x-3)+C.$

(6.) $\log\frac{x+2}{x+1}-\frac{2}{x+2}+C.$

(7.) $\frac{a+b}{3}\log\frac{x-1}{\sqrt{x^2+x+1}}+\frac{2b-a}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C.$

(8.) $\frac{x^2}{2}+x+\log\frac{(x-2)^6(x+3)^3}{x-1}+C.$

(9.) $\frac{a^2}{(a-b)^2}\log(x+a)-\frac{b^2}{a-b}\cdot\frac{1}{x+b}-\frac{(2a-b)b}{(a-b)^2}\log(x+b)+C.$

(10.) $-\frac{1}{5}\cdot\frac{1}{x+2}+\frac{4}{25}\log(x+2)-\frac{2}{25}\log(x^2+1)+\frac{3}{25}\tan^{-1}x+C.$

(11.) $\frac{2-2x-5x^2}{4x^2(x+1)}+\frac{1}{8}\log\frac{x^2-1}{x^2+1}+\log\frac{x+1}{x}-\frac{1}{4}\tan^{-1}x+C.$

(12.) $\frac{1}{4}\log(x-1)-\frac{3}{4}\log(x+1)+\frac{1}{4}\log(x^2+1)+C=\log c\frac{\sqrt{x^4-1}}{x+1}.$

(13.) $\frac{b^3}{3a^5}\left\{\log\frac{(x-a)\sqrt{x^2-ax+a^2}}{(x+a)\sqrt{x^2+ax+a^2}}-\sqrt{3}\left(\tan^{-1}\frac{2x-a}{a\sqrt{3}}+\tan^{-1}\frac{2x+a}{a\sqrt{3}}\right)\right\}+C$

(14.) $\log\frac{(x+1)(x-2)}{(x-3)}+\frac{1}{x+1}+C.$

CHAPTER III.

ON THE INTEGRATION OF IRRATIONAL FUNCTIONS*

The first class of integrals which we shall consider are comprehended under the general form

$$x^{\pm m}(a^2\pm x^2)^{\pm\frac{1}{2}}\,dx$$

The integrals belonging to this class are, for the most part, obtained by the method of parts. Let us first take those of the form

$$\frac{x^m\,dx}{\sqrt{x^2\pm a^2}}$$

1. Let $du=\frac{dx}{\sqrt{x^2\pm a^2}}$

Let $y=x+\sqrt{x^2\pm a^2}$ .... (1)

$\therefore$ $y-x=\sqrt{x^2\pm a^2}$ .... (2)

$$y^2-2xy+x^2=x^2\pm a^2$$

$$y^2-2xy=\pm a^2$$

$$-2xdy-2ydx=0$$

$$(y-x)\,dy=ydx$$

$$\frac{dy}{y}=\frac{dx}{y-x}$$

$$=\frac{dx}{\sqrt{x^2\pm a^2}}\text{ by (2)}$$

$$\therefore\int\frac{dx}{\sqrt{x^2\pm a^2}}=\int\frac{dy}{y}$$

* The limits and nature of the present work will not permit us to enter at any length upon a subject so extensive and intricate as the Integral Calculus. We shall therefore merely indicate the process by which some of the most useful integrals may be found, and refer the student who wishes to prosecute this subject to the masterly work of Lacroix.

$$= \log. y + C$$
$$= \log. (x + \sqrt{x^2 \pm a^2}) + C$$

2. Let $$du = \frac{xdx}{\sqrt{x^2 \pm a^2}}$$
$$u = \sqrt{x^2 \pm a^2} + C, \text{ by II. Chap. I. } Ex. \text{ 5.}$$

3. Let $du = \dfrac{x^2\,dx}{\sqrt{x^2 \pm a^2}}$, integrating by parts, we have

$$u = \int x \cdot \frac{x\,dx}{\sqrt{x^2 \pm a^2}}$$
$$= x\sqrt{x^2 \pm a^2} - \int dx\sqrt{x^2 \pm a^2}$$
$$= x\sqrt{x^2 \pm a^2} - \int \frac{dx\,(x^2 \pm a^2)}{\sqrt{x^2 \pm a^2}}$$ multiplying numerator and denominator by $\sqrt{x^2 \pm a^2}$
$$= x\sqrt{x^2 \pm a^2} \mp a^2\int\frac{dx}{\sqrt{x^2 \pm a^2}} - \int\frac{x^2\,dx}{\sqrt{x^2 \pm a^2}}$$
$$= x\sqrt{x^2 \pm a^2} \mp a^2\int\frac{dx}{\sqrt{x^2 \pm a^2}} - u$$
$$2u = x\sqrt{x^2 \pm a^2} \mp a^2\int\frac{dx}{\sqrt{x^2 \pm a^2}}$$
$$u = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\int\frac{dx}{\sqrt{x^2 \pm a^2}}$$
$$= \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log.(x + \sqrt{x^2 \pm a^2}) + C$$

4. Let $$du = \frac{x^3\,dx}{\sqrt{x^2 \pm a^2}}$$
$$u = \int x^2 \cdot \frac{xdx}{\sqrt{x^2 \pm a^2}}$$
$$= x^2\sqrt{x^2 \pm a^2} - 2\int xdx\sqrt{x^2 \pm a^2}$$
$$= x^2\sqrt{x^2 \pm a^2} - 2\int\frac{x\,dx\,(x^2 \pm a^2)}{\sqrt{x^2 \pm a^2}}$$
$$= x^2\sqrt{x^2 \pm a^2} \mp 2a^2\int\frac{xdx}{\sqrt{x^2 \pm a^2}} - 2u$$
$$= \frac{x^2}{3}\sqrt{x^2 \pm a^2} \mp \frac{2a^2}{3}\sqrt{x^2 \pm a^2} + C$$
$$= \sqrt{x^2 \pm a^2}\left\{\frac{x^2}{3} \mp \frac{2a^2}{3}\right\} + C$$

5. Let $$du = \frac{x^4\,dx}{\sqrt{x^2 \pm a^2}}$$
$$u = \int x^3 \cdot \frac{xdx}{\sqrt{x^2 \pm a^2}}$$
$$= x^3\sqrt{x^2 \pm a^2} - 3\int x^2\,dx\sqrt{x^2 \pm a^2}$$

$$= x^3 \sqrt{x^2 \pm a^2} \mp 3a^2 \int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} - 3u$$

$$= \frac{x^3}{4}\sqrt{x^2 \pm a^2} \mp \frac{3a^2}{4}\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} (u')$$

But by example (3)

$$u' = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log.\ (x + \sqrt{x^2 \pm a^2})$$

$$u = \frac{x^3}{4}\sqrt{x^2 \pm a^2} \mp \frac{3\,.\,1\,.\,a^2\,.\,x}{4\,.\,2\,.\,1}\sqrt{x^2 \pm a^2} \pm \frac{3\,.\,1\,.\,a^4}{4\,.\,2}\log.$$

$$(x + \sqrt{x^2 \pm a^2}) + C$$

$$= \sqrt{x^2 \pm a^2}\left\{\frac{x^3}{4} \mp \frac{3\,.\,1\,.\,a^2}{4\,.\,2\,.\,1}\,x\right\} \pm \frac{3\,.\,1\,.\,a^4}{4\,.\,2}\log.(x + \sqrt{x^2 \pm a^2}) + C$$

Similarly

6. $$\int \frac{x^5\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}\left\{\frac{x^4}{5} \mp \frac{4\,.\,a^2\,.\,x^2}{5\,.\,3} \pm \frac{4\,.\,2\,.\,a^4}{5\,.\,3\,.\,1}\right\} + C$$

7. $$\int \frac{x^6\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}\left\{\frac{x^5}{6} \mp \frac{5\,.\,a^2\,.\,x^3}{6\,.\,4} \pm \frac{5\,.\,3\,.\,a^4\,.\,x}{6\,.\,4\,.\,2}\right\} \pm \frac{5\,.\,3\,.\,1\,.\,a^6}{6\,.\,4\,.\,2}$$

$$\log.\ (x + \sqrt{x^2 \pm a^2}) + C$$

8. $$\int \frac{x^7\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}\left\{\frac{x^6}{7} \mp \frac{6\,.\,a^2\,.\,x^4}{7\,.\,5} \pm \frac{6\,.\,4\,.\,a^4\,.\,x^2}{7\,.\,5\,.\,3} \mp \frac{6\,.\,4\,.\,2\,.\,a^6}{7\,.\,5\,.\,3\,.\,1}\right\} + C$$

And generally, if

$$du = \frac{x^m\,dx}{\sqrt{x^2 \pm a^2}}$$

When m is even

$$u = \sqrt{x^2 \pm a^2}\left\{\frac{x^{m-1}}{m} \mp \frac{(m-1)x^{m-3}a^2}{m\,(m-2)} \pm \frac{(m-1)\,(m-3)x^{m-5}a^4}{m\,(m-2)(m-4)} \mp \dots\right.$$

$$\left.\frac{(m-1)\,(m-3)\dots 3\,.\,1\,xa^{m-2}}{m\,(m-2)\dots 4\,.\,2}\right\}$$

$$\pm \frac{(m-1)\,(m-3)\dots 5\,.\,3\,.\,1\,.\,a^m}{m\,.\,(m-2)\dots 4\,.\,2}\log.\ (x + \sqrt{x^2 \pm a^2}) + C$$

When m is odd

$$u = \sqrt{x^2 \pm a^2}\left\{\frac{x^{m-1}}{m} \mp \frac{(m-1)x^{m-3}a^2}{m\,(m-2)} \pm \frac{(m-1)\,(m-3)x^{m-5}a}{m\,(m-2)\,(m-4)} \mp \dots\right.$$

$$\left.\pm \frac{(m-1)\,(m-3)\dots 4\,.\,2\,.\,a^{m-1}}{m\,(m-1)\,(m-4)\dots 5\,.\,3\,.\,1}\right\} + C$$

Next, to integrate

$$\frac{x^m\,dx}{\sqrt{a^2 - x^2}}.$$

1. Let $du = \dfrac{dx}{\sqrt{a^2 - x^2}}$ then by VI. Chap. I.

$$u = \sin.^{-1}\frac{x}{a} + C, \text{ the radius being unity.}$$

2. Let $du = \dfrac{xdx}{\sqrt{a^2 - x^2}}$; then $u = -\sqrt{a^2 - x^2} + C$.

3\. Let $$du = \frac{x^2\,dx}{\sqrt{a^2 - x^2}}$$

$$= \int x \cdot \frac{xdx}{\sqrt{a^2 - x^2}}$$

$$= -x\sqrt{a^2 - x^2} + \int dx\sqrt{a^2 - x^2}$$

$$= -x\sqrt{a^2 - x^2} + \int \frac{dx\,(a^2 - x^2)}{\sqrt{a^2 - x^2}}$$

$$= -x\sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - u$$

$$= -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin.^{-1}\frac{x}{a} + C$$

4\. Let $$du = \frac{x^3\,dx}{\sqrt{a^2 - x^2}}$$

$$u = \int x^2 \frac{xdx}{\sqrt{a^2 - x^2}}$$

$$= -x^2\sqrt{a^2 - x^2} + 2\int xdx\sqrt{a^2 - x^2}$$

$$= -x^2\sqrt{a^2 - x^2} + 2a^2 \int \frac{xdx}{\sqrt{a^2 - x^2}} -$$

$$= -\frac{x^2}{3}\sqrt{a^2 - x^2} + \frac{2a^2}{3}\int \frac{xdx}{\sqrt{a^2 - x^2}}$$

$$= -\frac{x^2}{3}\sqrt{a^2 - x^2} - \frac{2a^2}{3}\sqrt{a^2 - x^2} + C$$

$$= -\sqrt{a^2 - x^2}\left\{\frac{x^2}{3} + \frac{2a^2}{3}\right\} + C$$

5\. Let $$du = \frac{x^4\,dx}{\sqrt{a^2 - x^2}}$$

$$u = \int x^3 \cdot \frac{xdx}{\sqrt{a^2 - x^2}}$$

$$= -x^3\sqrt{a^2 - x^2} + 3\int x^2dx\sqrt{a^2 - x^2}$$

$$= -x^3\sqrt{a^2 - x^2} + 3a^2 \int \frac{x^2dx}{\sqrt{a^2 - x^2}} - 3u$$

$$= -\frac{x^3}{4}\sqrt{a^2 - x^2} + \frac{3a^2}{4}\int \frac{x^2dx}{\sqrt{a^2 - x^2}} \quad (u')$$

But by example (3)

or, $$\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin.^{-1}\frac{x}{a}$$

$$\therefore \quad u = -\frac{x^3}{4}\sqrt{a^2 - x^2} - \frac{3a^2 \cdot x}{4 \cdot 2}\sqrt{a^2 - x^2} + \frac{3 \cdot 1 \cdot a^4}{4 \cdot 2}\sin.^{-1}\frac{x}{a}$$

$$= -\sqrt{a^2 - x^2}\left\{\frac{x^3}{4} + \frac{3a^2 \cdot x}{4 \cdot 2}\right\} + \frac{3 \cdot 1 \cdot a^4}{4 \cdot 2}\sin.^{-1}\frac{x}{a} + C$$

Similarly,

$$4. \int \frac{x^5\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}\left\{\frac{x^4}{5} + \frac{4\,a^2\,x^2}{5\,.\,3} + \frac{4\,.\,2\,.\,a^4}{5\,.\,3\,.\,1}\right\}$$

$$5. \int \frac{x^6\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}\left\{\frac{x^5}{6} + \frac{5\,a^2\,x^3}{6\,.\,4} + \frac{5\,.\,3\,.\,a^4\,x}{6\,.\,4\,.\,2}\right\}$$
$$+ \frac{5\,.\,3\,.\,1\,.\,a^6}{6\,.\,4\,.\,2}\sin.^{-1}\frac{x}{a} + C$$

Generally,

$$\int \frac{x^{2m+1}\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}\left\{\frac{x^{2m}}{2m+1} + \frac{2ma^2\,x^{2m-2}}{(2m+1)\,(2m-1)}\right.$$
$$\left. + \frac{2m\,(2m-2)\,a^4\,x^{2m-4}}{(2m+1)\,(2m-1)\,(2m-3)} + \ldots + \frac{2m\,(2m-2)\ldots 4\,.\,2\,.\,a^{2m}}{(2m-1)\,(2m-2)\ldots 3\,.\,1}\right\} + C$$

$$\int \frac{x^{2m}\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}\left\{\frac{x^{2m-1}}{2m} + \frac{(2m-1)a^2x^{2m-3}}{2m\,(2m-2)} + \frac{(2m-1)(2m-3)a^4x^{2m-5}}{2m\,(2m-2)\,(2m-4)} + \ldots\right.$$
$$\left. + \frac{(2m-1)\,(2m-3)\,(2m-5)\ldots 3\,.\,1\,.\,a^{2m-2}\,x}{2m\,(2m-2)\,(2m-4)\ldots 4\,.\,2}\right\}$$
$$+ \frac{(2m-1)\,(2m-3)\ldots 3\,.\,1\,.\,a^{2m}}{2m\,(2m-2)\ldots 4\,.\,2}\sin.^{-1}\frac{x}{a} + C$$

To integrate

$$\frac{dx}{x^n\sqrt{a^2+x^2}}$$

In all cases where the index of x is negative, assume $y = \frac{1}{x}$

1. Let $\quad du = \dfrac{dx}{x\sqrt{a^2+x^2}}$

Let $\quad y = \dfrac{1}{x} \qquad \therefore -dy = \dfrac{dx}{x^2}$ and $-\dfrac{dy}{y} = \dfrac{dx}{x}$

$$\therefore \quad du = \frac{-dy}{y\sqrt{a^2+\frac{1}{y^2}}}$$
$$= \frac{-dy}{\sqrt{a^2\,y^2+1}}$$
$$= -\frac{1}{a}\cdot\frac{dy}{\sqrt{\frac{1}{a^2}+y^2}}$$
$$u = -\frac{1}{a}\,.\log.\left(y+\sqrt{y^2+\frac{1}{a^2}}\right)$$
$$= -\frac{1}{a}\,.\log.\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+\frac{1}{a^2}}\right)$$
$$= -\frac{1}{a}\,.\log.\left(\frac{a+\sqrt{a^2+x^2}}{ax}\right)$$

2. Let $\quad du = \dfrac{dx}{x^2\sqrt{a^2+x^2}}$

Let $\quad y = \dfrac{1}{x} \qquad \therefore -dy = \dfrac{dx}{x^2}$

$$\therefore \quad du = \frac{-dy}{\sqrt{a^2 + \frac{1}{y^2}}}$$

$$= \frac{-y\,dy}{\sqrt{a^2 y^2 + 1}}$$

$$\therefore \quad u = -\frac{1}{a^2} \cdot \sqrt{a^2 y^2 + 1}$$

$$= -\frac{1}{a^2} \cdot \sqrt{a^2 \times \frac{1}{x^2} + 1}$$

$$= -\frac{\sqrt{a^2 + x^2}}{a^2\,x}$$

3. Let $\quad du = \dfrac{dx}{x^5 \sqrt{1 + x^2}}$

Let $\quad y = \dfrac{1}{x} \quad \therefore \; -dy = \dfrac{dx}{x^2}$ and $-y^3 dy = \dfrac{dx}{x^5}$

$$\therefore \quad du = \frac{-y^4\,dy}{\sqrt{1 + y^2}}$$

Integrating by parts,

$$u = \int - y^3 \cdot \frac{y dy}{\sqrt{1 + y^2}}$$

$$= -\sqrt{1 + y^2}\left\{\frac{y^3}{4} - \frac{3 \,.\, y}{4 \,.\, 2}\right\} - \frac{3.1}{4.2} \log.\left(y + \sqrt{1 + y^2}\right)$$

$$= -\sqrt{1 + x^2}\left\{\frac{1}{4.x^4} - \frac{3 \,.\, 1}{4.2.x^2}\right\} - \frac{3.1}{4.2} \log.\left(\frac{1 + \sqrt{1 + x^2}}{x}\right)$$

To integrate

$$\frac{dx}{x^m \sqrt{a^2 - x^2}}$$

The process is precisely analogous to that employed in the last case

1. Let $\quad du = \dfrac{ax}{x^4 \sqrt{1 - x^2}}$

Let $\quad y = \dfrac{1}{x} \quad \therefore -dy = \dfrac{dx}{x^2}$ and $-y^2\,dy = \dfrac{dx}{x^4}$

$$\therefore \quad du = \frac{-y^3\,dy}{\sqrt{y^2 - 1}}$$

Integrating by parts

$$u = \int -y^2 \cdot \frac{y dy}{\sqrt{y^2 - 1}}$$

$$= -\sqrt{y^2 - 1}\left\{\frac{y^2}{3} + \frac{2}{3}\right\} + C$$

$$= -\sqrt{1 - x^2}\left\{\frac{1}{3x^3} + \frac{2 \,.\, 1}{3 \,.\, x}\right\} + C$$

2. Let $\quad du = \dfrac{dx}{x^5 \sqrt{1 - x^2}}$

Let $\quad y = \frac{1}{x}$

$\therefore \quad du = \dfrac{-y^4\, dy}{\sqrt{y^2 - 1}}$

$$u = -\sqrt{y^2-1}\left\{\frac{y^3}{4} + \frac{3\,.\,y}{4\,.\,2}\right\} + \frac{3\,.\,1}{4\,.\,2}\log.\left(y + \sqrt{y^2-1}\right) + C$$

$$= -\sqrt{1-x^2}\left\{\frac{1}{4x^4} + \frac{3\,.\,1}{4\,.\,2\,.\,x^2}\right\} + \frac{3.1}{4.2}\log.\left(\frac{1+\sqrt{1-x^2}}{x}\right) + C$$

In like manner all integrals of the form

$$\frac{dx}{x^m\sqrt{x^2-a^2}}$$

may be determined.

To integrate

$$x^m\, dx\sqrt{a^2+x^2} \qquad x^m\, dx\sqrt{a^2-x^2}, \qquad x^m\, dx\sqrt{x^2-a^2}.$$

1 $\qquad du = x^2\, dx\sqrt{a^2+x^2}$

integrating by parts

$$u = \int x\,.\,x dx\sqrt{a^2+x^2}$$

$$= \frac{x}{3}(a^2+x^2)^{\frac{3}{2}} - \frac{1}{3}\int dx\,(a^2+x^2)^{\frac{3}{2}}$$

$$= \frac{x}{3}(a^2+x^2)^{\frac{3}{2}} - \frac{1}{3}\int dx\,(a^2+x^2)\sqrt{a^2+x^2}$$

$$= \frac{x}{3}(a^2+x^2)^{\frac{3}{2}} - \frac{a^2}{3}\int dx\sqrt{a^2+x^2} - \frac{1}{3}u$$

$$= \frac{x}{4}(a^2+x^2)^{\frac{3}{2}} - \frac{a^4}{4}\int\frac{dx}{\sqrt{a^2+x^2}} - \frac{a^2}{4}\int\frac{x^2\, dx}{\sqrt{a^2+x^2}}$$

which are both known forms.

$$= \frac{x}{4}(a^2+x^2)^{\frac{3}{2}} - \frac{a^4}{4}\log.(x+\sqrt{a^2+x^2}) - \frac{a^2}{4}$$

$$\left\{\frac{x}{2}\sqrt{a^2+x^2} - \frac{a^2}{2}\log.(x+\sqrt{a^2+x^2})\right\} + C$$

$$= \sqrt{a^2+x^2}\left\{\frac{x^3}{4} + \frac{a^2\,x}{4\,.\,2}\right\} - \frac{a^4}{4\,.\,2}\log.(x+\sqrt{a^2+x^2}) + C$$

The preceding integral may be found in a manner somewhat different

$$du = x^2\, dx\sqrt{a^2+x^2}$$

Multiply both numerator and denominator by $\sqrt{a^2+x^2}$

$$= \frac{x^2\, dx\,(a^2+x^2)}{\sqrt{a^2+x^2}}$$

$$\therefore \quad u = a^2\int\frac{x^2\, dx}{\sqrt{a^2+x^2}} + \int\frac{x^4\, dx}{\sqrt{a^2+x^2}}$$

which are both known forms.

2. $\qquad du = x^3\, dx\sqrt{a^2-x^2}$

$$= \frac{x^3\,dx\,(a^2 - x^2)}{\sqrt{a^2 - x^2}}$$

$$\therefore \qquad u = a^2 \int \frac{x^3\,dx}{\sqrt{a^2 - x^2}} - \int \frac{x^5\,dx}{\sqrt{a^2 - x^2}}$$

Which are known forms, and may be found by the method of parts.

3. $$du = x^4\,dx\,\sqrt{x^2 - a^2}$$

$$= \frac{x^4\,dx\,(x^2 - a^2)}{\sqrt{x^2 - a^2}}$$

$$u = \int \frac{x^6\,dx}{\sqrt{x^2 - a^2}} - a^2 \int \frac{x^4\,dx}{\sqrt{x^2 - a^2}}$$

which are known forms.

And similarly all integrals belonging to this class may be found

To integrate

$$x^{-m}\,dx\,\sqrt{x^2 \pm a^2}$$

And $$x^{-m}\,dx\,\sqrt{a^2 - x^2}$$

1. $$du = \frac{dx\,\sqrt{a^2 + x^2}}{x^2}$$

Let $$y = \frac{1}{x}$$

$$du = -\,dy\left(a^2 + \frac{1}{y^2}\right)^{\frac{1}{2}}$$

$$= -\frac{dy}{y}\sqrt{y^2 + a^2}$$

$$u = \int \frac{-\,y\,dy}{\sqrt{y^2 + a^2}} - a^2 \int \frac{dy}{y\sqrt{y^2 + a^2}}$$

which are known forms.

These integrals may in general be found more conveniently by multiplying both numerator and denominator by the irrational part

Thus,

2. $$du = \frac{dx\,\sqrt{a^2 + x^2}}{}$$

$$= \frac{dx\,(a^2 + x^2)}{x^2\sqrt{a^2 + x^2}}$$

$$u = a^2 \int \frac{dx}{x^2\sqrt{a^2 + x^2}} + \int \frac{dx}{\sqrt{a^2 + x^2}}$$

which are known.

3. $$du = \frac{dx\,\sqrt{a^2 - x^2}}{x^3}$$

$$= \frac{dx\,(a^2 - x^2)}{x^3\sqrt{a^2 - x^2}}$$

$$u = a^2 \int \frac{dx}{x^3 \sqrt{a^2 - x^2}} - \int \frac{dx}{x \sqrt{a^2 - x^2}}$$

The first is known, the second may be found as follows.

4. $$du = \frac{dx \sqrt{x^2 - a^2}}{}$$

$$= \frac{dx\ (x^2 - a^2)}{x^4 \sqrt{x^2 - a^2}}$$

$$u = \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} - a^2 \int \frac{dx}{x^4 \sqrt{x^2 - a^2}}$$

which are both known.

We have thus found all the integrals included under the general forms

$$\begin{cases} 1. & x^{\pm m}\, dx\, (x^2 \pm a^2)^{\pm\frac{1}{2}} \\ 2. & x^{\pm m}\, dx\, (a^2 \pm x^2)^{\pm\frac{1}{2}} \end{cases}$$

Let us now proceed to integrate

$$\begin{cases} 1. & x^{\pm m}\, dx\, (x^2 \pm a^2)^{\pm\frac{n}{2}} \\ 2. & x^{\pm m}\, dx\, (a^2 - x^2)^{+\frac{n}{2}} \end{cases}$$

where m is any odd number.

The principles are exactly the same as in the more simple cases, and therefore a very few examples will suffice.

1. $$du = \frac{x^3\, dx}{(a + bx^2)^{\frac{3}{2}}}$$

$$u = \int x^2 \,.\, xdx\, (a + bx^2)^{-\frac{3}{2}}$$

$$= -\frac{x^2}{b} \cdot \frac{1}{\sqrt{a + bx^2}} + \frac{2}{b} \int \frac{xdx}{\sqrt{a + bx^2}}$$

$$= -\frac{x^2}{b} \cdot \frac{1}{\sqrt{a + bx^2}} + \frac{2}{b^2} \sqrt{a + bx^2}$$

$$= \frac{1}{\sqrt{a + bx^2}} \left\{ \frac{x^2}{b} + \frac{2a}{b^2} \right\}$$

2. $$du = \frac{dx}{x^2 (a + bx^2)^{\frac{3}{2}}}$$

Let $$y = \frac{1}{x}$$

$$\therefore \quad du = \frac{-y^3\, dy}{(ay^2 + b)^{\frac{3}{2}}}$$

$$= \int -y^2 \,.\, ydy\, (ay^2 + b)^{-\frac{3}{2}}$$

$$= \frac{y^2}{a} \cdot \frac{1}{\sqrt{ay^2 + b}} - \frac{2}{a} \int \frac{y\, dy}{\sqrt{ay^2 + b}}$$

$$= \frac{y^2}{a} \cdot \frac{1}{\sqrt{ay^2 + b}} - \frac{2}{a^2} \cdot \sqrt{ay^2 + b}$$

$$= -\frac{1}{\sqrt{ay^2+b}}\left\{\frac{y^2}{a}+\frac{2b}{a^2}\right\}$$

$$= -\frac{1}{\sqrt{a+bx^2}}\left\{\frac{1}{ax}+\frac{2bx}{a^2}\right\}$$

3. $$du = \frac{dx}{x^3(1+x^2)^{\frac{3}{2}}}$$

Let $$y = \frac{1}{x}$$

$\therefore$ $$du = \frac{-y^4\,dy}{(1+y^2)^{\frac{3}{2}}}$$

$$u = \int -y^3 \,.\, ydy\,(1+y^2)^{-\frac{3}{2}}$$

$$= y^3(1+y^2)^{-\frac{1}{2}} - 3\int \frac{y^2\,dy}{\sqrt{1+y^2}}$$

$$= y^3 \cdot \frac{1}{\sqrt{1+y^2}} - 3\left\{\frac{y}{2}\sqrt{1+y^2} - \frac{1}{2}\log.(y+\sqrt{1+y^2})\right\} + C$$

$$= -\frac{1}{\sqrt{1+y^2}}\left\{\frac{y^3}{2}+\frac{3y}{2}\right\} + \frac{3}{2}\log.(y+\sqrt{1+y^2}) + C$$

$$= -\frac{1}{\sqrt{1+x^2}}\left\{\frac{1}{2x^2}+\frac{3}{2}\right\} - \frac{3}{2}\log.\left(\frac{\sqrt{1+x^2}-1}{x}\right) + C$$

4. $$du = \frac{x^4\,dx}{(a^2-x^2)^{\frac{3}{2}}}$$

$$u = \int x^3 \,.\, xdx\,(a^2-x^2)^{-\frac{3}{2}}$$

$$= x^3 \cdot \frac{1}{\sqrt{a^2-x^2}} - 3\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$$

$$= x^3 \cdot \frac{1}{\sqrt{a^2-x^2}} - 3\left\{-\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin.^{-1}\frac{x}{a}\right\} + C$$

$$= -\frac{1}{\sqrt{a^2-x^2}}\left\{\frac{x^3}{2}-\frac{3a^2x}{2}\right\} - \frac{3a^2}{2}\sin.^{-1}\frac{x}{a} + C$$

The next class of integrals which we shall consider is comprised under the general forms

1. $x^{\pm m}\,dx\,(x^2 \pm 2ax)^{\pm\frac{1}{2}}$

2. $x^{\pm m}\,dx\,(2ax - x^2)^{\pm\frac{1}{2}}$

To integrate

$$\frac{x^m\,dx}{\sqrt{x^2 \pm 2ax}}$$

1. $$du = \frac{dx}{\sqrt{x^2 \pm 2ax}}$$

Let $$y = x \pm a$$

$$y^2 = x^2 \pm 2ax + a^2$$

$$y^2 - a^2 = x^2 \pm 2ax$$

$$\therefore \int \frac{dx}{\sqrt{x^2+2ax}} = \int \frac{dy}{\sqrt{y^2-a^2}}$$

$$= \log.\ (y + \sqrt{y^2 - a^2}) + C$$
$$= \log.\ (x \pm a + \sqrt{x^2 \pm 2ax})$$

2. $$du = \frac{xdx}{\sqrt{x^2 \pm 2ax}}$$
$$= \frac{xdx \pm adx}{\sqrt{x^2 \pm 2ax}} \mp \frac{adx}{\sqrt{x^2 \pm 2ax}}$$
$$\therefore \quad u = \int \frac{xdx \pm adx}{\sqrt{x^2 \pm 2ax}} \mp a \int \frac{dx}{\sqrt{x^2 \pm 2ax}}$$
$$= \sqrt{x^2 \pm 2ax} \mp a \log.\ (x \pm a + \sqrt{x^2 \pm 2ax})$$

3. $$du = \frac{x^2\, dx}{\sqrt{2ax + x^2}} = \frac{x^{\frac{3}{2}}\, dx}{\sqrt{2a + x}}$$
$$u = \int x^{\frac{3}{2}} \cdot (2a + x)^{-\frac{1}{2}}\, dx$$
$$= 2x^{\frac{3}{2}} \sqrt{2a + x} - 3 \int x^{\frac{1}{2}}\, dx \sqrt{2a + x}$$
$$= 2x^{\frac{3}{2}} \sqrt{2a + x} - 3 \cdot 2 \cdot a \int \frac{x^{\frac{1}{2}}\, dx}{\sqrt{2a + x}} - 3u$$
$$= \frac{x^{\frac{3}{2}}}{2} \sqrt{2a + x} - \frac{3a}{2} \int \frac{xdx}{\sqrt{2ax + x^2}}$$
$$= \frac{x^{\frac{3}{2}}}{2} \sqrt{2a + x} - \frac{3a}{2} \{\sqrt{x^2 + 2ax} - a \log.(x + a + \sqrt{x^2 + 2ax})\}$$
$$= \frac{x}{2} \sqrt{2ax + x^2} - \frac{3a}{2} \sqrt{x^2 + 2ax} + \frac{3a^2}{2} \log.(x + a + \sqrt{x^2 + 2ax})$$
$$= \sqrt{2ax + x^2} \left\{ \frac{x}{2} - \frac{3a}{2} \right\} + \frac{3a^2}{2} \cdot \log.\ (x + a + \sqrt{2ax + x^2}) + C$$

4. Similarly if

$$u = \sqrt{x^2 - 2ax} \left\{ \frac{x^2}{3} + \frac{5ax}{2 \cdot 3} + \frac{5a^2}{2} \right\} + \frac{5 \cdot a^3}{2} \log.(x - a + \sqrt{x^2 - 2ax}) + C$$

Next to integrate
$$\frac{x^m\, dx}{\sqrt{2ax - x^2}}$$

1. $$du = \frac{dx}{\sqrt{2ax}}$$
$$u = \frac{1}{a} \text{versin.}^{-1}\, x \text{ to radius } a$$

2. $$du = \frac{xdx}{\sqrt{2ax - x^2}}$$
$$= -\frac{adx - xdx}{\sqrt{2ax - x^2}} + \frac{adx}{\sqrt{2ax - x^2}}$$

$$u = -\int \frac{adx - xdx}{\sqrt{2ax - x^2}} + a\int \frac{dx}{\sqrt{2ax - x^2}}$$

$$= -\sqrt{2ax - x^2} + \text{versin.}^{-1}\, x \text{ to radius } a + C.$$

3. $$du = \frac{x^2\, dx}{\sqrt{2ax - x^2}}$$

$$= x^{\frac{3}{2}} dx\, (2a - x)^{-\frac{1}{2}}$$

$$u = \int x^{\frac{3}{2}} \cdot (2a - x)^{-\frac{1}{2}}\, dx$$

$$= -2x^{\frac{3}{2}}\sqrt{2a - x} + 3\int x^{\frac{1}{2}}\, dx\sqrt{2a - x}$$

$$= -2x^{\frac{3}{2}}\sqrt{2a - x} + 3 \cdot 2 \cdot a\int \frac{xdx}{\sqrt{2ax - x^2}} - 3u$$

$$= -\frac{x^{\frac{3}{2}}}{2}\sqrt{2a - x} + \frac{3a}{2}\left\{-\sqrt{2ax - x^2} + a\text{ versin.}^{-1}\frac{x}{a}\right\} + C$$

$$= -\sqrt{2ax - x^2}\left\{\frac{x}{2} + \frac{3a}{2}\right\} + \frac{3a^2}{2}\text{versin.}^{-1}\frac{x}{a} + C$$

Next to integrate

$$\frac{dx}{x^m\sqrt{x^2 + 2ax}}$$

1. $$du = \frac{dx}{x\sqrt{x^2 + 2ax}}$$

Let $$y = \frac{1}{x}$$

$$du = \frac{-dy}{\sqrt{1 + 2ay}}$$

$$u = -\int (1 + 2ay)^{-\frac{1}{2}}\, dy$$

$$= -\frac{1}{a}\sqrt{1 + 2ay}$$

$$= -\frac{1}{ax}\sqrt{x^2 + 2ax}$$

2. $$du = \frac{dx}{x^2\sqrt{2\,ax + x^2}} \text{ Let } y = \frac{1}{x}$$

$$u = \int \frac{-ydy}{\sqrt{2ay + 1}}$$

$$= -\int y \cdot (2\,ay + 1)^{-\frac{1}{2}}\, dy$$

$$= -\frac{y}{a}\sqrt{2ay + 1} + \frac{1}{a}\int \sqrt{2ay + 1} \cdot dy$$

$$= -\frac{y}{a}\sqrt{2ay + 1} + \frac{1}{a}\int \frac{dy}{\sqrt{2ay + 1}} - 2u$$

$$= -\frac{y}{3a}\sqrt{2ay + 1} + \frac{1}{a^2}\sqrt{2ay + 1} + C$$

$$= -\sqrt{2ax + x^2}\left\{\frac{1}{3ax^2} - \frac{1}{a^2x}\right\} + C$$

3. Similarly if

$$du = \frac{dx}{x^4 \sqrt{x^2 + 2x}}$$

$$u = -\frac{\sqrt{x^2 + 2x}}{15x^3}(2x^2 - 2x + 3) + C$$

By a process analogous to the above we can find the integrals of

$$\frac{dx}{x^m \sqrt{2ax - x^2}}$$

Next to integrate

$$x^{\pm m}\, dx \sqrt{x^2 \pm 2ax}$$

1. $$du = dx \sqrt{2ax + x^2}$$

$$u = 2a \int \frac{xdx}{\sqrt{2ax + x^2}} + \int \frac{x^2\, dx}{\sqrt{2ax + x^2}} \text{ known forms.}$$

2. $$du = xdx \sqrt{2ax + x^2}$$

$$u = 2a \int \frac{x^2\, dx}{\sqrt{2ax + x^2}} + \int \frac{x^3\, dx}{\sqrt{2ax + x^2}} \text{ known forms.}$$

3. $$du = x^2\, dx \sqrt{x^2 - 2ax}$$

$$u = \int \frac{x^4\, dx}{\sqrt{x^2 - 2ax}} - 2a \int \frac{x^3\, dx}{\sqrt{x^2 - 2ax}} \text{ known forms.}$$

Next to integrate

1. $$du = dx \sqrt{2ax - x^2}$$

$$u = 2a \int \frac{xdx}{\sqrt{2ax - x^2}} - \int \frac{x^2\, dx}{\sqrt{2ax - x^2}}$$

and so on for the rest.

When m is negative, the integral may always be found by assuming $y = \frac{1}{x}$.

To integrate

$$du = \frac{dx}{\sqrt{a + bx + cx^2}}$$

$$= \frac{1}{\sqrt{c}} \cdot \frac{dx}{\sqrt{\frac{a}{c} + \frac{b}{c}x + x^2}}$$

$$= \frac{1}{\sqrt{c}} \cdot \frac{dx}{\sqrt{\alpha + \beta x + x^2}} \quad \text{Putting } \alpha = \frac{a}{c}, \text{ and } \beta = \frac{b}{c}$$

Let $$y = x + \frac{\beta}{2} \quad \therefore\ dy = dx$$

$$y^2 = x^2 + \beta x + \frac{\beta^2}{4}$$

$$\therefore y^2+\left(\alpha-\frac{\beta^2}{4}\right)=x^2+\beta x+\alpha$$

$$\therefore \qquad du=\frac{1}{\sqrt{c}}\cdot\frac{dy}{\sqrt{y^2+\left(\alpha-\frac{\beta^2}{4}\right)}}$$

$$=\frac{1}{\sqrt{c}}\cdot\frac{dy}{\sqrt{y^2\pm\delta^2}}$$ Putting, $\left(\alpha-\frac{\beta^2}{4}\right)=+\delta^2$ or $-\delta^2$ according as α is $>$ or $<\frac{\beta^2}{4}$

$$=\frac{1}{\sqrt{c}}\cdot\log.\left(y+\sqrt{y^2\pm\delta^2}\right)$$

$$=\frac{1}{\sqrt{c}}\cdot\log.\left(x+\frac{\beta}{2}+\sqrt{x^2+\beta x+\alpha}\right)$$

$$=\frac{1}{\sqrt{c}}\cdot\log.\left(x+\frac{b}{2c}+\sqrt{x^2+\frac{b}{c}x+\frac{a}{c}}\right)$$

$$=\frac{1}{\sqrt{c}}\left\{\log.\left(2cx+b+2\sqrt{c}\sqrt{a+bx+cx^2}\right)-\log.2c\right\}$$

2. Let $\quad du=\dfrac{dx}{x\sqrt{a+bx+cx^2}}$

$$y=\frac{1}{x}\quad\therefore\ -\frac{dy}{y}=\frac{dx}{x}$$

$$\therefore\qquad du=\frac{-dy}{y\sqrt{a+\frac{b}{y}+\frac{c}{y^2}}}$$

$$=\frac{-dy}{\sqrt{ay^2+by+c}}$$

$$=-\frac{1}{\sqrt{a}}\,\frac{dy}{\sqrt{y^2+\frac{b}{a}y+\frac{c}{a}}}$$

$$=-\frac{1}{\sqrt{a}}\cdot\frac{dy}{\sqrt{y^2+\beta y+\alpha}}$$ The same form as the last.

3. $\qquad du=\dfrac{dx}{\sqrt{1+x+x^2}}$

Let $\qquad y=x+\dfrac{1}{2}\quad\therefore\ dy=dx$

$$y^2=x^2+x+\frac{1}{4}$$

$$y^2+\frac{3}{4}=1+x+x^2$$

$$\therefore\qquad du=\frac{dy}{\sqrt{y^2+\frac{3}{4}}}$$

$$=\log.\left(y+\sqrt{y^2+\frac{3}{4}}\right)$$

$$= \log.\left(x + \frac{1}{2} + \sqrt{1 + x + x^2}\right)$$

$$= \log.(2x + 1 + 2\sqrt{1 + x + x^2}) - \log. 2$$

4. $$du = \frac{dx}{\sqrt{1 + x - x^2}}$$

Let $$y = x - \frac{1}{2} \quad \therefore\ dy = dx$$

$$y^2 = x^2 - x + \frac{1}{4}$$

$$y^2 - \frac{1}{4} = x^2 - x$$

$$\frac{1}{4} - y^2 = x - x^2$$

$$\frac{5}{4} - y^2 = 1 + x - x^2$$

$\therefore$ $$du = \frac{dy}{\sqrt{\frac{5}{4} - y^2}}$$

$\therefore$ $$u = \frac{2}{\sqrt{5}} \cdot \sin.^{-1} y \text{ to rad. } \frac{\sqrt{5}}{2}$$

$$= \frac{2}{\sqrt{5}} \cdot \sin.^{-1}\left(x - \frac{1}{2}\right) \text{ to rad. } \frac{\sqrt{5}}{2}$$

5. $$du = \frac{dx}{\sqrt{1 - x - x^2}}$$

Let $$y = x + \frac{1}{2} \quad \therefore\ dy = dx$$

$$y^2 = x^2 + x + \frac{1}{4}$$

$$y^2 - \frac{1}{4} = x + x^2$$

$$\frac{1}{4} - y^2 = -x - x^2$$

$$\frac{5}{4} - y^2 = 1 - x - x^2$$

$\therefore$ $$du = \frac{dy}{\sqrt{\frac{5}{4} - y^2}}$$

$\therefore$ $$u = \frac{2}{\sqrt{5}} \cdot \sin.^{-1} y \text{ to rad. } \frac{\sqrt{5}}{2}$$

$$= \frac{2}{\sqrt{5}} \cdot \sin.^{-1}\left(x + \frac{1}{2}\right) \text{ to rad. } \frac{\sqrt{5}}{2}$$

On Exponential Functions.

1°. If $X = f(a^x)$, then the function $X dx$, if we make $a^x = u$ will become $\dfrac{f(u)\, du}{\log. a}$

For example,

$$\frac{a^x\, dx}{\sqrt{1 + a^{nx}}} = \frac{1}{\log. a} \cdot \frac{du}{\sqrt{1 + u^n}}$$

2°. Differentiating Xe^x, we have $e^x dx\left(X + \dfrac{dX}{dx}\right)$ so that every exponential function in which the factor of $e^x dx$ is composed of two parts, one of which is the first differential co-efficient of the other, will be easily integrated. For example

$$\int e^x dx\,(3x^2 + x^3 - 1) = (x^3 - 1)\, e^x$$

In like manner, if we make $1 + x = z$, we shall find

$$\int \frac{e^x\, x dx}{(1 + x)^2} = \int \frac{e^z}{e}\left(\frac{dz}{z} - \frac{dz}{z^2}\right) = \frac{e^z}{ez} = \frac{e^x}{1 + x} + C$$

In every other case, however, we must have recourse to the method of integration by parts.

Ex. $\quad du = x^n dx \,.\, a^x$

$u = \int a^x\, dx \,.\, x^n$ and considering x^n in the first instance as constant.

$$= \frac{a^x\, x^n}{\log. a} - \frac{n}{\log. a} \int a^x\, x^{n-1}\, dx$$

Treating $a^x\, x^{n-1}\, dx$, &c. in the same manner, we shall finally have

$$u = a^x \left\{ \frac{x^n}{\log. a} - \frac{n\, x^{n-1}}{\log.^2 a} + \frac{n(n-1)x^{n-2}}{\log.^3 a} \cdots \pm \frac{1.2.3\,..\,n}{\log.^{n+1} a} \right\} + C$$

It is manifest that the same method is applicable to $Xa^x\, dx$, where X is any entire algebraical function of x.

But if the exponent n be negative, it is manifest that the exponent of x must go on increasing; and therefore, in the integration by parts we must consider a^x as constant in the first instance, in this manner, if

$$du = \frac{a^x\, dx}{x^n}$$

$$u = \int \frac{dx}{x^n} \,.\, a^x$$

$$= \frac{-a^x}{(n-1)\, x^{n-1}} + \frac{\log. a}{n - 1} \int \frac{a^x\, dx}{x^{n-1}}$$

Integrating $\displaystyle\int \frac{a^x\, dx}{x^{n-1}}$ in the same manner, we shall finally have

$$u \text{ or } \int \frac{a^x\, dx}{x^n} = \frac{-a^x}{n-1} \left\{ \frac{1}{x^{n-1}} + \frac{\log. a}{(n-2)x^{n-2}} + \frac{\log.^2 a}{(n-2)(n-3)x^{n-3}} + \cdots \right.$$

$$\left. \cdots + \frac{\log.^{n-2} a}{1.2.3\,..\,(n-2)x} \right\} + \frac{\log.^{n-1} a}{1.2.3\,..\,(n-1)} \int \frac{a^x\, dx}{x}$$

We cannot, however, proceed with our calculation beyond this point, because we should obtain a result $= \infty$

The integral of $\int \frac{a^x\, dx}{x}$ has not yet been discovered by analysts.

We can, however, approximate to it in the following manner

$$\frac{a^x}{x} = \frac{1}{x} + \log. a + \frac{\log.^2 a}{2} x + \frac{\log.^3 a}{1.2.3} x^2 + \ldots$$

Multiplying by dx and integrating each term

$$\int \frac{a^x\, dx}{x} = \log. x + x \log. a + \frac{x^2 \log.^2 a}{2.2} + \frac{x^3 \log.^3 a}{3.2.3} + \ldots + C$$

If n is fractional, one or other of the above methods will enable us to reduce the exponent of x until its value lies between 0 and 1, or -1, and we shall then be enabled to approximate to the required integral by series.

On Logarithmic Functions.

Let it be required to integrate

$$X dx \log.^n x$$

where X is any algebraic function of x.

If n is a positive whole number we may integrate by the method of parts, regarding $\log.^n x$ as constant in the first instance. We shall then have

$$\int X\, dx \log.^n x = \log.^n x \int X\, dx - n \int \left(\log.^{n-1} x \frac{dx}{x} \int X\, dx\right)$$

and since $\int X\, dx$ is supposed to be known by the principles already established, we perceive that the integration of the proposed function is reduced to that of one whose form is the same, and in which the exponent of the logarithm is reduced by unity. The same process is applicable to this new function, and thus the integration will be completed step by step.

Thus,

$$\int x^m\, dx \log.^n x = \frac{x^{m+1}}{m+1} \log.^n x - \frac{n}{m+1} \int \log.^{n-1} x\, x^m\, dx$$

But

$$\int x^m dx \log.^{n-1} x = \frac{x^{m+1}}{m+1} \log.^{n-1} x - \frac{n-1}{m+1} \int \log.^{n-2} x\, x^m\, dx$$

&c. &c. &c.

Adding the successive results obtained in this manner, we find,

$$\int x^m\, dx \log.^n x = x^{m+1} \left\{ \frac{\log.^n x}{m+1} - \frac{n \log.^{n-1} x}{(m+1)^2} + \frac{n(n-1) \log.^{n-2} x}{(m+1)^3} - \ldots \right\} + C$$

But if n be integral and negative, we perceive that, as in the case of exponential functions, in performing the integration by parts of $\int X \log.^n x\, dx$, we must in the first instance suppose X constant.

Since

$$\frac{dx}{x} \log.^n x = \frac{\log.^{n+1} x}{n+1}$$

we shall divide $X \log.^n x\, dx$ into the two factors $X\, x \,.\, \frac{dx}{x} \log.^n x$, hence

$$\int \frac{X\, dx}{\log.^n x} = \frac{X\, x}{-(n-1)} \log.^{-(n-1)} x + \frac{1}{n-1} \int \left\{ \log.^{-(n-1)} x . d(Xx) \right\}$$

a formula which manifestly attains the object in view.

In order, however, to understand the difficulties which occur, let us apply this to the quantity $\frac{x^m\, dx}{\log.^n x}$

$$\int \frac{x^m\, dx}{\log.^n x} = \frac{-x^{m+1}}{(n-1)\log.^{n-1} x} + \frac{m+1}{n-1} \int \frac{x^m\, dx}{\log.^{n-1} x}$$

repeating the calculation for this last term, and performing the successive operations in the same manner, we shall find upon adding the different results together

$$\int \frac{x^m\, dx}{\log.^n x} = -\frac{x^{m+1}}{n-1} \left\{ \frac{1}{\log.^{n-1} x} + \frac{m+1}{n-2} \cdot \frac{1}{\log.^{n-2} x} + \frac{(m+1)^2}{(n-2)(n-3)} \cdot \frac{1}{\log.^{n-3} x} + \ldots \right\} + \frac{(m+1)^{n-1}}{1.2.3\ldots(n-1)} \int \frac{x^m\, dx}{\log.\, x}$$

We cannot, however, proceed with our calculation beyond this point, because our result would become $= \alpha$

Let us, however, assume

$$x^{m+1} = z \qquad \therefore \qquad (m+1)\, x^m\, dx = dz$$

Whence

$$\frac{x^m\, dx}{\log.\, x} = \frac{dz}{\log.\, z}$$

$$= \frac{e^u\, du}{u} \text{ putting } u = \log.\, z$$

In this manner we reduce the proposed quantity to the function already treated of in the chapter on Exponential Functions, which can be integrated by approximation only.

When n is a fraction either positive or negative, one or other of the above methods will enable us to reduce the integral of $X\, dx \log.^n x$, to that of a function of the same form, in which the value of n lies between $+1$ and -1. We must then approximate to the value of the required integral by series.

On Circular Functions.

These may always be reduced to algebraic functions by assuming $\sin.\, \theta$ or $\cos.\, \theta = z$, but with a few exceptions we shall obtain the integrals of these quantities by the method of parts.

To integrate

(1) $$u = \int \frac{d\theta}{\sin.\, \theta}$$

Let $\cos.\theta = y$

$\therefore\quad \sin.\theta\, d\theta = -dy$

$$\therefore \int \frac{d\theta}{\sin.\theta} = \int \frac{-dy}{\sin.^2\theta}$$

$$= \int \frac{-dy}{1-y^2}$$

$$= \frac{1}{2}\cdot\log.\frac{1-y}{1+y}$$

$$= \frac{1}{2}\cdot\log.\frac{1-\cos.\theta}{1+\cos.\theta}$$

$$= \log.\sqrt{\frac{1-\cos.\theta}{1+\cos.\theta}}$$

$$= \log.\tan.\frac{\theta}{2}$$

To integrate*

(2) $$u = \int \frac{d\theta}{\cos.\theta}$$

Let $\sin.\theta = y$

$$\cos.\theta\, d\theta = dy$$

$$\frac{d\theta}{\cos.\theta} = \frac{dy}{\cos.^2\theta}$$

$$= \frac{dy}{1-y^2}$$

$$= \frac{1}{2}\cdot\log.\frac{1+y}{1-y}$$

$$= \frac{1}{2}\cdot\log.\frac{1+\sin.\theta}{1-\sin.\theta}$$

$$= \log.\sqrt{\frac{1+\sin.\theta}{1-\sin.\theta}}$$

$$= \log.\tan.\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

To integrate

(3) $$u = \int \frac{d\theta\cos.\theta}{\sin.\theta}$$

The numerator is the differential of the denominator

$$\therefore \int \frac{d\theta\cos.\theta}{\sin.\theta} = \int \frac{d\theta}{\tan.\theta} = \int d\theta\cot.\theta = \log.\sin.\theta$$

To integrate

(4) $$u = \int \frac{d\theta\sin.\theta}{\cos.\theta}$$

The numerator is as before the differential of the denominator

$$\int \frac{d\theta\sin.\theta}{\cos.\theta} = \int \frac{d\theta}{\cot.\theta} = \int d\theta\tan.\theta = -\log.\cos.\theta = \log.\frac{1}{\cos.\theta}$$

Hence adding the forms (3) and (4)

$$\int \frac{d\theta}{\sin.\theta \cos.\theta} = \log. \frac{\sin.\theta}{\cos.\theta} = \log. \tan.\theta$$

To integrate

$$du = d\theta \sin.^m \theta \cos.^n \theta$$
$$u = \int d\theta \sin.\theta \sin.^{m-1}\theta \cos.^n \theta$$

Proceeding by the method of parts, and supposing $\sin.^{m-1}\theta$ constant in the first instance.

$$u = -\frac{1}{n+1}\cos.^{n+1}\theta \sin.^{m-1}\theta + \frac{m-1}{n+1}$$
$$\int \sin.^{m-2}\theta \cos.\theta \, d\theta \cos.^{n+1}\theta$$
$$= -\frac{1}{n+1}\cos.^{n+1}\theta \sin.^{m-1}\theta + \frac{m-1}{n+1}$$
$$\int \sin.^{m-2}\theta \cos.^n\theta (1 - \sin.^2\theta) d\theta$$
$$= -\frac{1}{n+1}\cos.^{n+1}\theta \sin.^{m-1}\theta + \frac{m-1}{n+1}$$
$$\int \sin.^{m-2}\theta \cos^n\theta \, d\theta - \frac{m-1}{n+1}u$$
$$\therefore \quad u = -\frac{1}{n+m}\cos.^{n+1}\theta \sin.^{m-1}\theta + \frac{m-1}{m+n}$$
$$\int d\theta \sin.^{m-2}\theta \cos.^n\theta \quad \ldots\ldots\ldots\ldots (1)$$

&c. &c. &c.

Similarly, if we integrate for the cosine in the same manner as we have done for the sine, we shall have

$$\int d\theta \sin.^m\theta \cos.^n\theta = \frac{1}{m+n}\sin.^{m+1}\theta \cos.^{n-1}\theta + \frac{n-1}{m+n}$$
$$\int d\theta \sin.^m\theta \cos.^{n-2}\theta \quad \ldots\ldots\ldots\ldots (2)$$

&c. &c. &c

These integrals will $\therefore$ by successive reduction become $d\theta \cos.^n\theta$, $d\theta \sin.^m\theta$ or $d\theta \sin.\theta \cos.^n\theta$, $d\theta \cos.\theta \sin.^m\theta$, according as m or n are odd or even.

We have found

$$\int d\theta \sin.^m\theta \cos.^n\theta = -\frac{1}{m+n}.\sin.^{m-1}\theta\cos.^{n+1}\theta + \frac{m-1}{m+n}\int d\theta \sin.^{m-2}\theta \cos^n\theta \ldots (1)$$

integrating for the sine, and

$$\int d\theta \sin.^m\theta \cos.^n\theta = \frac{1}{m+n}.\sin.^{m+1}\theta \cos.^{n-1}\theta + \frac{n-1}{m+n}\int d\theta \sin.^m\theta \cos.^{n-2}\theta \ldots (2)$$

integrating for cosine.

Now, suppose m, or n, to be negative, making n negative in (1)

$$\int \frac{d\theta \sin.^m\theta}{\cos.^n\theta} = -\frac{1}{(m-n)}\cdot\frac{\sin.^{m-1}\theta}{\cos.^{n-1}\theta} + \frac{m-1}{m-n}\int \frac{d\theta \sin.^{m-2}\theta}{\text{cosin.}^n\theta} \ldots\ldots (3)$$

and $\therefore$ the integral will at length depend upon that of $\frac{d\theta}{\cos.^n \theta}$, or $\frac{d\theta \sin. \theta}{\cos.^n \theta}$ according as m is odd or even.

The formula (2) making n negative, and integrating for the cosine gives

$$\int \frac{d\theta \sin.^m \theta}{\cos.^n \theta} = \frac{1}{n-1} \cdot \frac{\sin.^{m+1}\theta}{\cos.^{n-1}\theta} - \frac{m-n+2}{n-1} \int \frac{d\theta \sin.^m \theta}{\cos.^{n-2}\theta} \ldots\ldots\ldots\ldots (4)$$

And the integral will then be reduced to $d\theta \sin.^m \theta$ or to $\frac{d\theta \sin.^m \theta}{\cos. \theta}$

If both m and n are negative, the integral becomes

$\int \frac{d\theta}{\sin.^m\theta \cos.^n \theta}$ Multiply both numerator and denominator by $\sin.^2 \theta + \cos.^2 \theta$

$$\therefore \int \frac{d\theta}{\sin.^m \theta \cos.^n \theta} = \int \frac{d\theta}{\sin.^{m-2} \theta \cos.^n \theta} + \int \frac{d\theta}{\sin.^m \theta \cos.^{n-2} \theta}$$

And by continuing the process the first of these fractions will be freed from the sine, and the second from the cosine, and the integral will be reduced to finding that of

$\frac{d\theta}{\cos.^n \theta}$, $\frac{d\theta}{\sin.^m \theta}$, or $\frac{d\theta}{\sin. \theta \cos.^n \theta} + \frac{d\theta}{\cos. \theta \sin.^n \theta}$ according as m and n are odd or even.

If m and n be equal as in $\sin. x \cos. x = \frac{1}{2} \sin. 2x$, making $2x = z$, the fraction becomes

$$\int \frac{d\theta}{\cos.^n \theta \sin.^n \theta} = 2^{n-1} \int \frac{dz}{\sin.^n z}$$

We have now seen that in integrating the formula

$\int d\theta \sin.^{\pm m} \theta \cos.^{\pm n} \theta$, we ultimately reduce it to one of the following forms.

$$\left.\begin{array}{ll} \int \sin.^m \theta \, d\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1') \\ \int \cos.^n \theta \, d\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2') \\ \int \frac{d\theta}{\sin.^m \theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3') \\ \int \frac{d\theta}{\cos.^n \theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4') \end{array}\right\} \text{ or } \begin{array}{l} \int \sin.^m \theta \cos. \theta \, d\theta \\ \int \cos.^n \theta \sin. \theta \, d\theta \\ \int \frac{d\theta \cos. \theta}{\sin.^m \theta} \\ \int \frac{d\theta \sin. \theta}{\cos.^n \theta} \end{array}$$

Now we may find the first form of these by making m and $n = 0$ in the formulæ (1), (2), (3), (4), of last page, which will then give

$$\int d\theta \sin.^m \theta = -\frac{1}{m} \cdot \cos. \theta \sin.^{m-1} \theta + \frac{m-1}{m} \int \sin.^{m-2} \theta \, d\theta$$

$$\int d\theta \cos.^n \theta = \frac{1}{n} \cdot \sin. \theta \cos.^{n-1} \theta + \frac{n-1}{n} \int \cos.^{n-2} \theta \, d\theta$$

$$\int \frac{d\theta}{\sin.^m \theta} = -\frac{1}{m-1} \cdot \frac{\cos. \theta}{\sin.^{m-1} \theta} + \frac{m-2}{m-1} \int \frac{d\theta}{\sin.^{m-2} \theta}$$

$$\int \frac{d\theta}{\cos.^n \theta} = \frac{1}{n-1} \cdot \frac{\sin. \theta}{\cos.^{n-1} \theta} + \frac{n-2}{n-1} \int \frac{d\theta}{\cos.^{n-2} \theta}$$

And these again will ultimately be reduced to one of the form

$$\frac{d\theta}{\sin.\ \theta}, \quad \frac{d\theta}{\cos.\ \theta}, \quad \frac{d\theta \sin.\ \theta}{\cos.\ \theta}, \quad \frac{d\theta \cos.\ \theta}{\sin.\ \theta}, \quad \frac{d\theta}{\sin.\ \theta \cos.\ \theta}$$

which have been already found, and there remains now only the integral to

$$\left.\begin{aligned}
\int \sin.^m \theta \cos.\ \theta\, d\theta &= \frac{\sin.^{m+1} \theta}{m+1} \\
\int \cos.^n \theta \sin.\ \theta\, d\theta &= -\frac{\cos.^{n+1} \theta}{n+1} \\
\int \frac{\cos.\ \theta\, d\theta}{\sin^m \theta} &= -\frac{1}{(m-1)\sin^{m-1} \theta} \\
\int \frac{d\theta \sin.\ \theta}{\cos.^n \theta} &= -\frac{1}{(n-1)\cos.^{n-1} \theta}
\end{aligned}\right\}$$

In practice when integrati any of these forms it will be found convenient if any of the quantities are in the denominator, to reduce the expression to a binomial algebraic integral, by assuming $y = \frac{1}{\sin.\ \theta}$ or $= \frac{1}{\text{c}\quad \text{s.}}$ as may happen.

We may here give the integrals of one or two remarkable functions belonging to this class.

To integrate

$$du = \frac{d\theta}{a + b \cos.\ \theta}.$$

Assume $\cos.\ \theta = \dfrac{1 - x^2}{1 + x^2}$ .... (1)

$$\therefore \quad \cos.^2 \theta = \frac{1 - 2x^2 + x^4}{1 + 2x^2 + x^4}$$

$$1 - \cos.^2 \theta = \frac{4x^2}{(1 + x^2)^2}$$

$$\therefore \quad \sin.\ \theta = \frac{2x}{(1 + x^2)} \quad \text{.. (2)}$$

From (1) $\cos.\ \theta = \dfrac{1 - x^2}{1 + x^2}$

$$\therefore \quad a + b \cos.\ \theta = \frac{a(1 + x^2) + b(1 - x^2)}{1 + x^2} \quad \text{........................ (3)}$$

Again,

$$\cos.\ \theta = \frac{1 - x^2}{1 + x^2}$$

$$\cos.\ \theta + x^2 \cos.\ \theta = 1 - x^2$$

$$x^2 (1 + \cos.\ \theta) = 1 - \cos.\ \theta$$

$$x = \sqrt{\frac{1 - \cos.\ \theta}{1 + \cos.\ \theta}}$$

$$= \tan.\ \frac{\theta}{2} \quad \text{.. (4)}$$

Since $\cos.\ \theta = \dfrac{1 - x^2}{1 + x^2}$

$$-\sin.\ \theta\, d\theta = \frac{-2x dx (1 + x^2) - 2x dx (1 - x^2)}{(1 + x^2)^2}$$

$$d\theta = \frac{4x dx}{(1+x^2)^2} \times \frac{1}{\sin.\ \theta}$$

$$= \frac{4x dx}{(1+x^2)^2} \times \frac{1+x^2}{2x} \text{ from equation (2)}$$

$$= \frac{2dx}{1+x^2}$$

$$\frac{d\theta}{a+b\cos.\ \theta} = \frac{2dx}{1+x^2} \times \frac{1+x^2}{a(1+x^2)+b(1-x^2)} \text{ from equation (3)}$$

$$= \frac{2dx}{(a+b)+(a-b)x^2}$$

$$= \frac{2dx}{\alpha+\beta x^2} \quad \text{Putting } a+b=\alpha,\ a-b=\beta$$

$$\int \frac{d\theta}{a+b\cos.\ \theta} = \frac{2}{\sqrt{\alpha\beta}} \tan.^{-1}\ x\sqrt{\frac{\beta}{\alpha}}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \cdot \tan.^{-1} \tan.\ \frac{\theta}{2}\sqrt{\frac{a-b}{a+b}} \text{ from equation (4)}$$

$$\frac{2}{\sqrt{a^2-b^2}} \tan.^{-1} \frac{(a-b)\tan.\ \frac{\theta}{2}}{\sqrt{a^2-b^2}}$$

To integrate

$$du = \frac{d\theta}{a+b\tan.\ \theta}$$

Let $\quad b\tan.\ \theta = z$

$\therefore\ a+b\tan.\ \theta = (a+z)$ .... (1)

But, since $b\tan.\ \theta = z$

$$b \cdot \frac{d\theta}{\cos.^2\ \theta} = dz$$

$$d\theta = \frac{1}{b} \cdot \frac{dz}{\sec.^2\ \theta}$$

$$= \frac{1}{b} \cdot \frac{dz}{1+\tan.^2\ \theta}$$

$$= \frac{1}{b} \cdot \frac{dz}{1+\frac{z^2}{b^2}}$$

$$= b \cdot \frac{dz}{z^2+b^2}$$

$$\therefore \int \frac{d\theta}{a+b\tan.\ \theta} = b\int \frac{d\theta}{(z+a)(z^2+b^2)}$$

Let

$$\frac{1}{(z+a)(z^2+b^2)} = \frac{A}{z+a} + \frac{Bz+C}{z^2+b^2} .$$

$$\begin{aligned}\therefore \quad 1 = {} & A \,.\, z^2 + A \,.\, b^2 \\ & + B \,.\, z^2 + Ba \,.\, z \\ & + C \,.\, a + C \,.\, z\end{aligned}$$

$$\therefore \quad A + B = 0 \quad \therefore \quad A = -B$$

$$C + Ba = 0 \quad \therefore \quad C = Aa$$

$$Ab^2 + Ca = 1 \quad \therefore \quad A = \frac{1}{a^2 + b^2} = -B$$

$$\therefore \quad C = \frac{a}{a^2 + b^2}$$

$$\therefore \frac{dz}{(z+a)(z^2+b^2)} = \frac{1}{a^2+b^2} \cdot \frac{dz}{z+a} + \frac{1}{a^2+b^2} \cdot \frac{adz - zdz}{b^2+z^2}$$

$$\therefore \int \frac{dz}{(z+a)(z^2+b^2)} = \frac{1}{a^2+b^2} \log.\,(z+a) + \frac{a}{a^2+b^2} \int \frac{dz}{b^2+z^2} - \frac{1}{a^2+b^2} \int \frac{zdz}{b^2+z^2}$$

$$= \frac{1}{a^2+b^2} \log.\,(z+a) + \frac{a}{a^2+b^2} \cdot \frac{1}{b} \tan.^{-1} \frac{z}{b}$$

$$- \frac{1}{a^2+b^2} \cdot \frac{1}{2} \log.\,(z^2+b^2)$$

$$\therefore \int \frac{d\theta}{a + b \tan.\,\theta} = \frac{b}{a^2+b^2} \log.(z+a) + \frac{a}{a^2+b^2} \cdot \tan.^{-1} \frac{z}{b} - \frac{b}{a^2+b^2} \log.\,\sqrt{z^2+b^2}$$

$$= \frac{b}{a^2+b^2} \log.\,(a + b \tan.\,\theta) + \frac{a}{a^2+b^2} \cdot \tan.^{-1}\,(\tan.\,\theta)$$

$$- \frac{b}{a^2+b^2} \cdot \log.\, b \sec.\,\theta$$

The integral of $\frac{d\theta}{a + b \cos.\,\theta}$ may be obtained very simply by an algebraic artifice.

$$\int \frac{d\theta}{a + b \cos.\,\theta} = \int \frac{d\theta}{a + b\left(\cos.^2 \frac{\theta}{2} - \sin.^2 \frac{\theta}{2}\right)}$$

$$= \int \frac{d\theta}{(a+b) \cos.^2 \frac{\theta}{2} + (a-b) \sin.^2 \frac{\theta}{2}}$$

$$= \frac{1}{a+b} \int \frac{\dfrac{d\theta}{\cos.^2 \frac{\theta}{2}}}{1 + \dfrac{a-b}{a+b} \tan.^2 \frac{\theta}{2}}$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \int \frac{\dfrac{1}{2} \cdot \sqrt{\dfrac{a-b}{a+b}} \cdot \dfrac{d\theta}{\cos.^2 \frac{\theta}{2}}}{1 + \dfrac{a-b}{a+b} \cdot \tan^2 \frac{\theta}{2}}$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan.^{-1} \frac{\sqrt{a-b}}{\sqrt{a+b}} \tan.\,\frac{\theta}{2}$$

On Integration by Series.

When the integral of a proposed function cannot be exactly determined, we must have recourse to approximations. Thus in order to find

$$\int X\, dx$$

where X is a function of x, we must develope X in a series according to ascending or descending powers of x, and then multiplying each term by dx integrate them in succession. For example, we know that

$$\int \frac{dx}{1+x^2} = \tan.^{-1} x$$

But if we develope $(1 + x^2)^{-1}$ we have

$$\frac{dx}{1+x^2} = dx\,(1 - x^2 + x^4 - x^6 + \ldots .)$$

Whence $\tan.^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots .$

Again,

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin.^{-1} x$$

But

$$\frac{dx}{\sqrt{1-x^2}} = dx\,(1 - x^2)^{-\frac{1}{2}}$$

$$= dx\left(1 + \frac{x^2}{2} + \frac{1 \,.\, 3 \,.\, x^4}{2 \,.\, 4} + \ldots\right)$$

$$\therefore \quad \sin.^{-1} x = x + \frac{x^3}{1 \,.\, 2 \,.\, 3} + \frac{3 \,.\, x^5}{2 \,.\, 4 \,.\, 5} + \frac{3 \,.\, 5 \,.\, x^7}{2 \,.\, 4 \,.\, 6 \,.\, 7} +$$

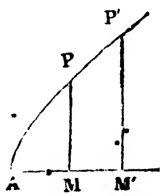
On the determination of Arbitrary Constants.

Let P be the integral of X dx a function of x, and C the arbitrary constant which we must add in order to render the result perfectly general, we have

$$\int X\, dx = P + C$$

So long as this calculation is altogether abstract, C may have any value whatever; but when we wish to apply this integral to the solution of some given problem the constant C ceases to be arbitrary and must answer certain conditions.

Thus, for example, if it be required to determine the area PP′ M′M = A included between the ordinates MP, M′P′, which correspond respectively to the abscissas a and b, since we have



$$\frac{dA}{dx} = y$$

$$A = \int y\,dx$$

$$= P + C$$

But since the required area $P + C$ commences when $x = AM = a$, A ought to be $= 0$ when we make $x = a$ in $P + C$, or

$$Q + C = 0$$

Q being the value which the function of x represented by P assumes when $x = a$, hence we find

$$C = -Q$$

whence the area $A = P - Q$.

It only now remains to substitute b for x, and we shall have tne area included within the prescribed limits. We shall have several examples in what follows.

CHAPTER IV.

APPLICATION OF THE INTEGRAL CALCULUS TO FINDING THE LENGTHS AND AREAS OF CURVES, AND THE SURFACES AND VOLUMES OF SOLIDS OF REVOLUTION.

I. The Rectification of Curves.

We have seen in p. 747 of the *differential calculus*, that if s represent the arc of a curve,

$$ds = dx\sqrt{1 + \frac{dy^2}{dx^2}} = \sqrt{dy^2 + dx^2}$$

and we shall now apply this formula to a few examples.

(1.) *To find the length of the arc of the common parabola.*

The equation is $y^2 = 4mx$, where $4m$ is the parameter.

$$\therefore \quad ydy = 2mdx \;\therefore\; \frac{dy}{dx} = \frac{2m}{y}$$

$$\therefore \quad ds = dx\sqrt{1 + \frac{dy^2}{dx^2}} = dx\sqrt{1 +}$$

$$\therefore \quad s = \frac{1}{2m}\int\frac{y^2dy}{\sqrt{y^2+4m^2}} + 2m\int\frac{dy}{\sqrt{y^2+4m^2}}$$

$$= \frac{y}{4m}\sqrt{y^2 + 4m^2} + m\log\left(y + \sqrt{y^2 + 4m^2}\right) + C$$

If we suppose the arc to be measured from the vertex; then when $y=0$, $s=0$, and $\therefore\; 0 = 0 + m\log 2m + C \;\therefore\; C = -m\log 2m$, and therefore

$$s = \frac{y\sqrt{y^2+4m^2}}{4m} + m\log\frac{y+\sqrt{y^2+4m^2}}{2m}.$$

(2.) *To rectify the circle.*

By the differential calculus we know that

$$d\tan^{-1}x = \frac{dx}{1+x^2} = (1 - x^2 + x^4 - x^6 + \ldots.)\,dx$$

$$\therefore\; s = \tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\ldots$$

$$= x\left(1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 + \ldots\ldots\right)\ldots\ldots \quad (A)$$

Assume now $\tan(a+b) = \frac{1}{e}$ $\qquad \therefore\ a + b = \tan^{-1}\frac{1}{e}$

$$\tan a = \frac{x}{y} \qquad \therefore \qquad a = \tan^{-1}\frac{x}{y}$$

$$\tan b = \frac{x'}{y'} \qquad \therefore \qquad b = \tan^{-1}\frac{x'}{y'}$$

$$\therefore\ \tan^{-1}\frac{1}{e} = \tan^{-1}\frac{x}{y} + \tan^{-1}\frac{x'}{y'} \ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}$$

But $\tan a = \tan(a + b - b) = \dfrac{\tan(a+b) - \tan b}{1 + \tan b \tan(a+b)}$

$$\therefore\ \frac{x}{y} = \frac{\frac{1}{e} - \frac{x'}{y'}}{1 + \frac{x'}{ey'}} \ldots\ldots\ldots\ldots \text{(C)}$$

Let $\frac{1}{e} = 1$, and $\frac{x'}{y'} = \frac{1}{5}$; then $\frac{x}{y} = \frac{1 - \frac{1}{5}}{1 + \frac{1}{5}} = \frac{2}{3}$, by eq. (C.)

$$\therefore\ \tan^{-1} 1 \text{ or } \frac{\pi}{4} = \tan^{-1}\frac{2}{3} + \tan^{-1}\frac{1}{5} \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Let $\frac{1}{e} = \frac{2}{3}$; and $\frac{x'}{y'} = \frac{1}{5}$; then $\frac{x}{y} = \frac{\frac{2}{3} - \frac{1}{5}}{1 + \frac{2}{15}} = \frac{7}{17}$

$$\therefore\ \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{7}{17} + \tan^{-1}\frac{1}{5}, \text{ by eq. (B)}$$

$$\therefore\ \frac{\pi}{4} = 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{7}{17} \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Let $\frac{1}{e} = \frac{7}{17}$; and $\frac{x'}{y'} = \frac{1}{5}$; then $\frac{x}{y} = \frac{\frac{7}{17} - \frac{1}{5}}{1 + \frac{7}{85}} = \frac{18}{92} = \frac{9}{46}$

$$\therefore\ \tan^{-1}\frac{7}{17} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{9}{46} \text{ by eq. (B)}$$

$$\therefore\ \frac{\pi}{4} = 3\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{9}{46} \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Let $\frac{1}{e} = \frac{9}{46}$; and $\frac{x'}{y'} = \frac{1}{5}$; then $\frac{x}{y} = \frac{\frac{9}{46} - \frac{1}{5}}{1 + \frac{9}{230}} = -\frac{1}{239}$

$$\therefore\ \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} \ldots\ldots\ldots\ldots\ldots \text{(4)}$$

Let $\frac{1}{e} = \frac{1}{239}$; and $\frac{x'}{y'} = \frac{1}{70}$; then $\frac{x}{y} = \frac{\frac{1}{239} - \frac{1}{70}}{1 + \frac{1}{16730}} = -\frac{169}{16731} = -\frac{1}{99}$

$$\therefore\ \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} \ldots\ldots\ldots \text{(5)}$$

In a similar manner we might obtain the following results:—

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = 3\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{11}$$

$$= 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}$$

$$= 8\tan^{-1}\frac{1}{10} - 4\tan^{-1}\frac{1}{515} - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\frac{1}{2} - 2\tan^{-1}\frac{1}{14} + 2\tan^{-1}\frac{1}{2786} - \tan^{-1}\frac{1}{10812186007}.$$

We may use any of these results for the rectification of the circle; but those are to be selected which are best adapted for facility of computation. We shall take the result in equation (5), and therefore by equation (A) we have

$$\frac{\pi}{4}=4\left\{\frac{1}{5}-\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}-\frac{1}{7}\cdot\frac{1}{5^7}+\frac{1}{9}\cdot\frac{1}{5^9}-\ldots\ldots\right\}$$
$$-\left\{\frac{1}{70}-\frac{1}{3}\cdot\frac{1}{70^3}+\frac{1}{5}\cdot\frac{1}{70^5}-\ldots\ldots\right\}$$
$$+\left\{\frac{1}{99}-\frac{1}{3}\cdot\frac{1}{99^3}+\frac{1}{5}\cdot\frac{1}{99^5}-\ldots\ldots\right\}$$
$$=\cdot 7895822394-\cdot 0142847425+\cdot 0101006665=\cdot 7853981634$$

$\therefore\ \pi=3\cdot 1415926536=$ semicircumference to radius unity.

Hence the circumference of a circle whose diameter is unity is $3\cdot 1415926536$, which is true as far as nine decimal places.

(3.) *To find the length of the arc of a cycloid.*

Here $y=\sqrt{2rx-x^2}+\text{vers}^{-1}x$, is the equation of the curve.

$$\therefore\ \frac{dy}{dx}=\frac{r-x}{\sqrt{2rx-x^2}}+\frac{r}{\sqrt{2rx-x^2}}=\sqrt{\frac{2r-x}{x}}$$

$$\therefore\ ds=dx\sqrt{1+\frac{dy^2}{dx^2}}=\sqrt{2r}\cdot\frac{dx}{\sqrt{x}}$$

$$=\sqrt{2r}\int x^{-\frac{1}{2}}dx=2\sqrt{2rx}+C$$

When $x=0$, $s=0$ $\therefore$ C $=0$, and when $x=2r$; then

semicycloidal arc $=2\sqrt{4r^2}=4r$

and the whole length of the cycloid is $=8r=4$ times the diameter of the generating circle.

(4.) *To find the length of the arc of an ellipse.*

Here $a^2y^2+b^2x^2=a^2b^2$; whence, if $\frac{a^2-b^2}{a^2}=1-\frac{b^2}{a^2}=e^2$

$$\therefore\ y^2=\frac{b^2}{a^2}(a^2-x^2)=(e^2-1)(x^2-a^2)$$

$$\therefore\ \frac{dy}{dx}=\frac{x\sqrt{e^2-1}}{\sqrt{x^2-a^2}}\quad\therefore\ ds=dx\sqrt{1+\frac{x^2(e^2-1)}{x^2-a^2}}$$

$$\therefore\ ds=\frac{dx\sqrt{a^2-e^2x^2}}{\sqrt{a^2-x^2}}=adv\,\frac{\sqrt{1-e^2v^2}}{\sqrt{1-v^2}},\ \text{where } x=av.$$

The numerator must now be developed by the binomial theorem, and the several terms of the series being multiplied by dv, divided by $\sqrt{1-v^2}$, and integrated between any proposed limits, will give the length of the elliptic arc required.

II. Areas of Curves.

(1.) *To find the area of a parabola.*

Here $y^2=4mx$ $\therefore$ $ydy=2mdx$.

But $d\text{A}=ydx=\frac{1}{2m}y^2dy$, by the differential calculus, p. 748.

$$\therefore\ \text{A}=\frac{1}{2m}\int y^2dy=\frac{1}{2m}\cdot\frac{y^3}{3}$$

$$=\frac{1}{6m}\cdot 4mx\cdot y=\frac{2}{3}xy+C$$

When $x=0$, $A=0$ $\therefore$ $C=0$, and

$\therefore$ area of parabola $=\frac{2}{3}\,xy=\frac{2}{3}$ of circumscribing rectangle.

(2.) *To find the area of a circle.*

The equation is $y^2+x^2=a^2$ $\therefore$ $y^2=a^2-x^2$

$$\therefore\ A=\int y\,dx=\int dx\sqrt{a^2-x^2}$$

$$=a^2\int\frac{dx}{\sqrt{a^2-x^2}}-\int\frac{x^2dx}{\sqrt{a^2-x^2}}$$

$$=a^2\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}-\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{xy}{2}+C.$$

When $x=0$, $A=0$ $\therefore$ $C=0$, and when $x=a$, $y=0$

$\therefore$ area of a quadrant $=\frac{a^2}{2}\cdot\frac{\pi}{2}=\frac{\pi a^2}{4}$

$\therefore$ area of a circle $=\pi a^2$.

(3.) *To find the area of an ellipse.*

$$A=\int y\,dx=\frac{b}{a}\int dx\sqrt{a^2-x^2}$$

$$=\frac{b}{a}\cdot\left\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}\right\}$$

$\therefore$ area of a quadrant $=\frac{ab}{2}\cdot\frac{\pi}{2}=\frac{1}{4}\pi ab$

$\therefore$ area of ellipse $=\pi ab$.

(4.) *To find the area of the cycloid.*

When the origin is at the vertex, the equation is

$$y=\text{vers}^{-1}x+\sqrt{2rx-x^2};$$

$$\therefore\ \text{area}=yx-\int x\,dy=yx-\int dx\sqrt{2rx-x^2}$$

$$=yx-2r\int\frac{x\,dx}{\sqrt{2rx-x^2}}+\int\frac{x^2dx}{\sqrt{2rx-x^2}}$$

$$=yx+2r\sqrt{2rx-x^2}-2r\,\text{vers}^{-1}x-\left(\frac{x}{2}+\frac{3r}{2}\right)\sqrt{2rx-x^2}$$

$$+\frac{3r}{2}\,\text{vers}^{-1}x$$

$$=\left(\frac{2x-r}{2}\right)\text{vers}^{-1}x+\frac{x+r}{2}\sqrt{2rx-x^2}+C.$$

When $x=0$, $A=0$ $\therefore$ $C=0$, and when $x=2r$,

semi-cycloidal area $=\frac{3r}{2}\cdot r\pi=\frac{3}{2}\pi r^2$

$\therefore$ cycloid $=3\pi r^2=3$ times area of generating circle.

(5.) *To find the area of the curve, whose equation is*

$$a^2(y^2-x^2)+(y^2+x^2)^2=0.$$

In order to transform this equation from rectangular to polar co-ordinates, we must put $y=r\sin\theta$, and $x=r\cos\theta$; then, by substitution in the proposed equation, we have

$$a^2r^2(\sin^2\theta - \cos^2\theta) + r^4 = 0$$
$$\therefore r^2 = a^2(\cos^2\theta - \sin^2\theta)$$
$$= a^2 \cos 2\theta, \text{ the polar equation.}$$

Again; let A′ denote the polar area, or space between the radius vector and the curve; then

$$A' = A - \frac{yx}{2} = \int y dx - \frac{yx}{2}$$

$$\therefore dA' = ydx - \frac{ydx + xdy}{2} = \frac{ydx - xdy}{2}$$

$$= \frac{(r^2 \sin^2\theta + r^2\cos^2\theta)\, d\theta}{2} = \frac{1}{2} r^2\, d\theta.$$

$$\therefore \quad A' = \frac{1}{2}\int r^2 d\theta$$

$$= \frac{1}{2} a^2 \int \cos 2\theta\, d\theta$$

$$= \frac{1}{4} a^2 \sin 2\theta + C.$$

And between the limits $r = a$ and $r = 0$, or between $\theta = 0$ and $\theta = \frac{\pi}{4}$,

$$\text{area of curve} = \frac{1}{4} a^2,$$

and if r make a complete revolution, the entire area will be $= a^2$.

III. Surfaces of Solids.

(1.) *To find the surface of a sphere.*

Here $y = \sqrt{2ax - x^2}$; $\frac{dy}{dx} = \frac{a - x}{\sqrt{2ax - x^2}}$; $1 + \frac{dy^2}{dx^2} = \frac{a^2}{2ax - x^2} = \frac{a^2}{y^2}$

$$\therefore S = 2\pi \int ydx \sqrt{1 + \frac{dy^2}{dx^2}} = 2\pi \int adx = 2\pi ax + C$$

$\therefore 0 = 0 + C \quad \therefore C = 0$

$\therefore$ surface of spherical segment $= 2\pi ax =$ circumf. $\times$ height of segment

$\therefore$ surface of sphere $= 2\pi a \cdot 2a = 4\pi a^2 =$ circumf. $\times$ diameter.

(2.) *To find the surface of a paraboloid.*

Here $y^2 = 4mx$; $\frac{dy}{dx} = \frac{2m}{y} \therefore \frac{dy^2}{dx^2} = \frac{4m^2}{y^2} = \frac{m}{x}$

$$\therefore S = 2\pi \int 2\sqrt{mx} \cdot dx \sqrt{\frac{x + m}{x}}$$

$$= 4\pi\sqrt{m} \int dx \sqrt{x + m}$$

$$= \frac{8}{3}\pi\sqrt{m}\,(x + m)^{\frac{3}{2}} + C$$

$$0 = \frac{8}{3}\pi m^2 + C \quad \therefore C = -\frac{8}{3}\pi m^2.$$

$$\text{Surface} = \frac{8}{3}\pi\sqrt{m}\left\{(x + m)^{\frac{3}{2}} - m^{\frac{3}{2}}\right\}.$$

(3.) *To find the surface of a cone, and also the surface of a conic frustum*

Put a = height of whole cone; r = radius of base of frustum; $a^2+r^2=c^2$;
b = height of top cone; r' = radius of top of frustum; $b^2+r'^2=c'^2$;

and taking the vertex as the origin of co-ordinates, we have $y=\frac{rx}{a}$, the equation of the line generating the surface; whence

$$S = 2\pi\int \frac{r}{a}\, x dx \sqrt{\frac{a^2+r^2}{a^2}}$$

$$= \frac{\pi r\sqrt{a^2+r^2}}{a^2} \cdot x^2 + C.$$

When $x=a$; surface of whole cone $= \pi r\sqrt{a^2+r^2} = \pi rc$

Sim. we have surface of top cone $= \pi r'\sqrt{b^2+r'^2} = \pi r'c'$

$\therefore$ surface of frustum of cone $= \pi\,(rc - r'c')$

$$= \pi\,(rc - rc' + r'c - r'c'),\ \text{since } rc' = r'c$$
$$= \pi\,(r+r')\,(c-c')$$
$$= (\pi r + \pi r')\,(c-c').$$

IV. Volumes of Solids.

(1.) *To find the content of a cone, and also that of a conic frustum.*

Let a = altitude of whole cone; r = radius of base of frustum
b = altitude of top cone; r' = radius of top of frustum

$\therefore a : b :: r : r'$

$\therefore y = \frac{r}{a} \cdot x$, and if V be the volume of the solid; then

$$V = \pi\int y^2 dx = \pi\,\frac{r^2}{a^2}\int x^2 dx = \frac{\pi r^2}{3a^2} \cdot x^3.$$

When $x=a$; then $V = \frac{\pi}{3}\, ar^2 = \frac{a}{3} \cdot \pi r^2$

$x=b$; then $V = \frac{\pi}{3} \cdot \frac{r^2}{a^2} \cdot b^3 = \frac{b}{3} \cdot \pi r'^2$

$\therefore$ volume of frustum $= \frac{\pi}{3}\,(ar^2 - br'^2)$

$$= \frac{\pi}{3}\left(ar^2 + arr' - br^2 + ar'^2 - brr' - br'^2\right)$$

since $ar' = br$, or $arr' = br^2$, and $ar'^2 = brr'$; whence

volume of frustum $= \frac{\pi}{3}\left\{(a-b)\,r^2 + (a-b)\,rr' + (a-b)\,r'^2\right\}$

$$= \frac{a-b}{3}\,(\pi r^2 + \pi rr' + \pi r'^2)$$

$$= \frac{\pi\,(a-b)}{3}\,(r^2 + rr' + r'^2).$$

(2.) *To find the volume of a sphere.*

Here $y^2 = 2ax - x^2$ $\therefore$ $y^2dx = 2axdx - x^2dx$

$$\therefore \ V = \pi\int y^2dx = 2a\pi\int x dx - \pi\int x^2 dx$$

$$= a\pi x^2 - \frac{\pi x^3}{3}$$

$$\therefore \qquad \text{segment} = \frac{\pi}{6}(6a-2x)x^2 = \frac{\pi}{6}(3d-2x)x^2, \text{ if } d = 2a.$$

$$\therefore \qquad \text{sphere} = \frac{\pi}{6}\,.\,d^3, \text{ by making } x = d.$$

(3.) *Find the volume of the paraboloid.*

Here $y^2 = 4mx$, is the equation to the generating parabola.

$$\therefore \ V = \pi\int y^2dx = 4m\pi\int x dx$$

$$= \frac{4m\pi x^2}{2} = \frac{\pi y^2 x}{2}$$

But $\pi y^2 x =$ volume of a cylinder, whose base $= \pi y^2$ and height $= x$

$\therefore$ volume of paraboloid $= \frac{1}{2}$ volume of circumscribing cylinder.

(4.) *Find the content of the prolate spheroid formed by the revolution of a semi-ellipse round its major axis.*

Here $y^2 = \frac{b^2}{a^2}(a^2-x^2)$ is the equation of the ellipse.

$$\therefore \ V = \frac{\pi b^2}{a^2}\int(a^2dx - x^2dx) = \pi b^2 x - \frac{\pi b^2}{3a^2}\,.\,x^3$$

$$= \frac{4}{3}\pi ab^2, \text{ integrating from } x = -a, \text{ and } x = +a.$$

(5.) *Find the content of the oblate spheroid formed by the revolution of a semi-ellipse round its minor axis.*

Interchanging x and y, we have $y^2 = \frac{a^2}{b^2}(b^2 - x^2)$

$$\therefore \ V = \frac{\pi a^2}{b^2}\int(b^2 - x^2)\,dx = \pi a^2 x - \frac{\pi a^2}{3b^2}\,x^3$$

$$= \frac{4}{3}\pi a^2 b, \text{ integrating from } x = -b \text{ to } x = +b.$$

Hence prolate spheroid : oblate spheroid :: $ab^2 : a^2b :: b : a$

$\therefore$ sphere on major axis : prolate spheroid :: $\frac{4}{3}\pi a^3 : \frac{4}{3}\pi ab^2 :: a^2 : b^2$

$\therefore$ oblate spheroid : sphere on minor axis :: $\frac{4}{3}\pi a^2 b : \frac{4}{3}\pi b^3 :: a^2 : b^2$.

MECHANICS.

DEFINITIONS AND FUNDAMENTAL NOTIONS.

1. *Mechanics* is the science which treats of the laws of rest and motion of bodies, whether solid or fluid, and is usually divided into the four following branches:—

(1.) *Statics*, which treats of the laws of forces in equilibrium.
(2.) *Dynamics*, which treats of the laws of motion of solid bodies.
(3.) *Hydrostatics*, of the laws of the equilibrium of fluid bodies.
(4.) *Hydrodynamics*, of the laws of motion of fluid bodies.

2. *Force* or *power* is the cause which produces, or tends to produce, motion in a body, or which changes, or tends to change, motion.

3. A *body* is a portion of matter limited in every direction, and is therefore of a determinate form and volume.

4. All bodies have a tendency to fall to the earth; and the force which they exert in consequence of this tendency is called their *weight*.

5. When forces are applied simultaneously to a body, and produce rest, they balance each other, or destroy each other's effects; and therefore such forces are said to be in *equilibrium*.

6. The *measure* of a force, in statics, is the weight which that force would support.

7 The *quantity of matter* of a body is proportional to its weight.

8. The *density* of a body is measured by the quantity of matter contained in a given space.

9. *Gravity* is that force by which a body endeavours to fall downwards.

10. *Specific gravity* is the relation of the weights of different bodies of equal magnitude, and is therefore proportional to the density of the body.

STATICS.

THE COMPOSITION AND EQUILIBRIUM OF FORCES ACTING ON A MATERIAL PARTICLE.

11. *Def.* The *resultant* of any number of forces is that single force which is equally effective with, or equivalent to, all the forces, and these forces are termed *component* or *constituent* forces.

PROP. I.

12. *To find the resultant of a given number of forces acting on a particle in the same straight line.*

The resultant of two or more forces acting on a particle in the same direction is equal to their sum, and acts in the same direction; but the resultant of two forces acting in opposite directions is equal to their difference, and acts in the direction of the greater component. Also, if several forces act in one direction, and others in a contrary direction, the resultant of all these forces will be equal to the excess of the sum of the forces acting in one direction, above the sum of those acting in the contrary direction, and it will act in the direction of the greater of these sums.

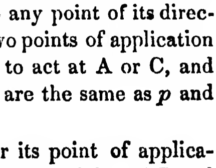
PROP. II.

13. *To find the resultant of two forces acting on a particle not in the same straight line.*

1. To find the *direction* of the resultant of two forces acting on a point.

When the forces are equal, it is obvious that the direction of the resultant will bisect the angle between the directions of the forces; or if the two forces be represented in magnitude and direction by two lines drawn from the point where they act, the diagonal of the rhombus described on these equal lines will be the direction of the resultant.

Assuming that the diagonal of a parallelogram described on the two lines representing the forces in magnitude and direction is the direction of the resultant; then if p, p_1 be any two unequal forces, and p, p_2 also two unequal forces, we can prove that the direction of the resultant of the two forces p and $p_1 + p_2$ is the diagonal of the parallelogram whose adjacent sides are p and $p_1 + p_2$.



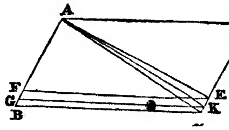
Let A be the point on which two forces p and p_1 act; AB, AC, their directions and proportional to them in magnitude. Complete the parallelogram BC, and draw the diagonal AD; then, by hypothesis, the resultant of p and p_1 acts in the direction of AD.

Again, produce AC to E, and take CE a fourth proportional to p_1, p_2, and AC; that is, make $p_1 . p_2 : : AC : CE$. Now, since the point of application of a force may be transferred to any point of its direction, without disturbing the equilibrium, so long as the two points of application are invariably connected, we may suppose the force p_2 to act at A or C, and therefore the forces p, p_1, p_2, in the lines AB, AC, CE, are the same as p and $p_1 + p_2$, in the lines AB and AE.

Now replace p and p_1 by their resultant, and transfer its point of application from A to D; then resolve this force at D into two, parallel to AB and AC; these resolved parts must evidently be p and p_1, where p acts in the direction DF and p_1 in the direction DG. Transfer these two forces p to C and p_1 to G; but by the hypothesis p and p_2 acting at C have a resultant in the direction CG; let, therefore, p and p_2 be replaced by their resultant, and transfer its point of application to G. But p_1 acts at G, and therefore by this process we have, without disturbing the equilibrium, removed the forces p and and $p_1 + p_2$, which acted at A to the point G; hence the resultant of p and

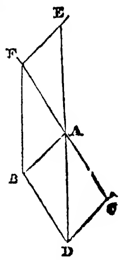
$p_1 + p_2$ acts in the direction of the diagonal AG, provided our assumption is correct. Now the hypothesis is correct for equal forces as p, p, and therefore it is true for forces $p, 2p$; consequently for $p, 3p$, and thus it is true for p, mp. Again, if it be true for p, mp, and p, mp, it is is also true for $2p, mp$; also for $3p, mp$, and thus it is true for np, mp, where n and m are positive integers.

We have now to show that the proposition is true for *incommensurable forces.* Let AB, AC represent two such forces, and complete the parallelogram BC. Then if their resultant do not act along AD, suppose it to act along AE, and draw EF parallel to BD. Divide AB into a number of equal parts, each less than DE; divide CD into parts equal to these, and let G be the last point of division of the former, which will obviously fall between B and F. Draw GK parallel to BD; then two forces represented by AC, AG, have a resultant in the direction AK, because they are commensurable; but this is nearer to AG than the resultant of the forces represented by AC, AB, which is absurd, since AB is greater than AG. In the same manner we may show that every direction besides AD leads to an absurdity, and therefore the resultant must act in the direction AD, whether the forces be commensurable or incommensurable.



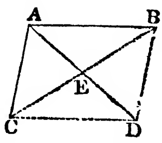
2. To find the *magnitude* of the resultant.

Let AB, AC be the direction of the given forces, AD that of their resultant; take AE in the prolongation of DA, and of such a length as to represent the *magnitude* of the resultant; then the forces represented by AB, AC, AE balance each other. Complete the parallelogram BE, and therefore AF is in the same straight line with AC, since the forces AB, AC, AE, balance each other; hence FD is a parallelogram, and therefore AD = FB = AE; that is, the resultant is represented in *magnitude* as well as in direction by the diagonal of the parallelogram.*



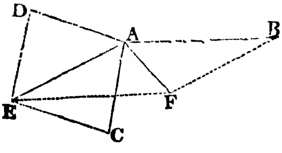
Cor. 1. The forces in the directions AB, AC, AD, are respectively proportional to the lines AB, AC, AD, and in these directions.

Cor. 2. The two oblique forces AB, AC, are equivalent to the single direct force AD, which may be compounded of these two, by drawing the diagonal of the parallelogram. Or, they are equivalent to the double of AE drawn to the middle of the line BC.



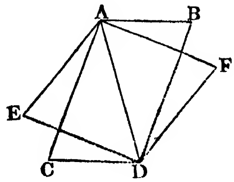
And thus any force may be compounded of two or more other forces; which is the meaning of the expression, *composition of forces.*

Example.—Suppose it were required to compound the three forces AB, AC, AD; or to find the direction and quantity of one single force, which shall be equivalent to, and have the same effect as if a body at A were acted on by three forces in the direction AB, AC, AD,



* The preceding demonstration of the *parallelogram of forces* is due to M. Duchayla, and is exceedingly simple and beautiful. Analytical demonstrations of this fundamental property have been given by Laplace, Pontécoulant, Poisson, and others; but want of room prevents us from giving them here.

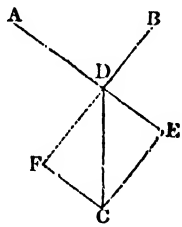
and proportional to these three lines. First, reduce the two, AC, AD, to one AE, by completing the parallelogram ADEC. Then reduce the two, AE, AB, to one AF, by the parallelogram AEFB. So shall the single force AF be the direction, and as the quantity, which shall of itself produce the same effect, as if all the three, AB, AC, AD, acted together.



Cor. 3. Any single direct force AD may be resolved into two oblique forces, whose quantities and directions are AB, AC, having the same effect, by describing any parallelogram whose diagonal may be AD; and this is called the *resolution of forces.* So the force AD may be resolved into the two AB, AC, by the parallelogram ABCD; or into the two AE, AF, by the parallelogram AEDF; and so on for any other two. And each of these may be resolved again into as many others as we please.

PROP. III.

14. *If three forces,* A, B, C, *acting together, keep one another in equilibrio, they will be proportional to the three sides* DE, CE, CD, *of a triangle, which are drawn parallel to the directions of the forces* AD, DB, CD.

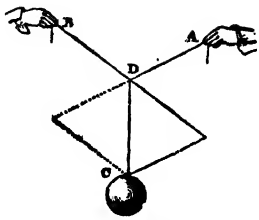


Produce AD, BD, and draw CF, CE, parallel to them. Then the force in CD is equivalent to the two AD, BD, by the supposition; but the force CD is equivalent to the two, ED and CE or FD; therefore, if CD represent the force C, ED will represent its opposite force A, and CE or FD its opposite force B; consequently, the three forces A, B, C, are proportional to DE, CE, CD, the three lines parallel to the directions in which they act.

Cor. 1. Because the three sides CD, CE, DE, are proportional to the sines of their opposite angles E, D ,C, therefore, the three forces, when in equilibrio, are proportional to the sines of the angles of the triangle made of their lines of direction; namely, each force proportional to the sine of the angle made by the directions of the other two.

Cor. 2. The three forces, acting against, and keeping one another in equilibrio, are also proportional to the sides of a triangle made by drawing lines either perpendicular to the directions of the forces, or forming any given angle with those directions. For, such a triangle is always similar to the former, which is made by drawing lines parallel to the directions; and therefore their sides are in the same proportion to one another.

Cor. 3. If any number of forces be kept in equilibrio by their actions against one another, they may be all reduced to two equal and opposite ones. For, by Cor. 2, Prop. II., any two of the forces may be reduced to one force acting in the same plane; then this last force and another may be likewise reduced to another force acting in their plane: and so on, till at last they be all reduced to the action of only two opposite forces, which will be equal, as well as opposite, because the whole are in equilibrio by the supposition.



Cor. 4. If one of the forces, as C, be a weight, which is sustained by two strings drawing in the directions DA, DB; then the force or tension of the string AD is to the weight C, or tension of the string DC, as DE to DC; and the force or tension of the string BD is to the weight C, or tension of CD, as CE to CD.

Cor. 5. Let f and f_1 be two forces acting simultaneously in directions making an angle ϕ; then in the triangle DEC we have

$$DE = f;\ EC = f_1 \text{ angle } DEC = \pi - \phi;$$

hence by the principles of trigonometry, we have

$$DC^2 = DE^2 + EC^2 - 2DE.EC \cos DEC;$$

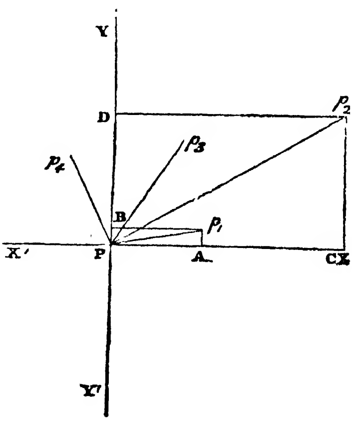
and therefore the magnitude of the resultant R is found from the equation

$$R = \sqrt{f^2 + f_1^2 - 2ff_1 \cos(\pi - \phi)} = \sqrt{f^2 + f_1^2 + 2ff_1 \cos \phi}.$$

PROP. IV.

15. *To find the resultant of several forces concurring in a point, and situated in the same plane.*

Let p_1, p_2, p_3, p_4, be any four forces acting on the point P, through which draw the axes of co-ordinates PX, PY at right angles to each other. Let Pp_1 represent the magnitude and direction of the force p_1, and draw p_1 B, p_1 A parallel to the axes XX^1 and YY^1. Then putting angle p_1 PX $= a_1$, we have the two rectangular forces PA, PB, equivalent to the given force p_1; but by trigonometry PA $=$ Pp_1 cos APp_1 $=$ $p_1 \cos a_1$, and PB $= p_1 \sin a_1$. In like manner, if a_2, a_3, a_4 be the angles which the direction of the forces p_2, p_3, p_4, make with PX, we shall have each of the proposed forces resolved into two others acting in the directions of the two axes, and therefore the sum, X, of all the component forces in direction PX, gives



$$X = p_1 \cos a_1 + p_2 \cos a_2 + p_3 \cos a_3 + p_4 \cos a_4 \ldots\ldots\ldots\ldots (1)$$

and the sum, Y, of all the other component forces in direction PY, gives

$$Y = p_1 \sin a_1 + p_2 \sin a_2 + p_3 \sin a_3 + p_4 \sin a_4 \ldots\ldots\ldots\ldots (2)$$

Hence the single force X, in direction PX, and the single force Y, in direction PY, may be substituted for the four given forces, and the resultant of the two forces X, Y, will be the resultant of the four forces p_1, p_2, p_3, p_4. But X and Y are two forces acting at right angles to each other, and their resultant, R, is the diagonal of the rectangle XY; hence we have

$$R = \sqrt{X^2 + Y^2} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Let R make an angle ϕ with the axis of X; then we have

$$\tan\phi = \frac{Y}{X};\ \cos\phi = \frac{X}{R};\ \sin\phi = \frac{Y}{R} \ldots\ldots\ldots\ldots\ldots \quad (4);$$

and any one of these three equations will give the position of the resultant.

In precisely the same manner may the magnitude and direction of any number of forces in the same plane be found.

Cor. 1. By means of a series of parallelograms the resultant of any number of forces may be found geometrically. For the diagonal of a parallelogram whose sides represent the first two forces will be their resultant, and this diagonal may be made the side of another parallelogram, having the third force for the other side, and so on. Or describe a polygon, whose sides beginning from the point, are successively equal and parallel to the given forces, and in the same direction; then the straight line which joins the point and the extremity of the last side completes the polygon, and represents the magnitude and direction of the resultant of the proposed forces.

Cor. 2. If three forces act on the same point in different places, and if the parallelopiped, whose adjacent edges represent these forces, be completed, its diagonal will represent their resultant both in magnitude and direction.

Cor. 3. Let $p_1, p_2, p_3, p_4, \ldots$ be any forces, and let each of these forces be resolved into three other forces in reference to three rectangular axes; then, collecting into one sum the component forces which act in the same axis, we can find the resultant of the three components thus obtained in the following manner:

Let $\alpha_1, \beta_1, \gamma_1$ be the angles which p_1 makes with the three axes
$\alpha_2, \beta_2, \gamma_2 \ldots\ldots\ldots\ldots\ldots p_2 \ldots\ldots\ldots\ldots\ldots$
$\alpha_3, \beta_3, \gamma_3 \ldots\ldots\ldots\ldots\ldots p_3 \ldots\ldots\ldots\ldots\ldots$
$\alpha_4, \beta_4, \gamma_4 \ldots\ldots\ldots\ldots\ldots p_4 \ldots\ldots\ldots\ldots\ldots$
&c. &c.

Then each of the given forces may be resolved into three others; viz.
p_1 into the three forces $p_1 \cos\alpha_1$, $p_1 \cos\beta_1$, $p_1 \cos\gamma_1$
p_2 into the three forces $p_2 \cos\alpha_2$, $p_2 \cos\beta_2$, $p_2 \cos\gamma_2$, and so on;

$$\text{hence } X = p_1\cos\alpha_1 + p_2\cos\alpha_2 + p_3\cos\alpha_3 + p_4\cos\alpha_4 + \ldots\ldots\ldots$$
$$Y = p_1\cos\beta_1 + p_2\cos\beta_2 + p_3\cos\beta_3 + p_4\cos\beta_4 + \ldots\ldots\ldots$$
$$Z = p_1\cos\gamma_1 + p_2\cos\gamma_2 + p_3\cos\gamma_3 + p_4\cos\gamma_4 + \ldots\ldots\ldots$$

hence $R = \sqrt{X^2+Y^2+Z^2}$ = magnitude of resultant.

And if α, β, γ, be the angles which R makes with each axis, we have

$$\cos\alpha = \frac{X}{R},\ \cos\beta = \frac{Y}{R},\ \cos\gamma = \frac{Z}{R}.$$

Examples for Practice.

16. *Ex.* 1. Let the four forces p_1, p_2, p_3, p_4, concurring in a point P, and situated in the same plane, be respectively denoted by the numbers 4, 6, 12, 10, and let the angles included by their directions be

$$p_1 P p_2 = 15^\circ, p_2 P p_3 = 30^\circ, p_3 P p_4 = 60^\circ;$$

required the magnitude and direction of the resultant of these forces.

We might assume any two rectangular axes whatever, PX, PY; but the solution will be simplified by taking one of the axes in the direction of one of

the given forces; let, therefore, the axis of X coincide with the direction of the force p_1; then we have

$$p_2\,PX = 15^\circ,\ \cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4} \text{ and } \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$p_3\,PX = 45^\circ,\ \cos 45^\circ = \tfrac{1}{2}\sqrt{2} \ \ldots\ldots \sin 45^\circ = \tfrac{1}{2}\sqrt{2}$$

$$p_4\,PX = 105^\circ,\ \cos 105^\circ = -\frac{\sqrt{6}-\sqrt{2}}{4} \ \ldots \sin 105 = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Hence

$$X = p_1+p_2\cos 15^\circ+p_3\cos 45^\circ+p_4\cos 105^\circ$$

$$= 4+\frac{3}{2}\sqrt{6}+\frac{3}{2}\sqrt{2}+6\sqrt{2}-\frac{5}{2}\sqrt{6}+\frac{5}{2}\sqrt{2} = 4+10\sqrt{2}-\sqrt{6}.$$

$$Y = p_2\sin 15^\circ+p_3\sin 45^\circ+p_4\sin 105^\circ$$

$$= \frac{3}{2}\sqrt{6}-\frac{3}{2}\sqrt{2}+6\sqrt{2}+\frac{5}{2}\sqrt{6}+\frac{5}{2}\sqrt{2} = 7\sqrt{2}+4\sqrt{6}.$$

$$\therefore R = \sqrt{X^2+Y^2} = \{(4+10\sqrt{2}-\sqrt{6})^2+(7\sqrt{2}+4\sqrt{6})^2\}^{\frac{1}{2}}$$

$$= 25\cdot184297 = \text{magnitude of the resultant.}$$

Also, $\tan RPX = \frac{Y}{X} = \frac{19\cdot697454}{15\cdot6926463} = 1\cdot2552028$; and hence angle $RPX = 51^\circ\,27'\,22'' =$ angle included by force p_1 and the resultant.

Ex. 2. Two forces, represented by 7 and 5, act at an angle of 60°; find their resultant, and the angle it makes with the less force.

Ans. $R = 10\cdot4403065$, and $\phi = 35^\circ\,30'$.

Ex. 3. The resultant of two forces is 24, and the angles it makes with them are 30° and 45°; find the component forces.

Ex. 4. Resolve a given force into two others, such that

(1.) Their sum shall be given, and act at a given angle.

(2.) Their difference shall be given, and act at a given angle.

Ex. 5. If a stream flows at the rate of two miles an hour, find the course which a boat, rowed at the rate of four miles an hour, must pursue, that it may pass directly across the stream.

Ex. 6. Two chords AB, AC of a circle, represent two forces; one of them, AB, is given; find the position of the other, when the resultant is a maximum.

Ex. 7. Three forces represented by 13, 14, 15, acting at a point, keep each other in equilibrium; find the angles which their directions make with each other. Ans. $112^\circ\,38'$, $120^\circ\,30'$, and $126^\circ\,52'$.

Ex. 8. Three forces p_1, p_2, p_3, act upon a given point and keep it at rest; given the magnitude and direction of p_1, the magnitude of p_3, and the direction of p_2, to find the magnitude of p_2, and the direction of p_3.

Ex. 9. A string 15 inches in length is attached at its extremities to two tacks, in the same horizontal line, at the distance of 10 inches from each other; a weight of 12lbs. is suspended between the tacks, by means of a string attached to the first, at the distance of 7 inches from one of its extremities; find the strain upon each tack.

Ex. 10. A cord PABQ passes over two small pulleys A, B, whose distance AB is 6 feet, and two weights of 4 and 3lbs., suspended at the extremities P and Q respectively, support a third weight W of 5lbs.; find the position of the point C to which the weight W is attached, when AB is inclined to the horizon at an angle of 30°.

Ex. 11. P and Q are two equal and given weights suspended by a string passing over three fixed points, A, B, C, given in position; find the actual pressure, and also the horizontal and vertical pressures on each of the three points A, B, C. Also, compare the pressures on A, B, C, when the angles at A, B, C are 150° 90°, 120° respectively.

ON THE MECHANICAL POWERS.

17. WEIGHT and POWER, when opposed to each other, signify the body to be moved, and the body that moves it; or the patient and agent. The power is the agent, which moves, or endeavours to move, the patient or weight.

18. A Machine, or Engine, is any mechanical instrument contrived to move bodies; and it is composed of the mechanical powers.

19. Mechanical Powers are certain simple machines, which are commonly employed for raising greater weights, or overcoming greater resistances, than could be effected by the natural strength without them. These are usually accounted six in number; namely, the Lever, the Pulley, the Wheel and Axle, the Wedge, the Inclined Plane, and the Screw.

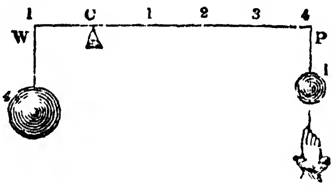
20. Centre of Motion, is the fixed point about which a body moves. And the Axis of Motion, is the fixed line about which it moves.

21. Centre of Gravity, is a certain point, upon which a body being freely suspended, it will rest in any position.

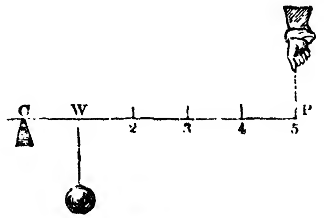
OF THE LEVER.

22. A Lever is any inflexible rod, bar, or beam which serves to raise weights, while it is supported at a point by a fulcrum or prop, which is the centre of motion. The lever is supposed to be void of gravity or weight, to render the demonstrations easier and simpler. There are three kinds of levers.

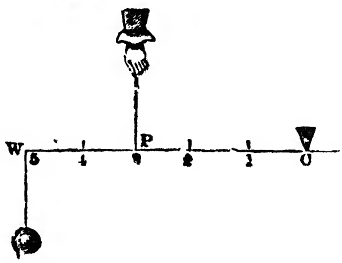
23. A Lever of the First Kind has the prop C between the weight W and the power P. And of this kind are balances, scales, crows, hand-spikes, scissors, pincers, &c.



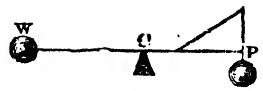
24. A Lever of the Second Kind has the weight between the power and the prop. Such as oars, rudders, cutting knives that are fixed at one end, &c.



25. A Lever of the Third Kind has the power between the weight and the prop. Such as tongs, the bones and muscles of animals, a man rearing a ladder, &c.



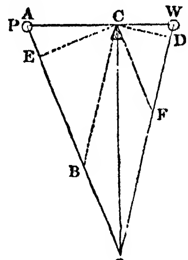
26. A Fourth Kind is sometimes added, called the Bended Lever. As a hammer drawing a nail.



27. In all these machines, the power may be represented by a weight, which is its most natural measure, acting downwards; but having its direction changed, when necessary, by means of a fixed pulley.

PROP. V.

28. *When the Weight and Power keep the Lever in equilibrio, they are to each other reciprocally as the Distances of their Lines of Direction from the Prop. That is,* P : W : : CD : CE; *where* CD *and* CE *are perpendicular to* WO *and* AO, *which are the Directions of the two Weights, or the Weight and Power* W *and* P



For, draw CF parallel to AO, and CB parallel to WO: Also, join CO, which will be the direction of the pressure on the prop C; for there cannot be an equilibrium unless the directions of the three forces all meet in, or tend to, the same point as O. Then, because these three forces keep each other in equilibrio, they are proportional to the sides of the triangle CBO or CFO, which are drawn in the direction of those forces; therefore, - - - P : W :: CF : FO or CB.
But, because of the parallels, the two triangles CDF, CEB are equiangular, therefore - CD : CE :: CF : CB.
Hence, by equality, - P : W :: CD : CE.

That is, each force is reciprocally proportional to the distance of its direction from the fulcrum.

And it will be found that this demonstration will serve for all the other kinds of levers, by drawing the lines as directed.

Corollary. 1. When the two forces act perpendicularly on the lever, as two weights, &c.; then, in case of an equilibrium, D coincides with W, and E with P; consequently then the above proportion becomes P : W :: CW : CP, or the distances of the two forces from the fulcrum, taken on the lever, are reciprocally proportional to those forces.

Corollary. 2. If any force P be applied to a lever at A; its effect on the lever, to turn it about the centre of motion C, is as the length of the lever CA, and the sine of the angle of direction CAE. For the perp. CE is as CA $\times$ sine of angle at A.

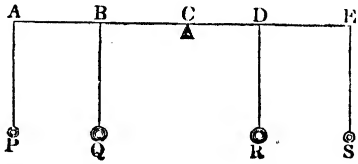
Corollary. 3. Because the product of the extremes is equal to the product of the means, therefore the product of the power by the distance of its direction, is equal to the product of the weight by the distance of its direction.

That is, $P \times CE = W \times CD$.

Corollary. 4. If the lever, with the weight and power fixed to it, be made to move about the centre C; the momentum of the power will be equal to the momentum of the weight; and their velocities will be in reciprocal proportion

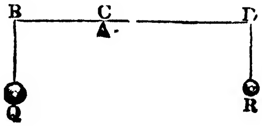
to each other. For the weight and power will describe circles whose radii are the distances CD, CE; and since the circumferences, or spaces described, are as the radii, and also as the velocities, therefore the velocities are as the radii CD, CE; and the momenta, which are as the masses and velocities, are as the momenta and radii; that is, as $P \times CE$ and $W \times CD$, which are equal by corol 3.

Corollary 5. In a straight lever, kept in equilibrio by a weight and power acting perpendicularly; then, of these three, the power, weight, and pressure on the prop, any one is as the distance of the other two.



Corollary 6. If several weights, P, Q, R, S, act on a straight lever, and keep it in equilibrio, then the sum of the products on one side of the prop, will be equal to the sum on the other, made by multiplying each weight by its distance; namely, $P \times AC + Q \times BC = R \times DC + S \times EC$.

For, the effect of each weight to turn the lever, is as the weight multiplied by its distance; and in the case of an equilibrium, the sums of the effects, or of the products on both sides, are equal.

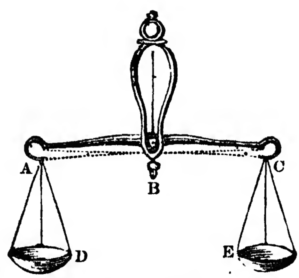


Corollary 7. Because, when two weights Q and R are in equilibrio,

$$Q : R :: CD : CB;$$

therefore, by composition, $Q + R : Q :: BD : CD$,
and, $Q + R : R :: BD : CB$.

That is, the sum of the weights is to either of them, as the sum of their distances is to the distance of the other.



29. Scholium.—Upon the foregoing principles depends the nature of scales and beams, for weighing all sorts of commodities. For, if the weights be equal, then will the distances be equal also, which gives the construction of the common scales, which ought to have these properties:

1*st*, The points of suspension of the scales and the centre of motion of the beam, ABC, must be in a straight line; 2*d*, The arms AB, BC must be of an equal length: 3*d*, That the centre of gravity be in the centre of motion B: 4*th*, That they be in equilibrio when empty: 5*th*, That there be as little friction as possible at the centre B. A defect in any of these properties, makes the scales either imperfect or false. But it often happens that the one side of the beam is made shorter than the other, and the defect covered by making that scale the heavier, by which means the scales hang in equilibro when empty; but when they are charged with any weights, so as to be still in equilibrio, those weights are not equal; but the de-

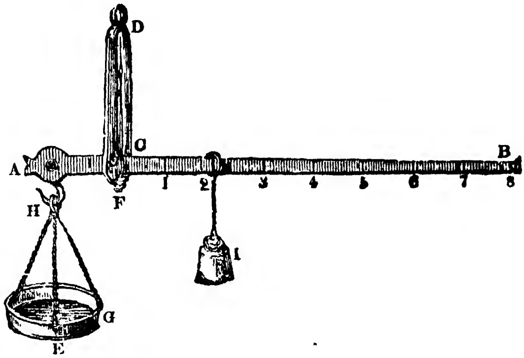
ceit will be shown by changing the weights to the contrary sides, for then the equilibrium will be immediately destroyed.

30. To find the true weight of any body by such a *false* balance :—First, weigh the body in one scale, and afterwards weigh it in the other; then the mean proportional between these two weights, will be the true weight required For, if any body b weigh W pounds or ounces in the scale D, and only w pounds or ounces in the scale E; then we have these two equations;

$$\begin{array}{rl} \text{namely,} & AB \,.\, b = BC \,.\, W, \\ \text{and,} & BC \,.\, b = AB \,.\, w; \\ \text{the product of the two is} & AB \,.\, BC \,.\, b^2 = AB \,.\, BC \,.\, Ww; \\ \text{hence, then} & b^2 = Ww, \\ \text{and} & b = \sqrt{Ww}, \end{array}$$

the mean proportional, which is the true weight of the body b.

31. The Roman Statera, or Steelyard, is also a lever, but of unequal brachia or arms, so contrived that one weight only may serve to weigh a great many, by sliding it backwards and forwards to different distances on the longer arm of the lever; and it is thus constructed:



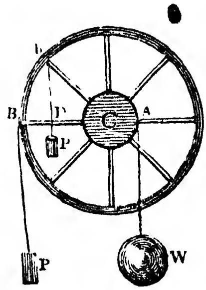
Let AB be the steelyard, and C its centre of motion, from whence the divisions must commence, if the two arms just balance each other: if not, slide the constant moveable weight I along from B towards C, till it just balance the other end without a weight, and there make a notch in the beam, marking it with a cypher 0. Then hang on at A a weight W equal to I, and slide I back towards B till they balance each other; there notch the beam, and mark it with 1. Then make the weight W double of I, and sliding I back to balance it, and there mark it with 2. Do the same at 3, 4, 5, &c., by making W equal to 3, 4, 5, &c. times I; and the beam is finished. Then, to find the weight of any body b by the steelyard; take off the weight W, and hang on the body b at A; then slide the weight I backwards and forwards till it just balance the body b, which suppose to be at the number 5; then is b equal to 5 times the weight of I. So, if I be 1 pound, then b is 5 pounds; but if I be 2 pounds, then b is 10 pounds; and so on.

OF THE WHEEL AND AXLE.

PROP. VI.

32. *In the Wheel and Axle; the Weight and Power will be in equilibrio, when the Power* P *is to the Weight* W, *reciprocally as the Radii of the Circles*

where they act; that is, as the Radius of the Axle CA, *where the Weight hangs, to the Radius of the Wheel* CB, *where the Power acts. That is,* P : W :: CA : CB.

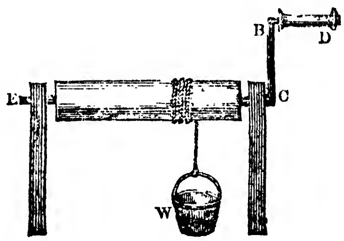


Here the cord, by which the power P acts, goes about the circumference of the wheel, while that of the weight W goes round its axle, or another smaller wheel, attached to the larger, and having the same centre C. So that BA is a lever moveable about the point C, the power P acting always at the distance BC, and the weight W at the distance CA; therefore P : W :: CA : CB.

Corollary. 1. If the wheel be put in motion; then the spaces moved being as the circumferences, or as the radii, the velocity of W will be to the velocity of P, as CA to CB; that is, the weight is moved as much slower, as it is heavier than the power; so that what is gained in power, is lost in time. And this is the universal property of all machines and engines.

Corollary. 2. If the power do not act at right angles to the radius C*b*, but obliquely; draw CD perp. to the direction of the power; then, by the nature of the lever, P : W :: CA : CD.

SCHOLIUM.

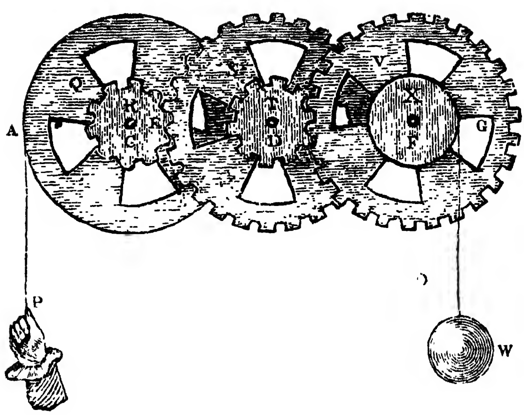


33. To this power belong all turning or wheel machines, of different radii. Thus, in the roller turning on the axis or spindle CE, by the handle CBD; the power applied at B is to the weight W on the roller, as the radius of the roller is to the radius CB of the handle.

34. And the same for all cranes, capstans, windlasses, and such like; the power being to the weight, always as the radius or lever at which the weight acts, to that at which the power acts; so that they are always in the reciprocal ratio of their velocities. And to the same principle may be referred the gimblet and augur for boring holes.

35. But all this, however, is on supposition that the ropes or cords, sustaining the weights, are of no sensible thickness. For, if the thickness be considerable, or if there be several folds of them, over one another, on the roller or barrel; then we must measure to the middle of the outermost rope, for the radius of the roller; or, to the radius of the roller and half the thickness of the cord, when there is but one fold.

36. The wheel-and-axle has a great advantage over the simple lever, in point of convenience. For a weight can be raised only a little way by the lever. But, by the continual turning of the wheel and roller, the weight may be raised to any height, or from any depth.



37. By increasing the number of wheels too, the power may be multiplied to any extent, making always the less wheels to turn greater ones, as far as we please; and this is commonly called Tooth and Pinion, the teeth of one circumference working in the rounds or Pinions of another, to turn the wheel. And then, in case of an equilibrium, the power is to the weight, as the continual product of the radii of all the axles, to that of all the wheels. So, if the power P turn the wheel Q, and this turn the small wheel or axle R, and this turn the wheel S, and this turn the axle T, and this turn the wheel V, and this turn the axle X, which raises the weight W; then

$$P : W :: CB . DE . FG : AC . BD . EF.$$

And in the same proportion is the velocity of W slower than that of P. Thus, if each wheel be to its axle, as 10 to 1; then $P : W :: 1^3 : 10^3$ or as 1 to 1000. So that a power of one pound will balance a weight of 1000 pounds; but then when put in motion, the power will move 1000 times faster than the weight.

38. If ropes are used for the action of the power and weight, we must consider the forces applied to the axes of the ropes. Hence if R, r denote the radii of the wheel and axle, and T, t half the thickness of the ropes, we have

$$P : W :: r+t : R+T.$$

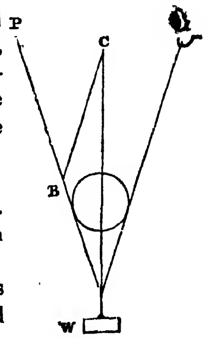
OF THE PULLEY.

39. A Pulley is a small wheel, commonly made of wood or brass, which turns about an iron axis passing through the centre, and fixed in a block, by means of a cord passed round its circumference, which serves to draw up any weight. The pulley is either single, or combined together, to increase the power. It is also either fixed or moveable, according as it is fixed to one place or moves up and down with the weight and power.

When a power sustains a weight by means of a *fixed* pulley, the power and weight are obviously equal; for if through the centre of the pulley a horizontal line be drawn, it will represent a lever of the first kind, whose prop or fulcrum is the fixed centre; hence the points where the power and weight act, are equally distant from the centre, and therefore the power must be equal to the weight. No mechanical advantage, however, is gained by the fixed pulley, though it is still of great utility in the raising of weights, both by changing the direction of the force, and also by enabling several persons to exert their united forces.

PROP. VII.

40 *In the single moveable pulley, and the strings parallel, the power is to the weight as* 1 : 2; *but if the strings produced make an angle* $= 2\phi$; then $P : W :: 1 : 2 \cos\phi$.



Through the centre of the pulley draw the vertical line WAC, and take AC to represent the weight W, where A is the point of intersection of the strings produced. Draw CB parallel to AH; then since the string is equally stretched throughout, we have $AB = BC$ and angle $BAC = \phi$; whence

$$P : W :: AB : AC :: \sin\phi : \sin 2\phi :: 1 : 2\cos\phi;$$

and when the strings are parallel $P:W::1:2$, for $\phi = 0$.

Cor. 1. If $w =$ weight of the moveable block, then $2P = W + w$.

Cor. 2. In the system where there are two blocks of pulleys, the one fixed and the other moveable, and the same rope passing round all the pulleys, then we have simply a combination of the preceding case; and therefore,

$$nP = W + w$$

where $n =$ number of strings at the moveable block and w, its weight. If the strings are not parallel, the cosine of the angle made with the vertical in each case must be introduced, as above.

Cor. 3. In the system where each pulley hangs by a separate string, we have merely a repetition of the single moveable pulley; and the strings being parallel, we get

$$2^n P = W + w_1 + 2w_2 + 2^2 w_3 + \ldots . 2^{n-1} w_n$$

where n is the number of *moveable* pulleys, and w_1, w_2, w_3, the weights of the pulleys including the blocks respectively.

For weight at $w_1 = W + w_1$

$$\therefore \text{ weight at } w_2 = \frac{W + w_1}{2} + w_2 = \frac{W}{2} + \frac{w_1}{2} + w_2$$

$$\ldots\ldots\ldots\ w_3 = \frac{W}{2^2} + \frac{w_1}{2^2} + \frac{w_2}{2} + w_3$$

$$\vdots$$

$$\text{hence weight at } w_n = \frac{W}{2^{n-1}} + \frac{w_1}{2^{n-1}} + \frac{w_2}{2^{n-2}} + \ldots\ldots \frac{w_{n-1}}{2} + w_n$$

$$\therefore\ P = \frac{1}{2} \text{ weight at } w_n$$

$$= \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \ldots\ldots \frac{w_{n-1}}{2^2} + \frac{w_n}{2}$$

$$\therefore\ 2^n P = W + w_1 + 2w_2 + 2^2 w_3 + \ldots\ldots 2^{n-1} w_n$$

When $w_1 = w_2 = w_3 = \&c.$; then

$$2^n P = W + w_1 (2^n - 1)$$

PROP. VIII.

41. *In the system of pulleys, where each string is attached to the weight, and the strings parallel, we have* $P : W :: 1 : 2^n - 1$, *where* n *is the number of*

Let w_1, w_2, w_3, &c. be the weights of the pulleys, and let the strings passing over the pulleys w_1, w_2, w_3, &c. be attached to the weight at the points p_1, p_2, p_3, &c.; then we have

tension of string at p_1 = weight supported at $p_1 =$ P
.... p_2 = ... $p_2 = 2P + w_1$
.... p_3 = ... $p_3 = 2^2P + 2w_1 + w_2$

and so on. Hence, if n be the number of pulleys, the whole weight supported is

$$\begin{aligned} W = &(1 + 2 + 2^2 + 2^3 + \ldots\ldots\ldots\ldots 2^{n-1}) P \\ &+ (1 + 2 + 2^2 + 2^3 + \ldots\ldots\ldots\ldots 2^{n-2}) w_1 \\ &+ (1 + 2 + 2^2 + 2^3 + \ldots\ldots\ldots\ldots 2^{n-3}) w_2 \\ &\quad\vdots \\ &+ (1 + 2) w_{n-2} \\ &+ w_{n-1} \\ = &(2^n-1) P + (2^{n-1}-1)w_1 + (2^{n-2}-1) w_2 + \ldots w_{n-1}. \end{aligned}$$

Cor. If the weights of the pulleys be neglected, we have $W = (2^n - 1) P$; hence it is manifest that the weights of the pulleys increase the weight supported, and the advantage is therefore on the side of the power.

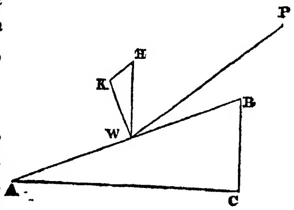
ON THE INCLINED PLANE.

42. THE *inclined plane* assists by its reaction in sustaining a heavy body.

PROP. IX.

43. *Let a weight* W *be supported on the inclined plane* AB, *by a power* P *acting in the direction* WP; *and let angle* BAC$=\alpha$, *and angle* BWP$=\beta$; *then*

$$P : W :: \sin\alpha : \cos\beta.$$



Draw WH perpendicular to the horizon, WK perpendicular to the plane AB, and HK parallel to WP; then the weight W is kept at rest by three forces, viz. the power P in direction HK, gravity in direction WH, and the reaction of the plane AB in direction WK; hence if WH be taken to represent the weight, we have

$$P : W :: HK : HW :: \sin KWH : \sin HKW$$
$$:: \sin BAC : \sin KWP :: \sin\alpha : \cos\beta;$$

because sin KWP = cos PWB, since BWK is a right angle.

Cor. 1. If p represent the pressure on the plane; then we have

$$P : p :: HK : KW :: \sin\alpha : \sin HWP :: \sin\alpha : \sin\left\{\frac{\pi}{2} - (\alpha+\beta)\right\}$$
$$:: \sin\alpha : \cos(\alpha+\beta).$$

Hence $W : P : p :: \cos\beta : \sin\alpha : \cos(\alpha+\beta)$

$$\text{or } \frac{W}{P} = \frac{\cos\beta}{\sin\alpha};\ \frac{W}{p} = \frac{\cos\beta}{\cos(\alpha+\beta)};\ \frac{P}{p} = \frac{\sin\alpha}{\cos(\alpha+\beta)}$$

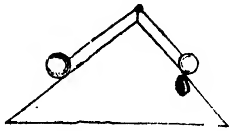
Cor. 2. When WP is parallel to the plane, $\beta = 0$; hence we have

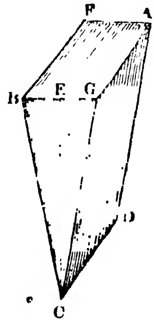
$$\frac{W}{P} = \frac{AB}{BC};\ \frac{W}{p} = \frac{AB}{AC};\ \frac{P}{p} = \frac{BC}{AC}$$

$$\text{or } W = P \operatorname{cosec} A = p \sec A.$$

Cor. 3. The power or relative weight that urges a body W down the inclined plane, is $= \frac{BC}{AB} \times W$, or the force with which it descends, or endeavours to descend, is as the sine of the angle A of inclination.

Cor. 4. Hence, if there be two planes of the same height, and two bodies be laid upon them proportional to the lengths of the planes, they will have equal tendencies to descend down the planes; and, consequently, they will mutually sustain each other if they be connected by a string acting parallel to the planes.





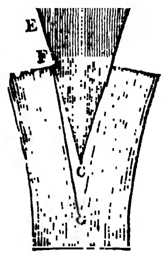
OF THE WEDGE.

44. The Wedge is a body of wood or metal, in form of a prism. AF or BG is the breadth of its back; CE its height; GC, BC its sides, and its end GBC is composed of two equal inclined planes, GCE, BCE.

PROP. X.

45. *When a wedge is in equilibrio; the power acting against the back, is to the force acting perpendicularly against either side, as the breadth of the back* AB, *is to the length of the side* AC *or* BC.

For, any three forces, which sustain one another in equilibrio, are as the corresponding sides of a triangle drawn perpendicular to the directions in which they act. But AB is perpendicular to the force acting on the back, to urge the wedge forward; and the sides AC, BC are perpendicular to the forces acting upon them; therefore the three forces are as AB, AC, BC.



Corollary. The force on the back,
Its effect in direct. perp. to AC,
And its effect parallel to AB,
Are as the three lines,

AB,
AC,
DC,
which are perp. to them.

And therefore the thinner a wedge is, the greater is its effect, in splitting any body, or in overcoming any resistance against the sides of the wedge.

46. Scholium.—But it must be observed, that the resistance, or the forces above mentioned, respect one side of the wedge only. For if those against both sides be taken in, then, in the foregoing proportions, we must take only half the back AD, or else we must take double the lines AC and DC. In the wedge, the friction against the sides is very great, at least equal to the force to be overcome, because the wedge retains any position to which it is driven; and therefore the resistance is doubled by the friction. But then the wedge has a great advantage over all the other powers, arising from the force of percussion or blow with which the back is struck, which is a force incomparably greater than any dead weight or pressure, such as is employed in other machines. And accordingly, we find it produces effects vastly superior to those of any other power; such as

the splitting and raising the largest and hardest rocks, the raising and lifting the largest ship, by driving a wedge below it, which a man can do by the blow of a mallet; and thus it appears that the small blow of a hammer, on the back of a wedge, is incomparably greater than any mere pressure, and will overcome it.

OF THE SCREW.

47. The Screw is one of the six mechanical powers, chiefly used in pressing or squeezing bodies close, though sometimes also in raising weights.

The screw is a spiral thread or groove cut round a cylinder, and everywhere making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, were unfolded and stretched into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder is to the distance between two threads of the screw; as is evident by considering, that, in making one round, the spiral rises along the cylinder the distance between the two threads.

PROP. XI.

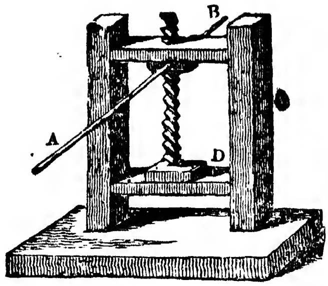
48. *The force of a power applied to turn a Screw round, is to the force with which it presses upwards or downwards, setting aside the friction, as the distance between two threads is to the circumference where the power is applied.*

The screw being an inclined plane, or half wedge, whose height is the distance between two threads, and its base the said circumference; and the force in the horizontal direction, being to that in the vertical one, as the lines perpendicular to them, namely, as the height of the plane, or distance of the two threads, is to the base of the plane, or circumference at the place where the power is applied; therefore the power is to the pressure, as the distance of two threads is to that circumference.

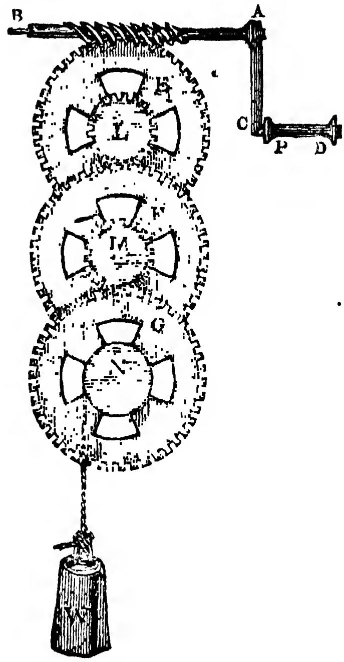
Corollary. When the screw is put in motion; then the power is to the weight which would keep it in equilibrio, as the velocity of the latter is to that of the former; and hence their two momenta are equal, which are produced by multiplying each weight or power by its own velocity. So that this is a general property in all the mechanical powers, namely, that the momentum of a power is equal to that of the weight which would balance it in equilibrio: or that each of them is reciprocally proportional to its velocity.

49. Scholium.—Hence we can easily compute the force of any machine turned by a screw. Let the annexed figure represent a press driven by a screw, whose threads are each a quarter of an inch asunder; and that the screw is turned by a handle of 4 feet long from A to B; then, if the natural force of a man, by which he can lift, pull, or draw, be 150 pounds; and it be re-

quired to determine with what force the screw will press on the board at D, when the man turns the handle at A and B with his whole force. The diameter AB of the power being 4 feet or 48 inches, its circumference is $48 \times 3{\cdot}1416$ or $150\frac{4}{5}$ nearly; and the distance of the threads being $\frac{1}{4}$ of an inch; therefore the power is to the pressure, as 1 to $603\frac{1}{5}$: but the power is equal to 150 lb.; therefore as $1 : 603\frac{1}{5} :: 150 : 90{,}480$; and consequently the pressure at D is equal to a weight of 90,480 pounds, independent of friction.



50. Again, if the endless screw AB be turned by a handle AC of 20 inches, the threads of the screw being distant half an inch each; and the screw turn a toothed wheel E, whose pinion L turns another wheel F, and the pinion M of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight the man will be able to raise, working at the handle C; supposing the diameters of the wheels to be 18 inches, and those of the pinions and barrel 2 inches; the teeth and pinions being all of a size.



Here $20 \times 3{\cdot}1416 \times 2 = 125{\cdot}664$, is the circumference of the power.

And $125{\cdot}664$ to $\frac{1}{2}$, or $251{\cdot}328$ to 1, is the force of the screw alone.

Also, 18 to 2, or 9 to 1 being the proportion of the wheels to the pinions; and as there are three of them, therefore 9^3 to 1, or 729 to 1 is the power gained by the wheels.

Consequently $251{\cdot}328 \times 729$ to 1, or $183218\frac{1}{9}$ to 1 nearly, is the ratio of the power to the weight, arising from the advantage of both the screw and the wheels.

But the power is 150 pounds; and therefore $150 \times 183218\frac{1}{9}$, or 27482716 pounds, is the weight the man can sustain, which is equal to 12269 tons weight.

But the power has to overcome, not only the weight, but also the friction of the screw, which is very great, in some cases equal to the weight itself, since it is sometimes sufficient to sustain the weight, when the power is taken off.

EXAMPLES ON THE PRINCIPLES OF THE MECHANICAL POWERS.

ON THE LEVER.

51. *Ex.* 1. The arms of a bent lever are to each other as 4 to 5, and are inclined at an angle of 135°. The lever rests upon a fulcrum at its angular point, and weights are suspended from the extremities of the two arms, such that the shorter arm rests in a horizontal position; what is the ratio of the weights? Ans. $8 : 5\sqrt{2}$ or $1 : \cdot 8838835$.

Ex. 2. The difference of the lengths of the arms of a lever is (a) inches; the same weight weighs (w) pounds at one end, and (w) ounces at the other; find the lengths of the arms. Ans. $\frac{a}{3}$ and $\frac{4a}{3}$.

Ex. 3. A lever three feet in length weighs 6lb.; what weight on the shorter arm will balance 12lb. on the longer, the fulcrum being one foot from the end? Ans. 27lb.

Ex. 4. The compound lever DK is composed of three levers of the first kind, DA, AB, BK, acting upon one another. The arms DC, CA of the first lever are respectively 8 and 6 inches; those of the second, AO, OB, are 12 and 2, and those of the third, BH, HK are 16 and 3; find the ratio of P, the power at D, to W, the weight suspended at K.

Ans. P : W : : 3 : 128.

Ex. 5. Suppose AB is a squared beam, or lever of oak, 30 feet long, each end being one foot square; what weight W at the end A would keep it in a horizontal position on a fulcrum C, 3 feet from that end, each cubic foot of the beam weighing 54lb.? Ans. 6480lb.

Ex. 6. AB is a uniform straight lever, 20 feet in length, and weighing 40lb.; and HBK, a flexible chain of the same length, and weight 130lb., is laid upon the lever in such a manner that it is kept in equilibrium on a fulcrum C, which is five feet from the end B; how much of the chain overhangs the end B? Ans. $20 - \frac{30}{13}\sqrt{26}$, or $8 \cdot 233032$ feet.

ON THE WHEEL AND AXLE.

52. *Ex.* 1. In a combination of four wheels and axles, each of the radii of the wheels is to each of the radii of the axles as 5 to 1; what power will balance a weight of 1875 pounds? Ans. 3 pounds.

Ex. 2. A power of 6lb. keeps in equilibrium a weight of 240lb., by means of a wheel and axle: the diameter of the axle is 6 inches; what is the radius of the wheel? Ans. 10 feet.

Ex. 3. In a combination of wheels and pinions, the circumference of each pinion is applied to the circumference of the next wheel, and the ratios of the radii of the wheels and pinions are $2:1$, $2^2:1$; $2^3:1$, and so on. Find the number of wheels, when the power is to the weight as $1:n$.

Ans. The number of wheels may be found from the quadratic equation

$$x^2 + x = \frac{2 \log n}{\log 2}, \text{ where } x = \text{number of wheels.}$$

On the Pulley.

53. *Ex.* 1. What power will sustain 40 pounds over five moveable pulleys?

Ans. $1\frac{1}{4}$lb.

Ex. 2. In a system of pulleys, where each pulley has a separate string passing over it, and fastened to the weight, $P : W :: 1 : 63$; what is the number of moveable pulleys? Ans. 5.

Ex. 3. In the same system, the number of *moveable* pulleys is 3, and the weight of each pulley 2lb.; what weight will a power of 60lb. support?

Ans. 922lb.

On the Inclined Plane.

54. *Ex.* 1. A power of 1lb. acting parallel to a plane supports a weight of 2lb.; what is the inclination of the plane? Ans. 30°.

Ex. 2. Two weights are fastened to the ends of a thread which moves freely over a pulley, and the thread makes angles $\frac{1}{3}\pi$ and β with the horizon when at rest; also one of the weights which is on a smooth plane is double of the other which hangs vertically; what is the inclination of the plane?

Ans. $\cot \phi = 2 \sec \beta - \tan \beta$, where ϕ = angle of inclination.

Ex. 3. A weight of 40 pounds acting parallel to the length, sustains another of 56 pounds on an inclined plane whose base is 340 feet; find the height and length of the plane.

Ans. Height $= \frac{425}{3}\sqrt{6}$ feet, and length $= \frac{595}{3}\sqrt{6}$ feet.

Of the Screw.

55. *Ex.* 1. The distance between two contiguous threads of a screw is 2 inches, and the arm to which P is applied is 20 inches; find the ratio of P to W when there is an equilibrium. Ans. $P : W :: 1 : 62{\cdot}832$.

Ex. 2. What must be the distance between the threads in a screw, that a man exerting a force of 50lb. at the end of an arm 18 inches in length, may press with a force of ten tons? Ans. ·25245 inches, or $\frac{1}{4}$in., nearly.

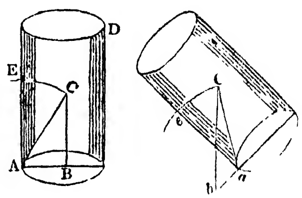
OF THE CENTRE OF GRAVITY.

56. THE Centre of Gravity of a body, is a certain point within it, upon which the body being freely suspended, it will rest in any position; and it will descend to the lowest place to which it can get, in other positions.

PROP. XII.

57. *If a perpendicular to the horizon, from the centre of gravity of any body, fall within the base of the body, it will rest in that position; but if the perpendicular fall without the base, the body will not rest in that position, but will tumble down.*

FOR, if CB be the perp. from the centre of gravity C, within the base: then the body cannot fall over towards A; because, in turning on the point A, the centre of gravity C would describe an arc which would rise from C to E; contrary to the nature of that centre, which only rests when in the lowest place. For the same reason, the body will not fall towards D. And therefore will stand in that position.

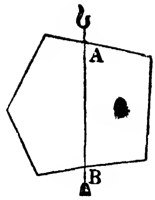


But if the perp. fall without the base, as Cb; then the body will tumble over on that side; because, in turning on the point a, the centre C descends by describing the centre arc Cc.

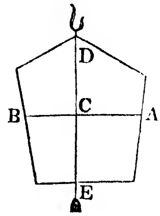
Corollary. 1. If a perpendicular, drawn from the centre of gravity, fall just on the extremity of the base, the body may stand; but any the least force will cause it to fall that way. And the nearer the perpendicular is to any side, or the narrower the base is, the easier it will be made to fall, or be pushed over that way; because the centre of gravity has the less height to rise: which is the reason that a globe is made to roll on a smooth plane by any the least force. But the nearer the perpendicular is to the middle of the base, or the broader the base is, the firmer it stands.

Corollary. 2. Hence, if the centre of gravity of a body be supported, the whole body is supported. And the place of the centre of gravity must be accounted the place of the body; for into that point the whole matter of the body may be supposed to be collected, and therefore all the force with which it endeavours to descend.

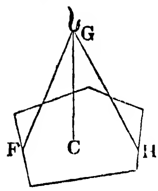
Corollary 3. From the property which the centre of gravity has, of always descending to the lowest point, is derived an easy mechanical method of finding that centre.



For if the body be hung up by any point A, and a plumb line AB be hung by the same point, it will pass through the centre of gravity; because that centre is not in the lowest point till it fall in the plumb line. Mark the line AB upon it. Then hang the body up by any other point D, with a plumb line DE, which will also pass through the centre of gravity, for the same reason as before; and therefore that centre must be at C where the two plumb lines cross each other.



Or, if the body be suspended by two or more cords, GF, GH, &c., then a plumb line from the point G will cut the body in its centre of gravity C.

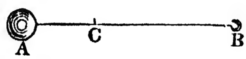


58. Likewise, because a body rests when its centre of gravity is supported, but not else; we hence derive another easy method of finding that centre mechanically. For, if the body be laid on the edge of a prism, and moved backwards and forwards till it rest, or balance itself; then is the centre of gravity just over the line of the edge. And if the body be then shifted into another position, and balanced on the edge again, this line will also pass by the centre of gravity, and consequently the intersection of the two will give the centre itself.

PROP. XIII.

59. *The common centre of gravity* C *of any two bodies* A, B, *divides the line joining their centres, into two parts, which are reciprocally as the bodies.*

That is, AC : BC :: B : A.



For, if the centre of gravity C be supported, the two bodies A and B will be supported, and will rest in equilibrio. But, by the nature of the lever, when two bodies are in equilibrio about a fixed point C, they are reciprocally as their distances from that point; therefore A : B :: CB : CA.

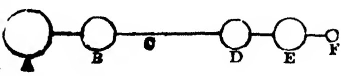
Corollary 1. Hence AB : AC :: A + B : B; or, the whole distance between the two bodies, is to the distance of either of them from the common centre, as the sum of the bodies is to the other body.

Corollary 2. Hence also, CA . A = CB . B; or, the two products are equal, which are made by multiplying each body by its distance from the centre of gravity.

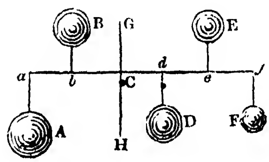
Corollary 3. As the centre C is pressed with a force equal to both the weights A and B, while the points A and B are each pressed with the respec-

tive weights A and B. Therefore, if the two bodies be both united in their common centre C, and only the ends A and B of the line AB be supported, each will still bear, or be pressed by the same weights A and B as before. So that, if a weight of 100 lb. be laid on a bar at C, supported by two men at A and B, distant from C, the one four feet, and the other 6 feet; then the nearer will bear the weight of 60 lb., and the farther only 40 lb. weight.

Corollary 4. Since the effect of any body to turn a lever about the fixed point C, is as that body and its distance from that point; therefore, if C be the common centre of gravity of all the bodies A, B, D, E, F, placed in the straight line AF; then is CA . A + CB . B = CD . D + CE . E + CF . F; or, the sum of the products on one side, equal to the sum of the products on the other, made by multiplying each body by its distance from that centre. And if several bodies be in equilibrium upon any straight lever, then the prop is in the centre of gravity.



Corollary 5. And although the bodies be not situated in a straight line, but scattered about in any promiscuous manner, the same property as in the last corollary still holds true, if perpendiculars to any line whatever *af* be drawn through the several bodies and their common centre of gravity, namely, that C*a* . A + C*b* . B = C*d* . D + C*e* . E + C*f* . F. For the bodies have the same effect on the line *af*, to turn it about the point C, whether they are placed at the points *a*, *b*, *d*, *e*, *f*, or in any part of the perpendiculars A*a*, B*b*, D*d*, E*e*, F*f*.

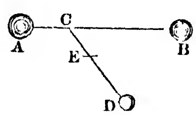


PROP. XIV.

60. *If there be three or more bodies, and, if a line be drawn from any one body* D *to the centre of gravity of the rest* C; *then the common centre of gravity* E *of all the bodies, divides the line* CD *into two parts in* E, *which are reciprocally proportional as the body* D *to the sum of all the other bodies.*

That is, CE : ED :: D : A + B, &c.

For, suppose the bodies A and B to be collected into their common centre of gravity C, and let their sum be called S. Then, by the last prop., CE : ED :: D : S or A + B, &c.



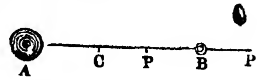
Corollary Hence we have a method of finding the common centre of gravity of any number of bodies; namely, by first finding the centre of any two of them, then the centre of that centre and a third, and so on for a fourth, or fifth, &c.

61. *If there be taken any point* P, *in the line passing through the centres of two bodies; then the sum of the two products of each body, multiplied by its*

distance from that point, is equal to the product of the sum of the bodies multiplied by the distance of their common centre of gravity C *from the same point* P.

That is, $PA . A + PB . B = PC . \overline{A + B}$.

For, by the 38th, $CA . A = CB . B$, that is, $\overline{PA - PC} . A = \overline{PC - PB} . B$, therefore, by adding



$$PA . A + PB . B = PC . \overline{A + B}.$$

Corollary. 1. Hence, the two bodies A and B have the same force to turn the lever about the point P, as if they were both placed in C, their common centre of gravity.

Or, if the line, with the bodies, move about the point P; the sum of the momenta of A and B, is equal to the momentum of the sum S or A + B placed at the centre C.

Corollary. 2. The same is also true of any number of bodies whatever, as will appear by cor. 4, prop. 38, namely, $PA . A + PB . B + PD . D$, &c $= PC . \overline{A + B + D}$, &c., where P is any point whatever in the line AC.

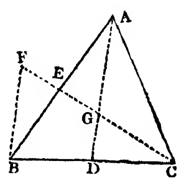
And, by cor. 5, prop. 38, the same thing is true when the bodies are not placed in that line, but any where in the perpendiculars passing through the points A, B, D, &c.; namely, $Pa . A + Pb . B + Pd . D$, &c., $= PC \times \overline{A + B + D}$, &c

Corollary. 3. And if a plane pass through the point P perpendicular to the line CP; then the distance of the common centre of gravity from that plane, is $PC = \frac{Pa . A + Pb . B + Pd . D, \&c.}{A + B + D, \&c.}$ that is, equal to the sum of all the forces divided by the sum of all the bodies. Or, if A, B, D, &c., be the several particles of one mass or compound body; then the distance of the centre of gravity of the body, below any given point P, is equal to the forces of all the particles divided by the whole mass or body, that is, equal to all the $Pa . A$, $Pb . B$, $Pd . D$, &c. divided by the body or sum of the particles A, B, D, &c.

PROP. XVI.

62. *To find the centre of gravity of a triangle.*

From any two of the angles draw lines AD, CE, to bisect the opposite sides; so will their intersection G be the centre of gravity of the triangle.



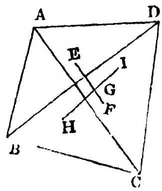
For, because AD bisects BC, it bisects also all its parallels, namely, all the parallel sections of the figure; therefore AD passes through the centres of gravity of all the parallel sections or component parts of the figure; and consequently the centre of gravity of the whole figure lies in the line AD. For the same reason, it lies also in the line CE. And consequently it is in their common point of intersection G.

Corollary. The distance of the point G, is $AG = \frac{2}{3}AD$, and $CG = \frac{2}{3}CE$; or $AG = 2GD$, and $CG = 2GE$.

For, draw BF parallel to AD, and produce CE to meet it in F. Then the triangles AEG, BEF are similar, and also equal, because $AE = BE$; consequently $AG = BF$. But the triangles CDG, CBF are also equiangular, and CB being $= 2CD$, therefore $BF = 2GD$. But BF is also $= AG$; consequently $AG = 2GD$ or $\frac{2}{3}AD$. In like manner, $CG = 2GE$ or $\frac{2}{3}CE$.

PROP. XVII.

63. *To find the centre of gravity of a trapezium.*



Divide the trapezium ABCD into two triangles, by the diagonal BD, and find E, F, the centres of gravity of these two triangles; then shall the centre of gravity of the trapezium lie in the line EF connecting them. And therefore if EF be divided, in G, in the alternate ratio of the two triangles, namely, EG : GF :: triangle BCD : triangle ABD, then G will be the centre of gravity of the trapezium.

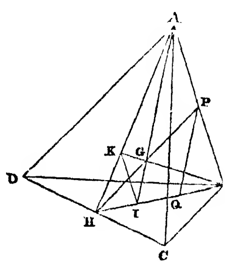
64. Or, having found the two points E, F, if the trapezium be divided into two other triangles BAC, DAC, by the other diagonal AC, and the centres of gravity H and I of these two triangles be also found; then the centre of gravity of the trapezium will also lie in the line HI.

So that, lying in both the lines, EF, HI, it must necessarily lie in their intersection G.

65. And thus we are to proceed for a figure of any greater number of sides, finding the centres of their component triangles and trapeziums, and then finding the common centre of every two of these, till they be all reduced into one only.

PROP. XVIII.

66. *To find the centre of gravity of a triangular pyramid.*



Let ABCD be a triangular pyramid, and to the point of bisection of DC draw AH, BH. Take $HK = \frac{1}{3}HA$ and $HI = \frac{1}{3}HB$; then K and I will be the centres of gravity of the surfaces ACD and BCD respectively. Join KI, AI, BK. Now if the pyramid be resolved into elements, by means of planes parallel to BCD, it is evident that the line AI must pass through the centre of gravity of the pyramid, since I is the centre of gravity of BCD. For the same reason, the centre of gravity of the pyramid is in the line BK, and because AI and BK are in one plane, the centre of gravity of the pyramid ABCD must be at G, the point of intersection of AI and BK.

By similar triangles AGB and KGI, we have

$$AG : GI :: AB : KI :: AH : KH :: 3 : 1$$

$$\therefore AG : AI :: 3 : 4 \text{ or } AG = \tfrac{3}{4}AI.$$

Cor. 1. Bisect AB in P and join HG and GP; then if PQ be drawn parallel to AI, we have BQ=QI=IH; but AI=2PQ and AI=4GI; hence PQ=2GI; and therefore HI : HQ : : IG : PQ; whence HGP is a straight line.

Cor. 2. Hence the centre of gravity of a triangular pyramid is the middle of the line joining the points of bisection of any two edges that do not meet.

Cor. 3. A solid bounded by plane surfaces may be divided by planes into a number of triangular pyramids; and if a plane be drawn parallel to the base, at a distance equal to $\frac{1}{4}$ of the altitude of the pyramid, then the centre of gravity of the whole pyramid must be in this plane, for that of each of the triangular pyramids is in this plane. Hence, the line joining the vertex of the pyramid and the centre of gravity of its base will cut the plane in the centre of gravity of the whole pyramid.

PROP. XIX.

67. *To find the centre of gravity of any body, or system of bodies.*

Let v_1, v_2, v_3, &c., denote the volumes of the material particles which compose the body or volume V; and x_1, y_1, z_1; x_2, y_2, z_2, &c.; their co-ordinates in reference to three rectangular axes; then if X, Y, Z denote the co-ordinates of the centre of gravity, we have (by Prop. XV., Cor. 3.)

$$X = \frac{v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 + \ldots}{V}$$

$$Y = \frac{v_1y_1 + v_2y_2 + v_3y_3 + v_4y_4 + \ldots}{V}$$

$$Z = \frac{v_1z_1 + v_2z_2 + v_3z_3 + v_4z_4 + \ldots}{V}$$

But to adapt these expressions to computation, we shall introduce the principles of the Differential and Integral Calculus, and then the preceding expressions will take the form

$$X = \frac{\int xdv}{\int dv}; \qquad Y = \frac{\int ydv}{\int dv}; \qquad Z = \frac{\int zdv}{\int dv},$$

where x, y, z denote the distances of the centre of gravity of dv from the three rectangular planes.

By means of these three equations, the determination of the centre of gravity may be effected; and when the figure is a plane surface, two of these equations are only required, since the centre of gravity is in the plane.

I. When the figure is a plane curve.

Here dv = differential of the arc $= \sqrt{dx^2 + dy^2} = ds$

$$\therefore\ X = \frac{\int xds}{s}; \qquad Y = \frac{\int yds}{s}$$

and if the arc be symmetrical on each side of the axis of x, we have $y = 0$, and then we have simply

$$X = \frac{\int xds}{s}$$

II. When the figure is a plane surface.

Here dv = differential of the area $= ydx$; hence

$$X = \frac{\int xydx}{\int ydx}; \qquad Y = \frac{\int y^2dx}{\int ydx}$$

And if the area is symmetrical on each side of the axis of x, we need only one equation, viz.

$$X = \frac{\int yxdx}{\int ydx}$$

III. For a surface of revolution round the axis of x.

Here one equation only is necessary; and since $dv =$ differential of the surface $= 2\pi yds$, we have

$$X = \frac{\int xyds}{\int yds}.$$

IV. For a solid of revolution round the axis of x.

Here $dv =$ differential of the volume $= \pi y^2 dx$, and hence

$$X = \frac{\int y^2 xdx}{\int y^2 dx}.$$

We shall now apply these formulæ to a few examples.

EXAMPLES.

68. *Ex.* 1. To find the centre of gravity of a circular arc.

Here the curve is symmetrical on each side of the axis of x, and the equation is $y^2 = 2ax - x^2$; hence we have

$$X = \frac{\int xds}{s} = \frac{\int x\sqrt{dy^2 + dx^2}}{s}$$

$$= \frac{a}{s} \cdot \int \frac{xdx}{\sqrt{2ax - x^2}} = \frac{a}{s}\left(a \text{ vers}^{-1}\frac{x}{a} - \sqrt{2ax - x^2}\right)$$

$$= \frac{a}{s}(s - y) = a - \frac{ay}{s}$$

that is, the distance of the centre of gravity of a circular arc from the vertex is $= a - \frac{ay}{s}$, and therefore the distance of the centre of gravity from the centre $= \frac{ay}{s}$.

Ex. 2. To find the centre of gravity of a cone.

Let x_1, x_2 represent any two parts of the axis of the cone, measuring from the vertex, and y_1, y_2 the radii of the circular sections of the cone corresponding to the altitudes x_1, x_2; then, if x denote any variable part of the axis, and y the radius of the corresponding circular section, we have

$x : x_1 : : y : y_1 \quad \therefore y^2 = x^2 . \frac{y_1^2}{x_1^2}$; where x_1, y_1 are constants; hence

$$X = \frac{\int y^2 xdx}{y^2 dx} = \frac{y_1^2}{x_1^2}\int x^3 dx \div \frac{y_1^2}{x_1^2}\int x^2 dx$$

$$= \frac{\int x^3 dx}{\int x^2 dx} = \frac{\frac{1}{4}x^4}{\frac{1}{3}x^3} = \frac{3}{4} \cdot \frac{x^4}{x^3} = \frac{3}{4}x = \text{distance from vertex.}$$

But integrating between the limits x_1 and x_2, we have

$$X = \frac{\frac{1}{4}(x_1^4 - x_2^4)}{\frac{1}{3}(x_1^3 - x_2^3)} = \frac{3}{4} \cdot \frac{(x_1^2 + x_2^2)(x_1^2 - x_2^2)}{(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2)}$$

$$= \frac{3}{4} \cdot \frac{(x_1^2 + x_2^2)(x_1 + x_2)}{x_1^2 + x_1x_2 + x_2^2} = \frac{3}{4} \cdot \frac{x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3}{x_1^2 + x_1x_2 + x_2^2}.$$

$$\therefore x_1 - X = x_1 - \frac{3}{4} \cdot \frac{x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3}{x_1^2 + x_1x_2 + x_2^2}$$

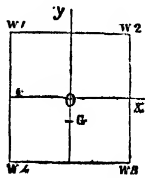
$$= \frac{(x_1 - x_2)}{4} \cdot \frac{x_1^2 + 2x_1x_2 + 3x_2^2}{x_1^2 + x_1x_2 + x_2^2}.$$

This expression gives the distance of the centre of gravity of the frustum of a cone from the greater end; hence, if R, r represent the radii of the greater and less ends of the frustum, and h its altitude, we have the distance of the centre of gravity $= \frac{h}{4} \cdot \frac{R^2 + 2Rr + 3r^2}{R^2 + Rr + r^2}$; and when $r = 0$, we have the distance of the centre of gravity of a cone from its base $= \frac{h}{4} =$ one-fourth of the altitude as found above.

Ex. 3. Four bodies, whose weights are w_1, w_2, w_3, w_4 pounds, are placed at the successive angles of a square whose side is $2a$ inches; required the position of their common centre of gravity, the square being considered without weight.

Take O, the centre of the square, as the origin of co-ordinate reference, and the two rectangular axes parallel and perpendicular to the sides; then we have

$$X = \frac{(w_2 + w_3 - w_1 - w_1)}{w_1 + w_2 + w_3 + w_4} a, \text{ and } y = \frac{w_1 + w_2 - w_3 - w_4}{w_1 + w_2 + w_3 + w_4} a.$$



Thus if $w_1 = 3$, $w_2 = 4$, $w_3 = 5$, $w_4 = 6$; and $2a = 12$ inches;

$\therefore X = 0$ and $y = -\frac{4}{18} \cdot 6 = -1\frac{1}{3} = OG =$ distance of centre of gravity below O on the axis of y.

Problems for Exercise.

69. *Ex.* 1. Find the centres of gravity of

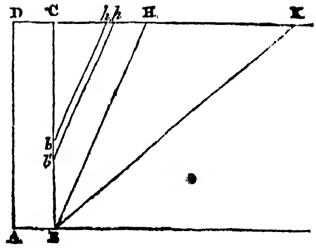
(1.) The common parabola and the paraboloid.
(2.) A semicircle, and the segment of a circle.
(3.) A hemispheroid, and a hemisphere.
(4.) The sector of a circle, and a spheric sector.
(5.) The surface of a spheric segment, and that of a cone.

Ex. 2. Two cones are placed with their equal bases in contact, and the altitude of the one is three times that of the other; find the position of their common centre of gravity.

Ex. 3. The surface generated by a plane line or curve revolving about an axis in the plane of the figure, is equal to the product of the generating line or curve, and the path described by its centre of gravity.

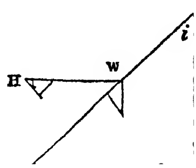
Ex. 4. The volume of the solid generated by the revolution of a plane figure about an axis in the plane of the figure, is equal to the product of the generating surface, and the path described by its centre of gravity.

Ex. 5. From a given rectangle ABCD of uniform thickness, to cut off a triangle CDO, so that the remainder, ABCO when suspended at O, shall hang with AB in a vertical position.



70. *To determine the horizontal thrust of the terrace, whose vertical section is* BCHK, *against the wall whose section is* ABCD, *and the momentum of the thrust to overturn the wall about the angle* A.

If it be required to support a terrace by a vertical wall, it must be constructed so as to counteract the horizontal thrust of the prismatic mass of earth which lies above the surface of a bank that would be itself supported. But this prismatic mass is partly supported by friction, and we must therefore ascertain how much of the horizontal thrust is counteracted by friction.



Suppose a weight W to be placed on a plane, inclined to the vertical at an angle i; and let H be the horizontal force, which, with the friction, just sustains the weight W. Resolve each of the forces W, H into two others, the one parallel and the other perpendicular to the plane; and those parallel to the plane act in opposite directions, while those perpendicular to the plane concur in direction; hence we have

$$\text{force parallel to the plane} = W\cos i - H\sin i$$
$$\text{force perpendicular to the plane} = W\sin i + H\cos i.$$

And the first of these forces must be precisely equal to the friction; that is, equal to a force that will just support the weight upon the plane; hence

$$W\cos i - H\sin i = fW\sin i + fH\cos i$$

$$\therefore\ H = \frac{\cos i - f\sin i}{\sin i + f\cos i}W = \frac{1 - f\tan i}{\tan i + f}.W$$

If then the weight W were sustained by a wall, the horizontal thrust of the weight W against the wall would be $\dfrac{1 - f\tan i}{\tan i + f}.W.$

Now to apply this to the investigation of the horizontal thrust of the prism BCH, we shall put $BC = a$, a variable part $Cb = x$, $bb' = dx$, and s the specific gravity of the earth. Then the area of $bb'hh' = xdx\tan i$, and its weight $= sxdx\tan i$; hence the horizontal thrust against bb' will be

$$\frac{1 - f\tan i}{\tan i + f}.sxdx\tan i = \frac{1 - f\tan i}{1 + f\cot i}.sxdx = sMxdx$$

where $M = \dfrac{1 - f\tan i}{1 + f\cot i}$; hence, integrating, we have

$\int_0^a sMxdx = \frac{1}{2}a^2 sM =$ whole horizontal thrust of triangle BCH.

Again, the length of the lever $Bb = a - x$, and the moment* of the thrust of the element $bb'hh' = sMx(a - x)dx = asMxdx - sMx^2dx$; hence

* If lines be expressed numerically, the product of a force acting on a lever, and the perpendicular from the axis of motion on its direction is called the *moment* of that force.

$\int_0^a as\mathrm{M}x\,dx - \int_0^a s\mathrm{M}x^2dx = \frac{1}{2}a^3s\mathrm{M} - \frac{1}{3}a^3s\mathrm{M} = \frac{1}{6}a^3s\mathrm{M} =$ moment of the whole horizonal thrust.

The expression $\frac{1-f\tan i}{1+f\cot i}$ will vanish when $\tan i = 0$, or $\tan i = \frac{1}{f}$; and between these limits there is a value which gives both the horizontal thrust and its moment a maximum. Let then

$$u = \frac{1-f\tan i}{1+f\cot i} = \text{maximum, and differentiating we have}$$

$$\frac{du}{di} = \frac{-f\sec^2 i\,(1+f\cot i) + f\operatorname{cosec}^2 i\,(1-f\tan i)}{(1+f\cot i)^2}$$

$$\therefore\ \sec^2 i\,(1+f\cot i) = \operatorname{cosec}^2 i\,(1-f\tan i)$$

$$\therefore\ \tan i = -f + \sqrt{1+f^2}$$

$$\text{Hence M} = \frac{1-f\tan i}{1+f\cot i} = \frac{\sec^2 i}{\operatorname{cosec}^2 i} = \tan^2 i = (-f+\sqrt{1+f^2})^2$$

$$\therefore\ \text{horizontal thrust} = \tfrac{1}{2}a^2s(-f+\sqrt{1+f^2})^2 = \tfrac{1}{2}a^2s\tan^2 i.$$

$$\text{and moment of thrust} = \tfrac{1}{6}a^3s(-f+\sqrt{1+f^2})^2 = \tfrac{1}{6}a^3s\tan^2 i.$$

71. The angle whose tangent is $-f+\sqrt{1+f^2}$ is just half of that whose tan. is $\frac{1}{f}$, or $\tan^{-1}(-f+\sqrt{1+f^2}) = \frac{1}{2}\tan^{-1}\frac{1}{f}$. For since $\tan 2i = \frac{2\tan i}{1-\tan^2 i}$; therefore $\frac{2(-f+\sqrt{1+f^2})}{1-(-f+\sqrt{1+f^2})^2} = \frac{2(-f+\sqrt{1+f^2})}{2f(-f+\sqrt{1+f^2})} = \frac{1}{f}$, and $\tan^{-1}\frac{1}{f}$ is the angle of the slope which the earth would naturally assume if unsustained by any wall.

For if i be the inclination of a plane to the vertical, and g the accelerating force of gravity; then the force g resolved into two, parallel and perpendicular to the plane, gives $g\cos i$ and $g\sin i$; hence the friction $= fg\sin i$, and being counteracted by the force $g\cos i$, we must have

$$g\cos i = fg\sin i,\ \text{or}\ \tan i = \frac{1}{f},\ \text{or}\ f = \cot i.$$

Hence if BK be the natural slope of loose earth, and BH bisect the angle KBC; then the prismatic mass CBH will exert the greatest force against the vertical wall BC.

72. In loose earth the natural slope is about 60° from the vertical, and in tenacious earth this angle is about 54°; hence in the former case $i = 30$; $\tan i = \tan 30° = \sqrt{\frac{1}{3}} = \frac{1}{3}\sqrt{3}$, and in the latter $i = 27°$, $\tan i = \tan 27° = \sqrt{\frac{1}{4}}$, nearly $= \frac{1}{2}$. Therefore, for loose earth, the horizontal thrust $= \frac{1}{6}a^2s$, and its moment $= \frac{1}{18}a^3s$, and for tenacious earth, the horizontal thrust is $\frac{1}{8}a^2s$, and its moment $= \frac{1}{24}a^3s$. Now put AB the breadth of the wall $= x$, BC $= a$, and the specific gravity $=$ S; then the moment of the resistance of the wall is $= \frac{1}{2}ax^2\mathrm{S}$, which, in the case of equilibrium, must be equal to the moment of the horizontal thrust; hence, for tenacious earth we have

$$\frac{1}{2}ax^2\mathrm{S} = \frac{1}{24}a^3s \quad \therefore\ \frac{x}{a} = \frac{1}{6}\sqrt{\frac{3s}{\mathrm{S}}}.$$

Ex. 1. Let S $= 2520$, and $s = 1600$; then we have

$$\frac{x}{a} = \frac{1}{6}\sqrt{\frac{3s}{\mathrm{S}}} = \frac{1}{6}\sqrt{\frac{4800}{2520}} = \frac{1}{3}\sqrt{\frac{10}{21}} = \cdot 23 = \frac{9}{40}\ \text{nearly.}$$

hence $x = \frac{9}{40}\cdot a = \frac{9}{40}$ of the height of the rectangular wall;

Ex. 2. Let the wall be triangular, as in the annexed figure, and let $x =$ its breadth; then the moment of the resistance will be $= \frac{2}{3}x \times \frac{1}{2} axS = \frac{1}{3}ax^2S$; hence we must have

$$\frac{1}{3} ax^2S = \frac{1}{24}a^3s \quad \therefore \quad \frac{x}{a} = \frac{1}{4}\sqrt{\frac{2s}{S}}.$$

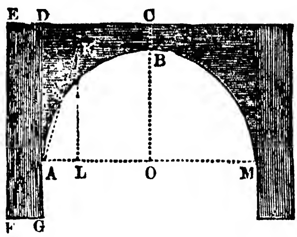
and S, s remaining as above, we have

$$\frac{x}{a} = \frac{1}{4}\sqrt{\frac{3200}{2520}} = \frac{1}{4}\sqrt{\frac{80}{63}} = \sqrt{\frac{5}{63}} = .282 \text{ nearly} = \frac{7}{25}.$$

$$. \ x = \frac{7}{25}\, a = \frac{7}{25} \text{ of the height of the triangular wall.}$$

PROP XXI.

78. *To determine the thickness of a pier necessary to support a given arch*



Let ABCD be half the arch, and DEFG the pier. From the centre of gravity K of the arch draw KL perpendicular to the horizon. Then the weight of the arch in direction KL will be to the horizontal push at A, in direction LA, as KL to LA.

For the weight of the arch in direction KL, the horizontal push or lateral pressure in direction LA, and the push in direction KA, will be as the three sides KL, LA, KA. So that, if A denote the weight or area of the arch; then $\frac{LA}{KL} . A$ will be its force at A in the direction LA; and $\frac{LA}{KL} . GA . A$ its effect on the lever GA to overset the pier, or to turn it about the point F.

Again, the weight or area of the pier, is as EF . FG; and therefore EF . FG . $\frac{1}{2}$FG, or $\frac{1}{2}$EF . FG^2, is its effect on the lever $\frac{1}{2}$FG, to prevent the pier from being overset; supposing the length of the pier, from point to point, to be no more than the thickness of the arch.

But that the pier and arch be in equilibrio, these two effects must be equal. Therefore we have $\frac{1}{2}EF . FG^2 = \frac{LA}{KL} . GA . A$, and consequently the thickness of the pier is $FG = \sqrt{\frac{2GA . AL}{EF . KL}} \times A$.

Example 1. Suppose the arc ABN to be a semicircle; and that DC or AO $= 45$, BC $= 6$, and GA $= 18$ feet. Then KL will be found $= 40$, AL $= 15$ nearly, and EF $= 69$; also, the area ABCD or A $= 704\frac{1}{2}$. Therefore FG $= \sqrt{\frac{2GA . AL}{EF . KL}} . A$ is $= \sqrt{\frac{36 . 15}{69 . 40}} . 704\frac{1}{2} = 11\frac{3}{4}$ nearly, which is the thickness of the pier.

Example 2. Suppose, in the segment ABN, AN $= 100$, OB $= 41\frac{1}{2}$, BC $= 6\frac{1}{2}$, and AG $= 10$. Then EF $= 58$, KL $= 35$, AL $= 15$ nearly, and ABCD or A $= 842$. Therefore FG $= \sqrt{\frac{2GA . AL}{EF . KL}} . A = \sqrt{\frac{20 . 15}{58 . 35}} . 842 = 11\frac{1}{13}$ nearly, the thickness of the pier in this case.

DYNAMICS.

Definitions and Principles.

1. A body is said to be *in motion* when it is continually changing its position in space.

2. Motion is said to be *uniform* when the spaces described in equal successive intervals of time are equal, and *variable* when these spaces are unequal.

3. The *velocity* of a body is the space it would describe in a unit of time, were the motion to become uniform at the commencement of that unit.

4. Motion is said to be *accelerated* when the velocity continually *increases*, and *retarded* when it continually *decreases;* and an *accelerating* or *retarding* force is said to be *uniform* or *variable*, according as the increments or decrements of velocity in equal times are *equal* or *unequal*.

5. The *momentum* or quantity of motion of a body is the sum of the motions of all its particles; and, as the motion of a particle is measured by its velocity, and the number of particles in a body constitutes its *mass;* hence the momentum will be equal to the product of the mass and velocity, when all the particles move with the same velocity.

6. *Inertia* is the opposition offered by a body to a change of state, either of rest or of motion, by the action of a force impressed upon it.

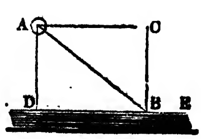
7. If a system of particles, $m, m_1, m_2, \ldots$ revolve round an axis, and $r, r', r'', \ldots$ be their respective distances from that axis; then $mr^2 + m_1r'^2 + m_2r''^2 + \ldots$ or $\Sigma(mr^2)$ is called the *moment of inertia* of the system.

ON THE COLLISION OF SPHERICAL BODIES.

PROP. I.

8. *If a spherical body strike or act obliquely on a plane surface, the force or energy of the stroke or action, is as the sine of the angle of incidence.*

Or, *the force on the surface is to the same if it had acted perpendicularly, as the sine of incidence is to radius.*



Let AB express the direction and the absolute quantity of the oblique force on the plane DE; or let a given body A, moving with a certain velocity, impinge on the plane at B; then its force will be to the action on the plane, as radius to the sine of the angle ABD, or as AB to BC, drawing BC perpendicular, and AC parallel to DE.

For, by Prop. II., the force AB is equivalent to the two forces AC, CB; of which the former AC does not act on the plane, because it is parallel to it. The plane is therefore only acted on by the direct force CB, which is to AB as the sine of the angle BAC, or ABD, to radius.

Corollary. 1. If a body act on another, in any direction, and by any kind of force, the action of that force on the second body, is made only in a direction perpendicular to the surface on which it acts.

For, the force in AB acts on DE only by the force CB, and in that direction.

Corollary. 2. If the plane DE be not absolutely fixed, it will move after the stroke, in the direction perpendicular to its surface. For it is in that direction that the force is exerted.

PROP. II.

9. *If one body A strike another body B, which is either at rest or moving towards the body A, or moving from it, but with a less velocity than that of A; then the momenta, or quantities of motion of the two bodies, estimated in any one direction, will be the very same after the stroke that they were before it.*

For, because action and re-action are always equal, and in contrary directions, whatever momentum the one body gains one way by the stroke, the other must just lose as much in that same direction; and therefore the quantity of motion in that direction, resulting from the motions of both the bodies, remains still the same as it was before the stroke.

Thus, if A with a momentum of 10, strike B at rest, and communicate to it a momentum of 4, in the direction AB. Then A will have only a momentum of 6 in that direction; which, together with the momentum of B, viz. 4, make up still the same momentum between them as before, namely, 10.

If B were in motion before the stroke, with a momentum of 5, in the same direction, and receive from A an additional momentum of 2; then the motion of A after the stroke will be 8, and that of B, 7, which between them make 15, the same as 10 and 5, the motions before the stroke.

Lastly, if the bodies move in opposite directions, and meet one another, namely, A with a motion of 10, and B, of 5; and A communicate to B a motion of 6 in the direction AB of its motion. Then, before the stroke, the whole motion from both, in the direction of AB, is 10 — 5 or 5; but after the stroke, the motion of A is 4 in the direction AB, and the motion of B is 6 — 5 or 1 in the same direction AB; therefore, the sum 4 + 1, or 5, is still the same motion from both as it was before.

PROP. III.

10. *The motion of bodies included in a given space, is the same, with regard to each other, whether that space be at rest, or move uniformly in a right line.*

For, if any force be equally impressed both on the body and the line in which it moves, this will cause no change in the motion of the body along the right line. For the same reason, the motions of all the other bodies, in their several directions, will still remain the same. Consequently, their motions among themselves will continue the same, whether the including space be at rest, or be moved uniformly forward; and therefore, their mutual actions on one another must also remain the same in both cases.

PROP. IV.

11. *If a hard or fixed plane be struck by either a soft or a hard unelastic body, the body will adhere to it; but if the plane be struck by a perfectly elastic body, it will rebound from it again with the same velocity with which it struck the plane.*

For, as the parts which are struck of the elastic body suddenly yield and give way by the force of the blow, and as suddenly restore themselves again with a force equal to the force which impressed them, by the definition of elastic bodies; the intensity of the action of that restoring force on the plane, will be equal to the force or momentum with which the body struck the plane. And, as action and re-action are equal and contrary, the plane will act with the same force on the body, and so cause it to rebound or move back again with the same velocity as it had before the stroke.

But hard or soft bodies, being devoid of elasticity, by the definition, having no restoring force to throw them off again, they must necessarily adhere to the plane struck.

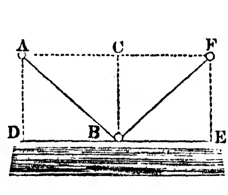
Corollary 1. The effect of the blow of the elastic body on the plane, is double to that of the unelastic one, the velocity and mass being equal in each.

For the force of the blow from the unelastic body, is as its mass and velocity, which is only destroyed by the resistance of the plane; but in the elastic body, that force is not only destroyed and sustained by the plane, but another also equal to it is sustained by the plane, in consequence of the restoring force, and by virtue of which the body is thrown back again with an equal velocity; and, therefore, the intensity of the blow is doubled.

Corollary 2. Hence, unelastic bodies lose, by their collision, only half the motion lost by elastic bodies, their mass and velocities being equal; for the latter communicate double the motion of the former.

PROP. V.

12. *If an elastic body* A *impinge on a firm plane* DE *at the point* B, *it will rebound from it in an angle equal to that in which it struck it; or the angle of incidence will be equal to the angle of reflection; namely, the angle* ABD *equal to the angle* FBE.



Let AB express the force of the body A in the direction AB; which let be resolved into the two AC, CB, parallel and perpendicular to the plane. Take BE and CF equal to AC, and draw BF. Now, action and re-action, being equal, the plane will resist the direct force CB by another BC equal to it, and in a contrary direction; whereas, the other AC, being parallel to the plane, is not acted on nor diminished by it, but still continues as before. The body is therefore reflected from the plane by two forces BC, BE, perpendicular and parallel to the plane, and therefore moves in the diagonal BF by composition. But, because AC is equal to BE or CF, and BC is common, the two triangles BCA, BCF, are mutually similar and equal; and consequently the angles at A and F are equal, as also their equal alternate angles ABD, FBE, which are the angles of incidence and reflection.

PROP. VI.

13. *Let* B *and* b *be two spherical bodies moving in the same direction with the velocities* V *and* v; *it is required to find the velocities of* B *and* b *after* B *has impinged on* b.

(1.) If the bodies are inelastic, it is obvious that the bodies will move on together after impact, because there is no elastic force to separate them. Let C be their common velocity after impact; then $BV + bv$ is the momentum of the system, and since it must remain unchanged after impact, we must have

$$(B + b)C = BV + bv \quad \therefore\ C = \frac{BV + bv}{B + b}.$$

$$\text{Velocity lost by } B = V - C = \frac{b(V - v)}{B + b}$$

$$\text{velocity gained by } b = C - v = \frac{B(V - v)}{B + b}.$$

$$\therefore\ B + b : b :: V - v : \text{velocity lost by B},$$

$$\text{and } B + b : B :: V - v : \text{velocity gained by } b.$$

(2.) If the bodies B and b are perfectly elastic, the velocity lost by B and gained by b will be the same as we have found above during the *compression* of their figures; but after the compression ceases, *elasticity* begins to act, and the bodies separate with exactly the same velocity as that with which they were compressed; therefore B will lose and b will gain as much velocity by the recovery of their figures as by their compression;

$$\text{hence velocity lost by } B = \frac{b(V - v)}{B + b} + \frac{b(V - v)}{B + b} = \frac{2b(V - v)}{B + b}$$

$$\text{velocity gained by } b = \frac{B(V - v)}{B + b} + \frac{B(V - v)}{B + b} = \frac{2B(V - v)}{B + b}$$

Or when the bodies are perfectly elastic, we have

$$B + b : 2b :: V - v : \text{velocity lost by B}.$$

$$B + b : 2B :: V - v : \text{velocity gained by } b.$$

(3.) If the bodies are not perfectly elastic, which is usually the case, then when elasticity begins to act, it produces effects proportionally less than perfect elasticity does. Let e denote the common elasticity of the bodies; then in consequence of the restoring force, B and b will be repelled with the velocities eV and ev respectively; hence

$$\text{velocity lost by } B = \frac{b(V - v)}{B + b} + \frac{be(V - v)}{B + b} = \frac{(1 + e)b(V - v)}{B + b}$$

$$\text{velocity gained by } b = \frac{B(V - v)}{B + b} + \frac{Be(V - v)}{B + b} = \frac{(1 + e)B(V - v)}{B + b}.$$

$$\therefore\ \text{velocity of B after impact} = V - \frac{(1 + e)b(V - v)}{B + b} = \frac{BV + bv - be(V - v)}{B + b}$$

$$\text{velocity of } b \text{ after impact} = v + \frac{(1 + e)B(V - v)}{B + b} = \frac{BV + bv + Be(V - v)}{B + b}.$$

The relative velocity of the bodies after impact is the difference of these velocities, and is hence $= e(V - v)$.

Cor. If the bodies are moving in contrary directions before impact, attention to the signs of the velocities will preserve the truth of the formulæ above deduced, and if the body b be at rest, its velocity will be zero, and the formulæ may be readily modified to this or any other case.

EXAMPLES.

(1.) An inelastic body B impinges directly on another inelastic body b, at rest, with a velocity of 10 feet per second; find the velocity after impact, when $B = 6$ and $b = 4$ ounces.

Here $v = 0$, and therefore we have, by the first case,

$$C = \frac{BV}{B + b} = \frac{6 \cdot 10}{6 + 4} = 6 \text{ feet per second, the velocity after impact.}$$

Or, thus:—

Let x = common velocity after impact; then
the velocity and momentum lost by B are $10 - x$ and $6(10 - x)$
the velocity and momentum gained by C are x and $4x$.
$\therefore\ 6(10 - x) = 4x$, and hence $x = 6$, as above.

(2.) $B = 10$lb. is moving with a velocity 20; with what velocity must $b = 6$lb meet B, that their common velocity after impact may be 10 in the direction of b's motion? Ans. $v = 60$ feet.

(3.) A sphere whose diameter is 2 inches impinges directly with a velocity of 0 feet per second on another sphere at rest, whose diameter is 4 inches; how will they move after impact?

(1) When the spheres are perfectly elastic.

(2) When their common elasticity is denoted by $\frac{15}{16}$.

Ans. The two-inch sphere moves backward with a velocity $= 7\frac{7}{9}$ feet, while the other moves forward with a velocity $= 2\frac{2}{9}$ feet. And when their elasticity is $\frac{15}{16}$, the velocities are respectively $7\frac{2}{9}$ and $2\frac{1}{72}$ in the same directions as in the first case.

14. *If a body* B *impinge on* b *at rest, and* b *on* b′ *at rest; the velocity communicated to* b′ *will be a maximum, when* b *is a mean proportional between* B *and* b′.

For the velocity communicated to $b = \dfrac{(1+e)BV}{B+b}$

and the velocity by b to $b' = \dfrac{(1+e)b}{b+b'} \cdot \dfrac{(1+e)BV}{B+b}$

$$= \frac{(1+e)^2 BV}{\dfrac{(B+b)(b+b')}{b}} = \text{maximum}.$$

$$\therefore\ \frac{(B+b)(b+b')}{b} = B + b' + b + \frac{Bb'}{b} = \text{minimum}.$$

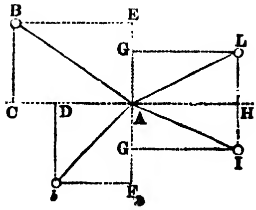
$$\therefore \text{ we have } u = b + \frac{Bb'}{b} = \text{minimum}.$$

$$\therefore\ \frac{du}{db} = 1 - \frac{Bb'}{b^2} = 0.$$

$$\therefore\ b^2 = Bb' \text{ and } b = \sqrt{Bb'}$$

PROP. VIII.

15. *If bodies strike one another obliquely, it is proposed to determine their motions after the stroke.*



If the two bodies B, b, move in the oblique directions BA, bA, and strike each other at A with velocities which are in proportion to the lines BA, bA; to find their motions after the impact. Let CAH represent the plane in which the bodies touch in the point of concourse; to which draw the perpendiculars BC, bD, and complete the rectangles CE, DF. Then the motion in BA is resolved into the two BC, CA; and the motion in bA is resolved into the two bD, DA; of which the antecedents BC, bD, are the velocities with which they directly meet, and the consequents CA, DA are parallel; therefore, by these the bodies do not impinge on each other, and consequently the motions, according to these directions, will not be changed by the impulse; so that the velocities with which the bodies meet, are as BC and bD, or their equals EA and FA. The motions, therefore, of the bodies B, b, directly striking each other with the velocities EA, FA, will be determined by Prop. VI. p. 837, according as the bodies are elastic or non-elastic; which being done, let AG be the velocity, so determined, of one of them, as A; and since there remains also in the body a force moving in the direction parallel to BE, with a velocity as BE; make AH equal to BE, and complete the rectangle GH; then the two motions in AH and AG, or HI; are compounded into the diagonal AI, which therefore will be the path and velocity of the body B after the stroke. And after the same manner is the motion of the other body b determined after the impact.

FUNDAMENTAL EQUATIONS OF MOTION.

PROP. IX.

16. *In uniform motion, the space* s *described with a velocity* v *in time* t *is*

$$s = tv.$$

For v is the space described in each unit of time, and t is the number of units of time; therefore the whole space described is

$$s = tv \quad \dots\dots\dots\dots \quad (1)$$

PROP. X.

17. *In uniformly accelerated or retarded motion, the velocity* v *generated by the force* f *in time* t *is*

$$v = ft.$$

For the velocity generated in each second is f, and hence in t seconds, it is ft, and therefore

$$v = ft \quad \dots\dots\dots\dots \quad (2)$$

Cor. If $u =$ velocity when $t = 0$, the velocity at the end of the time t is $v = u \pm ft$, the $+$ being used when the force is accelerating, and the $-$ when it is retarding.

PROP. XI.

18. *The space described from rest by a body, acted on by a uniformly accelerating force is* $s = \frac{1}{2}vt$.

Let s = space described, in the time t, by the action of the uniformly accelerating force f, and v = velocity acquired in time t. Divide t into n equal intervals, and therefore each $= \frac{t}{n}$; then velocity acquired in each interval $= \frac{v}{n}$, and the space described in time t with the velocity at the *end* of each interval continued uniform during that interval is

$$= \frac{vt}{n^2} + \frac{2vt}{n^2} + \frac{3vt}{n^2} + \ldots\ldots\ldots \frac{nvt}{n^2} = \frac{vt}{2}\left(1 + \frac{1}{n}\right)$$

and the space described in the time t with the velocity at the *beginning* of each interval continued uniform during that interval is

$$= o + \frac{vt}{n^2} + \frac{2vt}{n^2} + \ldots\ldots \frac{(n-1)vt}{n^2} = \frac{vt}{2}\left(1 - \frac{1}{n}\right).$$

Now s manifestly lies between these two spaces, being described with velocities intermediate to the velocities with which these are described. Increase n, the number of intervals without limit, and we have

$$s = \frac{vt}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Cor. 1. By Prop. X. $v = ft \quad \therefore \quad s = \frac{1}{2}ft^2 \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$

and $vt = 2s \quad \therefore \quad v^2t = 2fst$, or $v^2 = 2fs \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$

Cor. 2. The space described in t seconds $= \frac{1}{2}ft^2$

——————— $(t-1)$ ——— $= \frac{1}{2}f(t-1)^2$

$\therefore$ the space described in the t^{th} second $= \frac{1}{2}f(2t-1) \ldots\ldots\ldots \quad (6)$

Cor. 3. Hence the spaces described in equal successive portions of time are as the odd numbers, 1, 3, 5, 7, ...

Scholium.—If the body, instead of beginning to move from rest, be projected with velocity u; then the space described in time t is

$$s = ut \pm \frac{1}{2}ft^2$$

the + applying when the force accelerates, and the — when it retards the motion.

For ut is the space described with the velocity of projection alone, and therefore the space described from both causes is $s = ut \pm \frac{1}{2}ft^2 \ldots\ldots\ldots\ldots \quad (7)$

19. *If the accelerating or retarding force be variable, then* $ds = vdt$.

Let s_1 and s_2 be the spaces described in the times t_1 and t_2, where $t_1 = t - \theta$, and $t_2 = t + \theta$; then whatever is the nature of the motion, the space will be a function of the time; and hence by Taylor's theorem

$$s_1 = s - \frac{ds}{dt}\cdot\theta + \frac{d^2s}{dt^2}\cdot\frac{\theta^2}{1.2} - \frac{d^3s}{dt^3}\cdot\frac{\theta^3}{1.2.3} + \ldots\ldots$$

$$s_2 = s + \frac{ds}{dt}\cdot\theta + \frac{d^2s}{dt^2}\cdot\frac{\theta^2}{1.2} + \frac{d^3s}{dt^3}\cdot\frac{\theta^3}{1.2.3} + \ldots\ldots$$

$\therefore$ space in preceding time $\theta = s - s_1 = \frac{ds}{dt}\cdot\theta - \frac{d^2s}{dt^2}\cdot\frac{\theta^2}{1.2} + \frac{d^3s}{dt^3}\cdot\frac{\theta^3}{1.2.3} - .$

space in succeeding time $\theta = s_2 - s = \frac{ds}{dt}\cdot\theta + \frac{d^2s}{dt^2}\cdot\frac{\theta^2}{1.2} + \frac{d^3s}{dt^3}\cdot\frac{\theta^3}{1.2.3} +$

But the space described with the uniform velocity v in the time θ is $=v\theta$, which is always intermediate to the spaces $s - s_1$ and $s_2 - s$, however small θ may be; therefore we have

$$v\theta = \frac{ds}{dt} \cdot \theta, \text{ or } v = \frac{ds}{dt} \quad \ldots\ldots (8)$$

Cor. 1. In the same manner it is proved, that if f be the accelerating force at the end of the time t; then

$$f = \frac{dv}{dt} \quad \ldots\ldots (9)$$

Cor. 2. Hence, since $v = \frac{ds}{dt}$ by (8) and $dv = fdt$ by (9); therefore

$$vdv = fds. \quad \ldots\ldots (10)$$

Cor. 3. Again, $v = \frac{ds}{dt} \therefore \frac{dv}{dt} = \frac{d^2s}{dt^2}$, or $f = \frac{d^2s}{dt^2} \quad \ldots\ldots (11)$

20. In variable motion we have therefore the general equations

$$dv = fdt \ldots\ldots (a) \qquad vdv = fds \ldots\ldots (c)$$

$$ds = vdt \ldots\ldots (b) \qquad f = \frac{d^2s}{dt^2} \ldots\ldots (d)$$

which are applicable either to the case of the motion being accelerated or retarded; but in the latter case, dv is a negative quantity.

THE LAWS OF GRAVITY; THE DESCENT OF HEAVY BODIES; AND THE MOTION OF PROJECTILES IN FREE SPACE.

PROP. XIII.

21. *All the properties of motion delivered in Proposition* XI., *its corollaries and scholium, for constant forces, are true in the motions of bodies freely descending by their own gravity; namely, that the velocities are as the times, and the spaces as the squares of the times, or as the squares of the velocities.*

For, since the force of gravity is uniform, and constantly the same, at all places near the earth's surface, or at nearly the same distance from the centre of the earth, and that this is the force by which bodies descend to the surface; they therefore descend by a force which acts constantly and equally; consequently, all the motions freely produced by gravity, are as above specified by that proposition, &c.

22. Scholium.—Now it has been found, by numberless experiments, that gravity is a force of such a nature that all bodies, whether light or heavy, fall perpendicularly through equal spaces in the same time, abstracting from them the resistance of the air—as lead or gold and a feather, which in an exhausted receiver fall from the top to the bottom in the same time. It is also found, that the velocities acquired by descending are in the exact proportion of the times of descent; and farther, that the spaces descended are proportional to the squares of the times, and therefore to the squares of the velocities. And hence it follows that the weights, or gravities of bodies near the surface of the earth, are proportional to the quantities of matter contained in them; and that the spaces, times, and velocities generated by gravity, have the relations contained in the proposition to which we have above referred. Moreover, as it is found,

by accurate experiments, that a body, in the latitude of London, falls nearly $16\frac{1}{12}$ feet in the first second of time, and consequently that at the end of that time it has acquired a velocity double, or of $32\frac{1}{6}$ feet; hence it is obvious, if $\frac{1}{2}g$ denote $16\frac{1}{12}$ feet, the space fallen through in one second of time, or g the velocity generated in that time; then, because the velocities are directly proportional to the times, and the spaces to the squares of the times,

$$\text{therefore, } 1'' : t'' :: g : gt = v, \text{ the velocity,}$$
$$\text{and } 1^2 : t^2 :: \tfrac{1}{2}g : \tfrac{1}{2}gt^2 = s, \text{ the space.}$$

And hence, for the descents of gravity we have these general equations, namely,

$$s = \tfrac{1}{2}gt^2 = \frac{v^2}{2g} = \tfrac{1}{2}tv.$$
$$v = gt = \frac{2s}{t} = \sqrt{2gs}.$$
$$t = \frac{v}{g} = \frac{2s}{v} = \sqrt{\frac{2s}{g}}.$$
$$g = \frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s}.$$

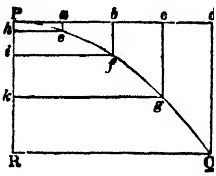
Hence, because the times are as the velocities, and the spaces as the squares of either, therefore

If the times be as the numbers	1, 2, 3, 4, 5, &c.
The velocities will also be as	1, 2, 3, 4, 5, &c.
And the spaces as their squares	1, 4, 9, 16, 25, &c.
And the space for each time as	1, 3, 5, 7, 9, &c.;

namely, as the series of the odd numbers, which are the differences of the squares denoting the whole spaces. So that, if the first series of natural numbers be seconds of time, namely,

The times in seconds	$1''$,	$2''$,	$3''$,	$4''$,	&c.
The velocity in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c.
The spaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c.
And the space for each second	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c.

These relations may be aptly represented by the abscisses and ordinates of a parabola. Thus, if PQ be a parabola, PR its axis, and RQ its ordinate; and Pa, Pb, Pc, &c., parallel to RQ, represent the times from the beginning, or the velocities, then ae, bf, cg, &c. parallel to the axis PR, will represent the spaces described by a falling body in those times; for, in a parabola, the abscisses Ph, Pi, Pk, &c., or ae, bf, cg, &c., which are the spaces described, are as the squares of the ordinates, he, if, kg, &c., or Pa, Pb, Pc, &c., which represent the times or velocities.



23. And because the laws for the destruction of motion are the same as those for the generation of it, by equal forces, but acting in a contrary direction; therefore,

1*st.* A body thrown directly upwards, with any velocity, will lose equal velocities in equal times.

2*nd.* If a body be projected upwards, with the velocity it acquired in any time by descending freely, it will lose all its velocity in an equal time, and will ascend just to the same height from whence it fell, and will describe equal spaces in equal times, both in rising and falling, but in an inverse order; and it will have equal velocities at any one and the same point of the line described, both in ascending and descending.

3rd. If bodies be projected upwards, with any velocities, the height ascended to, will be as the squares of those velocities, or as the squares of the times of ascending, till they lose all their velocities.

24. When the body, instead of being permitted to fall from rest, is projected upwards or downwards with a given velocity u; then by Art, 17 and the scholium to Prop. XI. we have

$$v = u \pm gt \qquad\qquad s = tu \pm \tfrac{1}{2}gt^2$$

where the — must be employed when the projection is vertically upwards, and the + when the projection is vertically downwards.

PROP. XIV.

25. *Let two weights* W *and* w *hang over a fixed pulley, to determine their motion, neglecting the inertia of the pulley, and the weight of the string.*

Here the moving force of W is Wg, and that of $w = -wg$; also the mass resisting motion is $W + w$; hence the accelerating force on $W = \dfrac{(W-w)g}{W+w}$, and therefore we have

$$v = \frac{W-w}{W+w}gt \text{ . and } s = \frac{W-w}{W+w}.\tfrac{1}{2}gt^2.$$

EXAMPLES.

(1.) Find the space descended by a body in 7 seconds of time, and the velocity acquired.

$$s = \tfrac{1}{2}gt^2 = 16\tfrac{1}{12} \times 7^2 = 16\tfrac{1}{12} \times 49 = 788\tfrac{1}{12}ft = \text{space descended.}$$
$$v = gt \ = 32\tfrac{1}{6} \ \times 7 \ = \ldots\ldots\ldots = 225\tfrac{1}{6}ft. \ = \text{velocity acquired.}$$

(2.) A body is projected vertically from the bottom of a tower 200 feet in height, with a velocity of 120 feet per second; in what time will it reach the top, and what will be its velocity at that time? Also, to what height above the top of the tower will it rise?

$$\text{Here } 200 = tu - \tfrac{1}{2}gt^2 = 120\,t - 16\tfrac{1}{12}t^2$$
$$\therefore\ t^2 - \frac{120}{16\frac{1}{12}}\,t = -\frac{200}{16\frac{1}{12}}.$$
$$\therefore\ t = 2{\cdot}513'' \text{ or } 4{\cdot}948'',$$

the former of these values of t is the time at which the body in its *ascent* passes the top of the tower; and the latter, the time at which the body in its *descent* passes the top.

Again, $v = u - gt = 120 - 32\frac{1}{6} \times 2{\cdot}513'' = 39{\cdot}165$ feet per second, the velocity of the body at the top of the tower.

When $v = 0$; then $gt = u$, or $t = \dfrac{u}{g}$, and therefore we have

$$s = tu - \tfrac{1}{2}gt^2 = \frac{u}{g}\left(u - \tfrac{1}{2}u\right) = \frac{u^2}{2g} = \frac{120^2}{64\frac{1}{3}} = 223{\cdot}83 \text{ feet.}$$

Hence height above top = 223·83 − 200 = 23·83 feet.

(3.) Find the time of generating a velocity of 100 feet per second, and the whole space descended. Ans. $3\frac{21}{193}''$ time, $155\frac{85}{193}$ ft. space.

(4.) Find the time of descending 400 feet, and the velocity at the end of that time. Ans. $4\frac{76}{77}''$ time, $160\frac{32}{77}$ velocity.

(5.) A body is projected downwards with a velocity of 30 feet per second: what space will it describe in 6 seconds? Ans. 759 feet.

(6.) A body is projected upwards from the top of a tower 200 feet in height, with a velocity of 40 feet per second, at the same time that another is projected upwards from the bottom of the tower with the velocity of 90 feet per second; where will they meet? Ans. $97\frac{1}{3}$ feet below the top of the tower.

(7.) Two weights W and w weighing 8 and 5 pounds respectively, hang freely over a pulley; how far will W descend after the commencement of motion in 2 seconds? Ans. $14\frac{11}{13}$ feet.

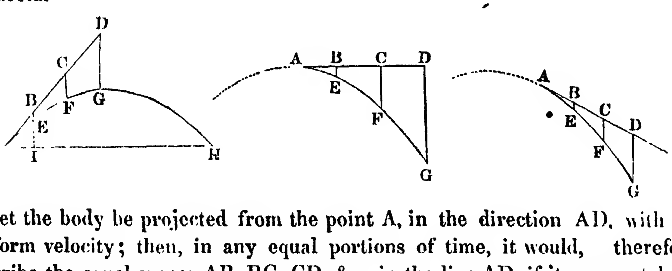
(8.) Two weights W and w hang over a pulley, and $W = 2\,w$; find the space through which a body will descend by the force of gravity, whilst W descends 2 feet. Ans. 6 feet.

(9.) The space described by a heavy body in the 4th second, is to the space described in the last second except 4, as 1 to 3; find the whole space described. Ans. 3618 feet 9 inches.

(10.) A body has fallen from A to B, when another body is let fall from C; how far will the latter body descend before it is overtaken by the former?
Ans. If $AB = a$, and $AC = b$; then space descended $= \dfrac{(b - a)^2}{4\,a}$.

PROP. XV.

26. *If a body be projected in free space, either parallel to the horizon, or in an oblique direction, by the force of gun-powder, or any other impulse; it will, by this motion, in conjunction with the action of gravity, describe the curve line of a parabola.*



Let the body be projected from the point A, in the direction AD, with any uniform velocity; then, in any equal portions of time, it would, therefore, describe the equal spaces AB, BC, CD, &c., in the line AD, if it were not drawn continually down below that line by the action of gravity. Draw BE, CF, DG, &c., in the direction of gravity, or perpendicular to the horizon, and equal to the spaces through which the body would descend by its gravity, in the same times in which it would uniformly pass over the corresponding spaces AB, AC, AD, &c., by the projectile motion. Then, since by these two motions, the body is carried over the space AB, in the same time as over the space BE, and the space AC in the same time as the space CF, and the space AD in the same time as the space DG, &c.; therefore, by the composition of motions, at the end of those times, the body will be found respectively in the points E, F, G, &c.; and con

sequently the real path of the projectile will be the curve line AEFG, &c. But the spaces AB, AC, AD, &c., described by uniform motion, are as the times of description; and the spaces BE, CF, DG, &c., described in the same times by the accelerating force of gravity, are as the squares of the times; consequently, the perpendicular descents are as the squares of the spaces in AD, that is BE, CF, DG, &c., are respectively proportional to AB^2, AC^2, AD^2, &c.; which is the property of the parabola. Therefore, the path of the projectile is the parabolic line AEFG, &c., to which AD is a tangent at the point A.

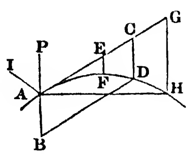
Corol. 1. The horizontal velocity of a projectile is always the same constant quantity, in every point of the curve; because the horizontal motion is in a constant ratio to the motion in AD, which is the uniform projectile motion. And the constant horizontal velocity, is in proportion to the projectile velocity, as radius to the cosine of the angle DAH, or angle of elevation or depression of the piece above or below the horizontal line AH.

Corol. 2. The velocity of the projectile in the direction of the curve, or of its tangent at any point A, is as the secant of its angle BAI of direction above the horizon. For the motion in the horizontal direction AI is constant, and AI is to AB, as radius to the secant of the angle A; therefore, the motion at A in AB, is every where as the secant of the angle A.

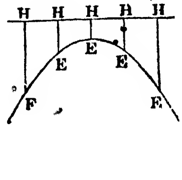
Corol. 3. The velocity in the direction DG of gravity, or perpendicular to the horizon at any point G of the curve, is to the first uniform projectile velocity at A, or point of contact of a tangent, as 2GD to AD. For, the times in AD and DG being equal, and the velocity acquired by freely descending through DG being such as would carry the body uniformly over twice DG in an equal time, and the spaces described with uniform motions being as the velocities, therefore the space AD is to the space 2DG, as the projectile velocity at A, to the perpendicular velocity at G.

PROP. XVI.

27. *The velocity in the direction of the curve, at any point of it, as A, is equal to that which is generated by gravity in freely descending through a space which is equal to one-fourth of the parameter of the diameter of the parabola at that point.*



Let PA or AB be the height due to the velocity of the projectile at any point A, in the direction of the curve or tangent AC, or the velocity acquired by falling through that height; and complete the parallelogram ACDB. Then is CD = AB or AP, the height due to the velocity in the curve at A; and CD is also the height due to the perpendicular velocity at D, which must be equal to the former: but, by the last corol., the velocity at A is to the perpendicular velocity at D, as AC to 2CD; and as these velocities are equal, therefore AC or BD is equal to 2CD, or 2AB; and hence AB or AP is equal to $\frac{1}{2}$BD, or $\frac{1}{4}$ of the parameter of the diameter AB.



Corol. 1. Hence it appears, if from the directrix of the parabola which is the path of the projectile, several lines HE be drawn perpendicular to the directrix, or parallel to the axis; that the velocity of the projectile in the direction of the curve, at any point E, is always equal to the velocity acquired by a body falling freely through the perpendicular line HE.

Corol. 2. If a body after falling through the height PA (last fig. but one) which is equal to AB, and when it arrives at A, have its course changed, by reflection from a firm plane AI, or otherwise, into any direction AC, without altering the velocity; and if AC be taken $= 2AP$ or $2AB$, and the paral-lelogram be completed; the body will describe the parabola passing through the point D.

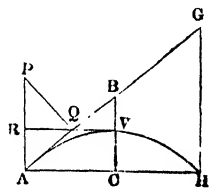
Corol. 3. Because $AC = 2AB$ or $2CD$ or $2AP$, therefore $AC^2 = 2AP \times 2CD$ or $AP.4CD$; and because all the perpendiculars EF, CD, GH, are as AE^2, AC^2, AG^2, therefore also $AP.4EF = AE^2$, and $AP.4GH = AG^2$, &c.; and, because the rectangle of the extremes is equal to the rectangle of the means of four proportionals, therefore, always

It is $AP : AE \quad AE : 4EF$,
And $AP : AC \quad AC : 4CD$,
And $AP : AG \quad AG : 4GH$,

And so on.

PROP. XVII.

28. *Having given the direction, and the impetus, or altitude due to the first velocity of a projectile; to determine the greatest height to which it will rise, and the random or horizontal range.*

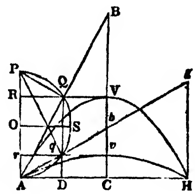


Let AP be the height due to the projectile velocity at A, AG the direction, and AH the horizon. Upon AG let fall the perpendicular PQ, and on AP the perpendicular QR; so shall AR be equal to the greatest altitude CV, and 4QR equal to the horizontal range AH. Or, having drawn PQ perpendicular to AG, take $AG = 4AQ$, and draw GH perpendicular to AH; then AH is the range.

For, by the last corollary, $AP : AG :: AG : 4GH$;
And, by similar triangles, $AP : AG :: AQ : GH$,
Or, ... $AP : AG :: 4AQ : 4GH$;

therefore $AG = 4AQ$; and, by similar triangles, $AH = 4QR$.

Also, if V be the vertex of the parabola, then AB or $\frac{1}{2}AG = 2AQ$, or $AQ = QB$; consequently, $AR = BV$, which is $= CV$ by the property of the parabola.



Corol. 1. Because the angle Q is a right angle, which is the angle in a semicircle, therefore, if upon AP, as a diameter, a semicircle be described, it will pass through the point Q.

Corol. 2. If the horizontal range and the projectile velocity be given, the direction of the piece, so as to hit the object H, will be thus easily found: Take $AD = \frac{1}{4}AH$, draw DQ perpendicular to AH, meeting the semicircle described on the diameter AP, in Q and q; then AQ or Aq will be the direction of the piece. And hence it appears, that there are two directions AB, Ab, which, with the same projectile velocity, give the very same horizontal range AH. And these two directions make equal angles qAD, QAP, with AH and AP, because the arc PQ = the arc Aq.

Corol. 3. Or, if the range AH, and direction AB, be given; to find the altitude and velocity or impetus. Take $AD = \frac{1}{4}AH$, and erect the perpendi-

cular DQ, meeting AB in Q; so shall DQ be equal to the greatest altitude CV Also, erect AP perpendicular to AH, and QP to AQ; so shall AP be the height due to the velocity.

Corol. 4. When the body is projected with the same velocity, but in different directions; the horizontal ranges AH will be as the sines of double the angles of elevation. Or, which is the same, as the rectangle of the sine and cosine of elevation; for AD or RQ, which is $\frac{1}{4}$AH, is the sine of the arc AQ, which measures double the angle QAD of elevation.

And when the direction is the same, but the velocities different, the horizontal ranges are as the square of the velocities, or as the height AP, which is as the square of the velocity; for the sine AD or RQ or $\frac{1}{4}$AH is as the radius, or as the diameter AP.

Therefore, when both are different, the ranges are in the compound ratio of the squares of the velocities, and the sines of double the angles of elevation.

Corol. 5. The greatest range is when the angle of elevation is 45°, or half a right angle; for the double of 45 is 90, which has the greatest sine. Or the radius OS, which is $\frac{1}{4}$ of the range, is the greatest sine.

And hence the greatest range, or that at an elevation of 45°, is just double the altitude AP which is due to the velocity, or equal to 4VC. And consequently, in that case, C is the focus of the parabola, and AH its parameter. Also, the ranges are equal, at angles equally above and below 45°.

Corol. 6. When the elevation is 15°, the double of which, or 30°, has its sine equal to half the radius; consequently, then its range will be equal to AP, or half the greatest range at the elevation of 45°; that is, the range at 15°, is equal to the impetus or height due to the projectile velocity.

Corol. 7. The greatest altitude CV, being equal to AR, is as the versed sine of double the angle of elevation, and also as AP or the square of the velocity. Or as the square of the sine of elevation, and the square of the velocity; for the square of the sine is as the versed sine of the double angle.

Corol. 8. The time of flight of the projectile, which is equal to the time of a body falling freely through GH or 4CV, four times the altitude, is therefore as the square root of the altitude, or as the projectile velocity and sine of the elevation.

92. Scholium.—From the last proposition and its corollaries, may be deduced the following set of theorems, for finding all the circumstances of projectiles on horizontal planes, having any two of them given. Thus, let e denote the elevation, R the horizontal range, t the time of flight, v the projectile velocity; h, the greatest height of the projectile; $g = 32\frac{1}{6}$ feet, and a the impetus or altitude due to the velocity v; then

$$R = 2a \sin 2e = \frac{v^2}{g} \sin 2e = \tfrac{1}{2}gt^2 \cot e = 4h \cot e$$

$$v = \sqrt{2ag} = \sqrt{gr \operatorname{cosec} 2e} = \tfrac{1}{2}gt \operatorname{cosec} e = \operatorname{cosec} e\sqrt{2gh}$$

$$t = \frac{2v}{g} \sin e = \frac{2 \sin e}{g}\sqrt{2ag} = \sqrt{\frac{2r \tan e}{g}} = 2\sqrt{\frac{2h}{g}}$$

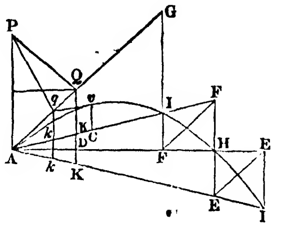
$$h = a \sin^2 e = \tfrac{1}{2}a \operatorname{vers} 2e = \tfrac{1}{4}r \tan e = \frac{v^2 \sin^2 e}{2g} = \tfrac{1}{8}gt^2.$$

And from any of these, the angle of direction may be found. Also, in these

theorems, $\frac{1}{2}g$ may in many cases be taken $= 16$, without the small fraction $\frac{1}{12}$; which will be near enough for common use.

PROP. XVIII.

30. *To determine the range on an oblique plane, having given the impetus or velocity, and the angle of direction.*



Let AE be the oblique plane, at a given angle, either above or below the horizontal plane AH; AG the direction of the piece, and AP the altitude due to the projectile velocity at A.

By the last proposition, find the horizontal range AH to the given velocity and direction; draw HE perpendicular to AH, meeting the oblique plane in E; draw EF parallel to AG, and FI parallel to HE; so shall the projectile pass through I, and the range on the oblique plane will be AI. For if AH, AI, be any two lines terminated at the curve, and IF, HE, parallel to the axis; then is EF parallel to the tangent AG.

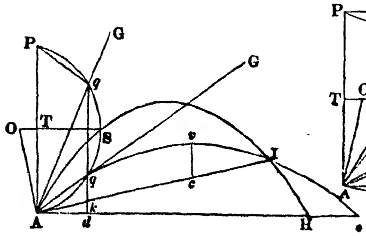
31. *Otherwise, without the horizontal range.* Draw PQ perpendicular to AG, and QD perpendicular to the horizontal plane AF, meeting the inclined plane in K; take AE = 4AK, draw EF parallel to AG, and FI parallel to AP or DQ; so shall AI be the range on the oblique plane. For AH = 4AD, therefore EH is parallel to FI, and so on, as above.

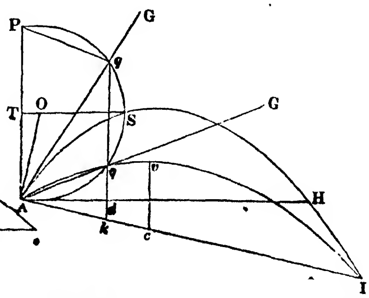
OTHERWISE.

32. Draw Pq making the angle APq = the angle GAI; then take AG = 4Aq, and draw GI perpendicular to AH. Or, draw qk perpendicular to AH, and take AI = 4Ak. Also, kq will be equal to cv, the greatest height above the plane.

For, by corol. 2, Prop. XVI. AP : AG :: AG : 4GI;
and, by similar triangles, AP : AG :: Aq : GI,
or . . . . AP : AG :: 4Aq : 4GI;
therefore AG = 4Aq; and, by similar triangles, AI = 4Ak.

Also, qk, or $\frac{1}{4}$GI, is = to cv.





Corol. 1. If AO be drawn perpendicular to the plane AI, and AP be bisected by the perpendicular STO; then with the centre O describing a circle through A and P, the same will also pass through q, because the angle GAI,

formed by the tangent AI and AG, is equal to the angle APq, which will therefore stand on the same arc Aq.

Cor. 2. If there be given the range and velocity, or the impetus, the direction will hence be easily found, thus: Take $Ak = \frac{1}{4}AI$, draw kq perpendicular to AH, meeting the circle described with the radius AO in two points q and q; then Aq or Aq will be the direction of the piece. And hence it appears, that there are two directions, which, with the same impetus, give the very same range AI. And these two directions make equal angles with AI and AP, because the arc Pq is equal the arc Aq. They also make angles with a line drawn from A through S, because the arc Sq is equal to the arc Sq.

Cor. 3. Or, if there be given the range AI, and the direction Aq; to find the velocity or impetus. Take $Ak = \frac{1}{4}AI$, and erect kq perpendicular to AH, meeting the line of direction in q; then draw qP making the angle AqP = angle Akq; so shall AP be the impetus, or the altitude due to the projectile velocity.

Cor. 4. The range on an oblique plane, with a given elevation, is directly as the rectangle of the cosine of the direction of the piece above the horizon, and the sine of the direction above the oblique plane, and reciprocally as the square of the cosine of the angle of the plane above or below the horizon. For in the triangles APq and Akq we have

$$\begin{aligned}
AP : Aq &: : \sin AqP : \sin APq : : \sin Akq : \sin qAI \\
&: : \sin Akd : \sin qAI : : \cos HAI : \sin qAI \\
\text{and } Aq : Ak &: : \sin Akq : \sin Aqk : : \sin Akd : \sin PAq \\
&: : \cos HAI : \cos qAH \\
\therefore AP : Ak &: : \cos^2 HAI : \cos qAH \sin qAI \\
AI = 4Ak &= 4AP . \frac{\cos qAH \sin qAI}{\cos^2 HAI} = \text{range on oblique plane.}
\end{aligned}$$

33. The range is the greatest when Ak is the greatest; that is, when kq touches the circle in the middle point S; and then the line of direction passes through S, and bisects the angle formed by the oblique plane and the vertex. Also, the ranges are equal at equal angles above and below this direction for the maximum.

Cor. 5. The greatest height cv or kq of the projectile above the plane is

$$= AP . \frac{\sin^2 qAI}{\cos^2 IAH} = AP \sin^2 qAI \sec^2 IAH.$$

$$\begin{aligned}
\text{For } AP : Aq &: \sin AqP : \sin APq : : \cos HAI : \sin qAI, \\
\text{and } Aq : kq &: \sin Akq : \sin qAI : : \cos HAI : \sin qAI. \\
\therefore AP : kq &: \cos^2 HAI : \sin^2 qAI. \\
\therefore kq &= AP \sin^2 qAI \sec^2 HAI.
\end{aligned}$$

Cor. 6. The time of flight in the curve $AvI = \frac{2 \sin qAI}{\cos HAI}\sqrt{\frac{2AP}{g}}$. For the time of describing the curve is equal to the time of falling freely through GI or $4kq = 4AP . \frac{\sin^2 qAI}{\cos^2 IAH}$; hence by the laws of gravity, the time of flight $= \sqrt{\frac{2s}{g}} = \frac{2 \sin qAI}{\cos HAI}\sqrt{\frac{2AP}{g}}$.

34. Scholium.—From the foregoing corollaries may be collected the following theorems, in which i denotes the inclination of the plane to the horizon, e the

angle of elevation above the horizon, and the other letters as in the former equations.

$$R = \frac{2v^2}{g} \cdot \frac{\cos e \sin (e-i)}{\cos^2 i} = \frac{v^2}{g} \cdot \frac{\sin (2e-i) - \sin i}{\cos^2 i} = 4a \cdot \frac{\cos e \sin (e-i)}{\cos^2 i}$$

$$= \tfrac{1}{2}gt^2 \cdot \frac{\cos e}{\sin (e-i)} \qquad = 4h \cdot \frac{\cos e}{\sin (e-i)}$$

$$h = a \cdot \frac{\sin^2 (e-i)}{\cos^2 i} = \frac{v^2}{2g} \cdot \frac{\sin^2 (e-i)}{\cos^2 i} = \tfrac{1}{4}R \cdot \frac{\sin (e-i)}{\cos e} = \tfrac{1}{8}gt^2$$

$$v = \sqrt{2ag} = \cos i \sqrt{\frac{gR}{2 \cos e \sin (e-i)}} = \tfrac{1}{2}gt \cdot \frac{\cos i}{\sin (e-i)} = \sqrt{2gh} \cdot \frac{\cos i}{\sin (e-i)}$$

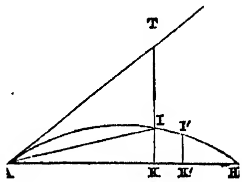
$$t = \frac{2 \sin (e-i)}{\cos i} \sqrt{\frac{2a}{g}} = \frac{2v}{g} \cdot \frac{\sin (e-i)}{\cos i} = \sqrt{\frac{2R}{g} \cdot \frac{\sin (e-i)}{\cos i}} = 2\sqrt{\frac{2h}{g}}.$$

Also $\sin (2e - i) = \frac{gR}{v^2} \cos^2 i + \sin i$; whence e may be found.

35. The principal properties of a projectile in a non-resisting medium, may be very elegantly deduced, by the method adopted in the following proposition.

PROP. XIX.

36. *A body is projected from a point* A, *with a velocity* v, *in the direction* AT, *making an angle* e *with the horizon; it is required to find where it will strike the plane* AI, *passing through the point of projection, and making an angle* i *with the horizontal plane* AH.



Let t=time of flight, or the time in which the body describes the path AVI.

$R =$ AI, the range on the oblique plane AI.

$g = 32\frac{1}{6}$ feet, the accelerating force of gravity.

$\because\ tv =$ AT $=$ space described in t seconds by velocity of projection.

$\frac{1}{2}gt^2 =$ TI $=$ space described by force of gravity in t seconds.

$$\text{Hence } \frac{AT}{AI} = \frac{\sin AIT}{\sin ATI} = \frac{\sin AIK}{\cos TAH} = \frac{\cos IAK}{\cos TAH} = \frac{\cos i}{\cos e}$$

$$\frac{tv}{R} = \frac{\cos i}{\cos e} \quad \ldots\ldots\ldots\ldots\ldots \quad (a)$$

$$\text{Again } \frac{AT}{TI} = \frac{\cos IAK}{\sin TAI} = \frac{\cos i}{\sin (e-i)}$$

$$\therefore \frac{2tv}{gt^2} = \frac{\cos i}{\sin (e-i)} \ldots\ldots\ldots\ldots \quad (b)$$

From equations (a) and (b) we have

$$R = tv \cdot \frac{\cos e}{\cos i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$t = \frac{2v}{g} \cdot \frac{\sin (e-i)}{\cos i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\left.\begin{aligned} \because R &= \frac{2v^2}{g} \cdot \frac{\cos e \sin (e-i)}{\cos^2 i} \\ &= \frac{v^2}{g} \cdot \frac{\sin (2e-i) - \sin i}{\cos^2 i} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

If a = altitude through which a body must fall from rest to acquire the velocity of projection; then $v^2 = 2ag$, and

$$R = 4a \cdot \frac{\cos e \sin (e - i)}{\cos^2 i} = 2a \cdot \frac{\sin (2e - i) - \sin i}{\cos^2 i} \ldots\ldots\ldots\ldots (4)$$

$$t = \frac{2\sqrt{2ag}}{g} \cdot \frac{\sin (e - i)}{\cos i} = \sqrt{\frac{2a}{g}} \cdot \frac{2 \sin (e - i)}{\cos i} \ldots\ldots\ldots\ldots (5)$$

Cor. 1. When the plane AI is horizontal; then $i = 0$, and we have

$$R = \frac{2v^2}{g} \cdot \cos e \sin e = \frac{v^2}{g} \sin 2e = 2a \sin 2e \ldots\ldots\ldots\ldots (6)$$

$$t = \frac{2v}{g} \cdot \sin e = \sqrt{\frac{2a}{g}} \cdot 2 \sin e \ldots\ldots\ldots\ldots (7)$$

Cor. 2. If AK $= x$ and KI $= y$; then we have

$$y = KT - TI = x \tan e - \tfrac{1}{2}gt^2.$$

But $x = AK = AT \cos TAK = tv \cos e$, and eliminating t by these two equations we have

$$y = x \tan e - \frac{gn^2}{2x^2 \cos^2 e} = x \tan e - \frac{x^2}{4\,a \cos^2 e}$$

$$= x \tan e - \frac{g}{2\,v^2} \cdot x^2 \sec^2 e = x \tan e - \frac{1}{4\,a} \cdot x^2 \sec^2 e \,.\,. (8)$$

This is the equation to the curve, and is of great advantage in the solution of equations in reference to projectiles.

PRACTICAL RULES IN PROJECTILES.

I. The velocity varies nearly as the square root of the charge, when the shot are the same; that is, if V and v are the velocities, and C and c the charges; then

$$\frac{V}{v} = \frac{\sqrt{C}}{\sqrt{c}} \text{ nearly.}$$

II. With equal charges, the velocity varies inversely, as the square root of the weight; that is, if B and b are the weights of two shots; then

$$\frac{V}{v} = \frac{\sqrt{b}}{\sqrt{B}} \text{ nearly.}$$

III. When unequal shot are projected with unequal charges, then

$$\frac{V}{v} = \frac{\sqrt{C}}{\sqrt{c}} \cdot \frac{\sqrt{b}}{\sqrt{B}} \text{ nearly.}$$

IV. If the charges are proportional to the weight of the shot; then the velocities will be the same for all shot. For let $C = m\,B$ and $c = m\,b$; then we have

$$\frac{V}{v} = \frac{\sqrt{C}}{\sqrt{c}} \cdot \frac{\sqrt{b}}{\sqrt{B}} = \frac{\sqrt{m\,B}}{\sqrt{m\,b}} \cdot \frac{\sqrt{b}}{\sqrt{B}} = 1 \;\therefore\; V = v.$$

V. It has been found by experiment, that if the charge be $\frac{1}{3}$ of the weight of the shot; then the velocity is 1600 feet per second nearly.

Let $c = \frac{1}{3}b$ and $v = 1600$; then we have

$$\frac{V}{v} = \frac{\sqrt{C}}{\sqrt{c}} \cdot \frac{\sqrt{b}}{\sqrt{B}} = \frac{\sqrt{C}}{\sqrt{\frac{1}{3}b}} \cdot \frac{\sqrt{b}}{\sqrt{B}} = \sqrt{\frac{3C}{B}} \;\therefore\; V = 1600\sqrt{\frac{3C}{B}}.$$

VI. With the same elevation, the range varies as the charge; that is, if R and R_1 are the ranges, then

$$\frac{R}{R_1} = \frac{V^2}{v^2} = \frac{C}{c}$$

VII. When the plane is horizontal, and the velocity the same, the range varies as the sine of twice the elevation; that is, if e and e_1 are the elevations, then $R : R_1 :: \sin 2e : \sin 2e_1$.

EXAMPLE IN PROJECTILES.

Find the velocity and angle of elevation, that the projectile may pass through the two given points I, I′; supposing AK = 300, AK′ = 400, KI = 60, and K′I′ = 40 feet.

$$\text{Here we have } y = x \tan e - \frac{g}{2v^2} \cdot x^2 \sec^2 e \ldots\ldots\ldots\ldots (1)$$

$$y_1 = x \tan e - \frac{g}{2v^2} x_1^2 \sec^2 e \ldots\ldots\ldots\ldots (2)$$

$$\therefore \text{ by (1) } \frac{g \sec^2 e}{2v^2} = \frac{\tan e}{x} - \frac{y}{x^2} \ldots\ldots\ldots\ldots (3)$$

$$\text{by (2) } \frac{g \sec^2 e}{2v^2} = \frac{\tan e}{x_1} - \frac{y_1}{x_1^2} \ldots\ldots\ldots\ldots (4)$$

$$\therefore \tan e \left(\frac{1}{x} - \frac{1}{x_1}\right) = \frac{y}{x^2} - \frac{y_1}{x_1^2}.$$

$$\therefore \tan e = \frac{x_1^2 y - x^2 y_1}{x x_1 (x_1 - x)}$$

$$= \frac{400^2.60 - 300^2.40}{300.400.100} = \tfrac{1}{2}.$$

$$\therefore e = \tan^{-1}\tfrac{1}{2} = 26^\circ\ 33'\ 54'' = \text{angle of elevation.}$$

From (1) $v = x \sec e \sqrt{\dfrac{\frac{1}{2}g}{x \tan e - y}} = 25\sqrt{g} = 141{\cdot}79$ feet, the velocity.

Also, if in equation (1) we make $y = 0$, we have the range

$$AH = x = \frac{v^2}{g} \sin 2e = 625 \sin 2e = 500 \text{ feet.}$$

To determine the greatest height of the projectile above the horizontal plane, we must find the maximum value of y from equation (1); hence by differentiating the equation

$$y = x \tan e - \frac{g}{2v^2} \cdot x^2 \sec^2 e$$

$$\text{we have } \frac{dy}{dx} = \tan e = \frac{g}{v^2} x \sec^2 e = 0 \ldots\ldots\ldots\ldots (5)$$

$$\therefore x = \frac{v^2}{g} \cos^2 e \tan e = 2\,a \sin e \cos e = a \sin 2e \ldots (6)$$

which is evidently half the whole range; therefore putting this value for x in the equation of the curve we have for y

$$y = 2a \sin^2 e - a \sin^2 e = a \sin^2 e \ldots\ldots\ldots\ldots (7)$$

$$= \frac{v^2}{2g} \cdot \sin^2 e = \frac{625}{2} \sin^2 e = 62{\cdot}5 \text{ feet.}$$

ADDITIONAL EXAMPLES FOR EXERCISE.

EXAMPLE 1. If a ball of 1lb. acquire a velocity of 1600 feet per second, when fired with $5\frac{1}{3}$ ounces of powder; it is required to find with what velocity

each of the several kinds of shells will be discharged by the full charges powder, viz.

Nature of the shells in inches ...	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs.	196	90	48	16	8
Charge of the powder in lbs. ...	9	4	2	1	$\frac{1}{2}$
Ans. The velocities in lbs.	594	584	595	693	693

EXAM. 2.—If a shell be found to range 1000 yards, when discharged at an elevation of 45°; how far will it range when the elevation is 30° 16′, the charge of powder being the same? Ans. 2612 feet, or 871 yards.

EXAM. 3.—The range of a shell at 45° elevation, being found to be 3750 feet; at what elevation must the piece be set, to strike an object at the distance of 2810 feet, with the same charge of powder? Ans. at 24° 16′, or at 65° 44′.

EXAM. 4.—With what impetus, velocity, and charge of powder, must a 13 inch shell be fired at an elevation of 32° 12′, to strike an object at the distance of 3250 feet? Ans. impetus 1802, velocity 340, charge 2·95 lb.

EXAM. 5.—A shell being found to range 3500 feet, when discharged at an elevation of 25° 12′; how far then will it range at an elevation of 36° 15′, with the same charge of powder? Ans. 4332 feet.

EXAM. 6.—If, with a charge of 9 lbs. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases? Ans. $6\frac{3}{4}$ lbs. of powder.

EXAM. 7.—What will be the time of flight for any given range, at the elevation of 45°?

Ans. The time in seconds is $\frac{1}{4}$ the square root of the range in feet.

EXAM. 8.—In what time will a shell range 3250 feet at an elevation of 32°? Ans. $11\frac{1}{4}$ seconds, nearly.

EXAM. 9.—How far will a shot range on a plane which ascends 8° 15′, and another which descends 8° 15′; the impetus being 3000 feet, and the elevation of the piece 32° 30′?

Ans. 4244 feet on the ascent, and 6745 feet on the descent.

EXAM. 10.—How much powder will throw a 13 inch shell 4244 feet on an inclined plane which ascends 8° 15′, the elevation of the mortar being 32° 30′? Ans. 4·92535 lb. or 4 lb. 15 oz. nearly.

EXAM. 11.—At what elevation must a 13 inch mortar be pointed, to range 6745 feet, on a plane which descends 8° 15′; the charge being $4\frac{15}{16}$ lb. of powder? Ans. 32° 8′.

EXAM. 12.—In what time will a 13 inch shell strike a plane which rises 8° 30′, when elevated 45°, and discharged with an impetus of 2304 feet?

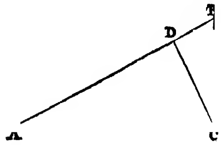
Ans. $11\frac{2}{3}$ seconds.

37. Suppose in ricochet firing AK=1600 feet, KI=12 feet, and KH=200 feet; required the elevation and velocity, that the projectile may just clear I and hit H.

DESCENTS ON INCLINED PLANES.

PROP. XX.

38. *Let* i = *inclination of a plane to the horizon; then* f *the force accelerating down the plane is a uniform force, and* f = g *sin* i.



Let AB be the inclined plane, and angle BAC = i. Draw CD at right angles to AB; then if the force of gravity be represented by BC, the effective part of it which accelerates the body down the plane is BD; hence we have

$f =$ BD = BC sin BCD = BC sin A = $g \sin i$.

Cor. 1. Let l = length of plane AB, and h = height BC; then we have $f = g \sin i = g . \frac{h}{l}$, and if this value of f be substituted for g in the several expressions deduced in Prop. XIII. for bodies falling freely, we shall have

$$s = \tfrac{1}{2}gt^2 \sin i = \frac{v^2}{2g \sin i} = \tfrac{1}{2}tv \ldots\ldots (1)$$

$$v = gt \sin i = \frac{2h}{t} = \sqrt{2gl \sin i} \ldots\ldots (2)$$

$$t = \frac{v}{g \sin i} = \frac{2h}{v} = \sqrt{\frac{2l}{g \sin i}} \ldots\ldots (3)$$

Cor. 2. Let u = velocity with which a body is projected up or down the plane; then we have as before

$$s = tu \mp \tfrac{1}{2}gt^2 \sin i = \frac{u^2 \mp v^2}{2g \sin i} \ldots\ldots (4)$$

$$v = u \mp gt \sin i \ldots\ldots (5)$$

Cor. 3. If $i = o$, and R = constant resistance to motion; then

$$s = tu - \tfrac{1}{2}\mathrm{R}t^2 = \frac{u^2 - v^2}{2\mathrm{R}} \ldots\ldots (6)$$

$$v = u - \mathrm{R}t \ldots\ldots (7)$$

When the motion ceases $v = o$, and $t = u \div \mathrm{R}$.

PROP. XXI.

39. *The velocity acquired by a body descending freely down an inclined plane* AB, *is to the velocity acquired by a body falling perpendicularly, in the same time, as the height of the plane* BC *is to its length* AB.

For by Cor. 1. Prop. XX. we have $v = gt \sin i$, and if the body fall vertically, then $v_1 = gt$; hence we have

$$v : v_1 :: gt \sin i : gt :: \sin i : 1$$

$$:: \frac{\mathrm{BC}}{\mathrm{BA}} : 1 :: \mathrm{BC} : \mathrm{BA} :: h : l.$$

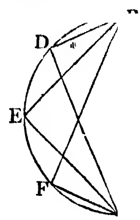
Cor. 1. Hence it is very evident that the velocities are as the times of descending from rest; that the spaces descended are as the squares of the velocities, or squares of the times; and that if a body be thrown up an inclined plane, with

the velocity it acquired in descending, it will lose all its motion, and ascend to the same height, in the same time, and will repass any point of the plane with the same velocity as it passed it in descending.

Cor. 2. Hence also, the space descended down an inclined plane, is to the space descended perpendicularly, in the same time, as the height of the plane CB, to its length AB, or as the sine of inclination to radius. For the spaces described by any forces, in the same time, are as the forces, or as the velocities.

Cor. 3. Consequently, the velocities and spaces descended, by bodies down different inclined planes, are as the sines of elevation of the planes.

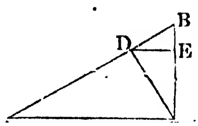
Cor. 4. If CD be drawn perpendicular to AB; then while a body falls freely through the perpendicular space BC, another body will in the same time descend down the part of the plane BD. For, by similar triangles BC : BD : : BA : BC, that is, as the spaces descended by cor. 2.



Or, in any right-angled triangle BDC, having its hypotenuse BC perpendicular to the horizon, a body will descend down any of its three sides BD, BC, DC, in the same time. And therefore, if upon the diameter BC a circle be described, the times of descending down any chords, BD, BE, BF, DC, EC, FC, &c. will be all equal, and each equal to the time of falling freely through the perpendicular diameter BC.

PROP. XXII.

40. *The time of descending down the inclined plane* BA, *is to the time of falling through the height of the plane* BC, *as the length* BA *to the height* BC.



For by Cor. 1. Prop. xx. we have $t = \sqrt{\frac{2l}{g \sin i}}$

d when the body falls vertically, $t_1 = \sqrt{\frac{2h}{g}}$; hence we have

$$t : t_1 :: \sqrt{\frac{2l}{g \sin i}} : \sqrt{\frac{2h}{g}} :: \sqrt{\frac{l}{\sin i}} : \sqrt{h}$$

$$\sqrt{\frac{l^2}{h}} : \sqrt{h} \quad :: \quad l : h.$$

Cor. 1. If i_1 = inclination of another plane of the same height BC, and l_1 its length; then if t_1 be the time of descending down this plane, we have

$$t \,.\, t_1 :: \frac{v}{g \sin i} : \frac{v}{g \sin i_1} :: \sin i_1 : \sin i$$

$$:: \frac{h}{l_1} : \frac{h}{l} :: l : l_1$$

that is, the times of descending down different planes of the same height, are to one another as the lengths of the planes.

PROP. XXIII.

41. *A body acquires the same velocity in descending down any inclined plane* BA, *as by falling perpendicularly through the height of the plane* BC.

For by Cor. 1, Prop. xx. we have $v = \frac{2h}{t}$, and by the formulæ for descents by gravity we get $v = \frac{2h}{t}$; hence the velocities acquired are equal.

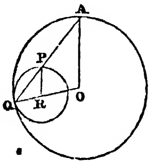
Cor. 1. Hence, the velocities acquired by bodies descending down any planes, from the same height to the same horizontal line, are equal.

Cor. 2. If the velocities be equal, at any two equal altitudes D, E; they will be equal, at all other equal altitudes A, C.

Cor. 3. Hence, also, the velocities acquired by descending down any planes, are as the square roots of the heights.

PROP. XXIV.

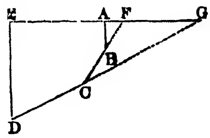
42. *To find the plane of quickest descent from a point within a vertical circle, to its circumference.*



Let P be the given point, O the centre of the given circle, and OA vertical. Join AP and produce it to meet the given circle in Q; then PQ is the plane of quickest descent. For join QO and draw PR parallel to AO. Then since AO = OQ; therefore PR = RQ, and a circle described from centre R with radius RP or RQ, will pass through P, and touch internally the given circle at Q; hence the time down the plane PQ will be less than that down any other plane, from P to the circumference of the given circle; because every plane from P to the circumference of the circle, whose centre is O, except PQ, will fall partly without the circle, whose centre is R.

PROP. XXV.

43. *If a body descend down any number of contiguous planes* AB, BC, CD, *it will at last acquire the same velocity as a body falling perpendicularly through the same height* ED, *supposing the velocity not altered by changing from one plane to another.*



Produce the planes DC, CB, to meet the horizontal line EA produced in F and G. Then by Prop. XXIII. the velocity at B is the same, whether the body descend through AB or FB. And therefore the velocity at C will be the same, whether the body descend through ABC or through FC, which is also again, by Prop. XXIII. the same as by descending through GC. Consequently, it will have the same velocity at D, by descending through the planes AB, BC, CD, as by descending through the plane GD; supposing no obstruction to the motion by the body impinging on the planes at B and C; and this again is the same velocity as by descending through the same perpendicular height ED.

Cor. 1. If the lines ABCD, &c., be supposed indefinitely small, they will form a curve line, which will be the path of the body: from which it appears that a body acquires also the same velocity in descending along any curve, as in falling perpendicularly through the same height.

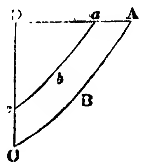
Cor. 2. Hence, also, bodies acquire the same velocity by descending from the same height, whether they descend perpendicularly, or down any planes, or down any curve or curves. And if their velocities be equal, at any one height, they will be equal at all other equal heights. Therefore, the velocity acquired by descending down any lines or curves, are as the square roots of the perpendicular heights.

Corol. 3. And a body, after its descent through any curve, will acquire a velocity which will carry it to the same height through an equal curve, or through any other curve, either by running up the smooth concave side, or by being retained in the curve by a string, and vibrating like a pendulum: also, the velocities will be equal, at all equal altitudes; and the ascent and descent will be performed in the same time, if the curves be the same.

PROP. XXVI.

44. *The times in which bodies descend through similar parts of similar curves* ABC, abc, *placed alike, are as the square roots of their lengths.*

That is, the time in AC is to the time in *ac*, as $\sqrt{AC}$ to $\sqrt{ac}$.

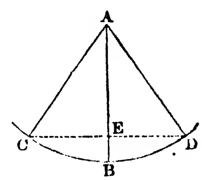


For, as the curves are similar, they may be considered as made up of an equal number of corresponding parts, which are every where, each to each, proportional to the whole; and as they are placed alike, the corresponding small similar parts will also be parallel to each other. But the time of describing each of these pairs of corresponding parallel parts, by article 39, are as the square roots of their lengths, which, by the supposition, are as $\sqrt{AC}$ to $\sqrt{ac}$, the roots of the whole curves. Therefore, the whole times are in the same ratio of $\sqrt{AC}$ to $\sqrt{ac}$.

Corol. 1. Because the axes DC, Dc, of similar curves, are as the lengths of the similar parts AC, *ac*, therefore the times of descent in the curves AC, *ac*, are as $\sqrt{DC}$ to $\sqrt{Dc}$, the square roots of their axes.

Corol. 2. As it is the same thing, whether the bodies run down the smooth concave side of the curves, or be made to describe those curves by vibrating like a pendulum, the lengths being DC, Dc; therefore, the times of the vibration of pendulums, in similar arcs of any curves, are as the square roots of the lengths of the pendulums.

45. SCHOLIUM.—Having, in the last corollary, mentioned the pendulum, it may not be improper here to add some remarks concerning it.

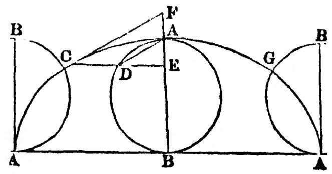


A pendulum consists of a ball, or any other heavy body B, hung by a fine string or thread, moveable about a centre A, and describing the arc CBD; by which vibration the same motion happens to this heavy body, as would happen to any body descending by its gravity along the spherical superficies CBD, if that superficies was perfectly hard and smooth. If the pendulum be carried to the situation AC, and then let fall, the ball in descending will describe the arc CB, and in the point B it will have that velocity which is acquired by descending through CB, or by a body falling freely through EB. This velocity will be sufficient to cause the ball to ascend through an equal arc BD, to the same height D from whence it fell at C: having there lost all its motion, it will again begin to descend by its own gravity; and in the lowest point B it will acquire the same velocity as before, which will cause it to reascend to C; and thus, by ascending and descending, it will perform continual vibrations in the circumference CBD. And if the motions of pendulums met with no resistance from the air, and if there were no friction at the

centre of motion A, the vibrations of pendulums would never cease. But from those obstructions, though small, it happens, that the velocity of the ball in the point B is a little diminished in every vibration; and, consequently, it does not return precisely to the same points C or D, but the arcs described continually become shorter and shorter, till at length they grow insensible; unless the motion be assisted by a mechanical contrivance, as in clocks, called a maintaining power.

DEFINITION.

46. If the circumference of a circle be rolled on a right line, beginning at any point A, and continued till the same point A arrive at the line again, making just one revolution, and thereby measuring out a straight line ABA equal to the circumference of the circle, while the point A in the circumference traces out a curve line ACAGA: then this curve is called a cycloid; and some of its properties are contained in the following lemma:



LEMMA.

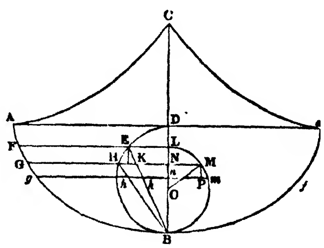
47. If the generating or revolving circle be placed in the middle of the cycloid, its diameter coinciding with the axis AB, and from any point there be drawn the tangent CF, the ordinate CDE perpendicular to the axis, and the chord of the circle AD; then the chief properties are these:

The right line CD = the circular arc AD;
The cycloidal arc AC = double the chord AD;
The semi-cycloid ACA = double the diameter AB, and
The tangent CF is parallel to the chord AD.

PROP. XXVII.

48. *When a pendulum vibrates in a cycloid, the time of one vibration, is to the time in which a body falls through half the length of the pendulum, as the circumference of a circle is to its diameter.*

Let ABa be the cycloid; DB its axis, or the diameter of the generating semicircle DEB; CB = 2DB the length of the pendulum, or radius of curvature at B. Let the ball descend from F, and, in vibrating describe the arc FBf. Divide FB into innumerable small parts, one of which is Gg; draw FEL, GM, gm, perpendicular to DB. On LB describe the semicircle LMB, whose centre is O; draw MP parallel to DB; also draw the chords BE, BH, EH, and the radius OM.



Now, the triangles BEH, BHK, are similar; therefore, BK : BH :: BH : BE, or BH^2 = BK . BE, or BH = $\sqrt{BK . BE}$. Also, the similar triangle MmP,

MON, give $Mp : Mm :: MN : MO$. And, by the nature of the cycloid, Hh is equal and parallel to Gg.

If another body descend down the chord EB, it will have the same velocity as the ball in the cycloid has at the same height. So that Kk and Gg are passed over with the same velocity, and consequently the time in passing them will be as their lengths Gg, Kk, or as Hh to Kk, or BH to BK by similar triangles, or $\sqrt{BK . BE}$ to BK, or $\sqrt{BE}$ to $\sqrt{BK}$, or as $\sqrt{BL}$ to $\sqrt{BN}$ by similar triangles.

That is, the time in Gg : time in $Kk :: \sqrt{BL} : \sqrt{BN}$.

Again, the time of describing any space with a uniform motion, is directly as the space, and reciprocally as the velocity; also, the velocity in K or Kk, is to the velocity at B, as $\sqrt{EK}$ to $\sqrt{EB}$, or as $\sqrt{LN} : \sqrt{LB}$; and the uniform velocity for EB is equal to half that at the point B, therefore the time in Kk : time in EB $:: \dfrac{Kk}{\sqrt{LN}} : \dfrac{EB}{\frac{1}{2}\sqrt{LB}}$ :: (by sim. tri.) $\dfrac{Nn}{\sqrt{LN}} : \dfrac{LB}{\frac{1}{2}\sqrt{LB}} :: Nn$ or $Mp : 2\sqrt{BL . LN}$.

That is, the time in Kk : time in EB :: $Mp : 2\sqrt{BL . LN}$.

But it was, time in Gg : time in $Kk :: \sqrt{BL} : \sqrt{BN}$; therefore,

By comp., time in Gg : time in EB :: $Mp : 2\sqrt{BN . NL}$ or 2NM.

But, by sim. tri., Mm : 2OM or BL : Mp : 2NM.

Therefore, time in Gg : time in EB :: Mm : BL.

Consequently, the sum of all the times in all the Gg's is to the time in EB, or the time in DB, which is the same thing, as the sum of all the Mm's is to LB;

That is, the time in Fg : time in DB :: Lm : LB.
And the time in FB : time in DB :: LMB : LB.
Or the time in FBf : time in DB :: 2LMB : LB.

That is, the time of one whole vibration,
is to the time of falling through half CB,
as the circumference of any circle,
is to its diameter.

Corol. Hence all the vibrations of a pendulum in a cycloid, whether great or small, are performed in the same time; which time is to the time of falling through the axis, or half the length of the pendulum, as 3·1416 to 1, the ratio of the circumference to its diameter; and hence that time is easily found thus. Put $p = 3{\cdot}1416$, and l the length of the pendulum, also $\frac{1}{2}g$ the space fallen through by a heavy body in 1″ of time:

Then $\sqrt{\tfrac{1}{2}g} : \sqrt{\tfrac{1}{2}l} :: 1'' : \sqrt{\dfrac{l}{g}}$ the time of falling through $\frac{1}{2}l$,

Therefore, $1 : p :: \sqrt{\dfrac{l}{g}} : p\sqrt{\dfrac{l}{g}}$, which therefore is the time of one vibration of the pendulum.

49. And if the pendulum vibrate in the small arc of a circle; because that small arc nearly coincides with the small cycloidal arc at the vertex B; therefore the time of vibration in the small arc of a circle, is nearly equal to the time of vibration in the cycloidal arc; and consequently the time of vibration in a small circular arc is equal to $p\sqrt{\dfrac{l}{g}}$, where l is the radius of the circle.

50. So that, if one of these, g or l, be found by experiment, this theorem will give the other. Thus, if $\frac{1}{2}g$ or the space fallen through by a heavy body in 1″ of time, be found, then this theorem will give the length of the seconds pendulum. Or, if the length of the seconds pendulum be observed by experiment, which is the easier way; this theorem will give $\frac{1}{2}g$ the descent of gravity in 1″. Now, in the latitude of London, the length of a pendulum which vibrates seconds, has been found to be $39\frac{1}{8}$ inches; and this being written for l in the theorem, it gives $p\sqrt{\frac{39\frac{1}{8}}{g}} = 1''$; and hence is found $\frac{1}{2}g = \frac{1}{2}p^2l = \frac{1}{2}p^2 \times 39\frac{1}{8} = 193{\cdot}07$ inches $= 16\frac{1}{12}$ feet, for the descent of gravity in 1″; which it has also been found to be very exactly, by many accurate experiments.

SCHOLIUM.

51. Hence is found the length of a pendulum that shall make any number of vibrations in a given time. Or, the number of vibrations that shall be made by a pendulum of a given length. Thus, suppose it were required to find the length of a half seconds pendulum, or a quarter seconds pendulum; that is, a pendulum to vibrate twice in a second, or 4 times in a second. Then, since the time of vibration is as the square root of the length,

Therefore $1 : \frac{1}{2} :: \sqrt{39\frac{1}{8}} : \sqrt{l}$,

Or $1 : \frac{1}{4} :: 39\frac{1}{8} : \frac{39\frac{1}{8}}{4} = 9\frac{3}{4}$ inches nearly, the length of the half seconds pendulum.

And $1 : \frac{1}{16} :: 39\frac{1}{8} : 2\frac{7}{16}$ inches, the length of the quarter seconds pendulum

Again, if it were required to find how many vibrations a pendulum of 80 inches long will make in a minute,

Here $\sqrt{80} : \sqrt{39\frac{1}{8}} :: 60''$ or $1' : 60\sqrt{\frac{39\frac{1}{8}}{80}} = 7\frac{1}{2}\sqrt{31{\cdot}3} = 41{\cdot}95987$, or almost 42 vibrations in a minute.

52. In these propositions, the thread is supposed to be very fine, or of no sensible weight, and the ball very small, or all the matter united in one point; also, the length of the pendulum, is the distance from the point of suspension, or centre of motion, to this point, or centre of the small ball. But if the ball be large, or the string very thick, or the vibrating body be of any other figure; then the length of the pendulum is different, and is measured, from the centre of motion, not to the centre of magnitude of the body, but to such a point, as that if all the matter of the pendulum were collected into it, it would then vibrate in the same time as the compound pendulum; and this point is called the Centre of Oscillation, which will be treated of in what follows.

The pendulum may be applied to three several important purposes.

(1.) To measure portions of time, or to subdivide the units we derive from astronomical phenomena, into smaller and equal portions.

(2.) To determine the measure of the force of gravity, at different places, and under different circumstances; and thus to enable us to infer the variation in the apparent intensity that is due to the centrifugal force; and the variation in the actual intensity at the surface, that is due to the figure of the earth. Hence the figure of the earth may be determined.

(3.) The standard unit from which all lineal measures are taken, is the length

of a pendulum vibrating seconds of mean time in the latitude of London, in a vacuum at the level of the sea, Fah. thermometer being at 62°, and the barometer at 30 inches.

OF THE CENTRES OF PERCUSSION, OSCILLATION, AND GYRATION.

53. THE Centre of Percussion of a body, or a system of bodies, revolving about a point, or axis, is that point, which striking an immoveable object, the whole mass shall not incline to either side, but rest as it were in equilibrio, without acting on the centre of suspension.

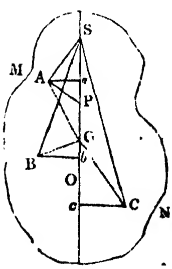
54. The Centre of Oscillation is that point, in a vibrating body, in which if any body be placed, or if the whole mass be collected, it will perform its vibrations in the same time, and with the same angular velocity, as the whole body, about the same point or axis of suspension.

55. The Centre of Gyration, is that point, in which, if the whole mass be collected, the same angular velocity will be generated in the same time, by a given force acting at any place, as in the body or system itself.

56. The angular motion of a body, or system of bodies, is the motion of a line connecting any point and the centre or axis of motion; and is the same in all parts of the same revolving body. And in different, unconnected bodies, each revolving about a centre, the angular velocity is as the absolute velocity directly, and the distance from the centre inversely; so that, if their absolute velocities be as their radii or distances, the angular velocities will be equal.

PROP. XXVIII.

57. *To find the centre of percussion of a body, or system of bodies.*



LET the body revolve about an axis passing through any point S in the line SGO, passing through the centres of gravity and percussion, G and O. Let MN be the section of the body, or the plane in which the axis SGO moves. And conceive all the particles of the body to be reduced to this plane, by perpendiculars let fall from them to the plane; a supposition which will not affect the centres G, O, nor the angular motion of the body.

Let A be the place of one of the particles, so reduced; join SA, and draw AP perpendicular to AS, and Aa perpendicular to SGO: then AP will be the direction of A's motion, as it revolves about S; and the whole mass being stopped at O, the body A will urge the point P forward, with a force proportional to its quantity of matter and velocity; or to its matter and distance from the point of suspension S; that

is, as $A . SA$; and the efficacy of this force in a direction perpendicular to SO, at the point P, is as $A . Sa$, by similar triangles; also, the effect of this force on the lever, to turn it about O, being as the length of the lever, is as $A . Sa . PO = A . Sa . \overline{SO - SP} = A . Sa . SO - A . Sa . SP = A . Sa . SO - A . SA^2$. In like manner, the forces of B and C, to turn the system about O, are as,

$$B . Sb . SO - B . SB^2, \text{ and,}$$
$$C . Sc . SO - C . SC^2, \text{ \&c.}$$

But, since the forces on the contrary sides of O destroy one another, by the definition of this force, the sum of the positive parts of these quantities, must be equal to the sum of the negative parts,

that is, $A . Sa . SO + B . Sb . SO + C . Sc . SO$, &c. $=$ - - -
$A . SA^2 + B . SB^2 + C . SC^2$, &c.;

and hence $SO = \frac{A . SA^2 + B . SB^2 + C . SC^2 \text{ \&c.}}{A . Sa + B . Sb + C . Sc \text{ \&c.}}$, which is the distance of the centre of percussion below the axis of motion.

And here it must be observed that, if any of the points a, b, &c. fall on the contrary side of S, the corresponding product $A . Sa$, or $B . Sb$, &c. must be made negative.

Corol. 1. Since, by cor. 3, pr. 15, $A + B + C$, &c. or the body $b \times$ the distance of the centre of gravity, SG, is $= A . Sa + B . Sb + C . Sc$, &c. which is the denominator of the value of SO; therefore the distance of the centre of percussion is $SO = \frac{A . SA^2 + B . SB^2 + C . SC^2 \text{ \&c.}}{SG \times \text{body } b}$

Corol. 2. Since, by Geometry, theor. 36, 37,

$$\text{it is } SA^2 = SG^2 + GA^2 - 2SG . Ga,$$
$$\text{and } SB^2 = SG^2 + GB^2 + 2SG . Gb,$$
$$\text{and } SC^2 = SG^2 + GC^2 + 2SG . Gc, \text{ \&c.}$$

and, by cor. 5, pr. 13, the sum of the last terms is nothing, namely,

$$-2SG . Ga + 2SG . Gb + 2SG . Gc, \text{ \&c.} = 0;$$

therefore the sum of the others, or $A . SA^2 + B . SB^2$, &c. . . . . .

$$= \overline{A + B, \text{ \&c.}} . SG^2 + A . GA^2 + B . GB^2 + C . GC^2, \text{ \&c.}$$
$$\text{or} = b . SG^2 + A . GA^2 + B . GB^2 + C . GC^2, \text{ \&c.}$$

which being substituted in the numerator of the foregoing value of SO, gives,

$$SO = \frac{b . SG^2 + A . GA^2 + B . GB^2 + \text{\&c.}}{b . SG}$$

$$\text{or } SO = SG + \frac{A . GA^2 + B . GB^2 + C . GC^2, \text{ \&c.}}{b . SG}$$

Corol. 3. Hence, the distance of the centre of percussion always exceeds the distance of the centre of gravity, and the excess is always $GO = \frac{A . GA^2 + B . GB^2, \text{ \&c.}}{b . SG}$.

58. And hence also, $SG . GO = \frac{A . GA^2 + B . GB^2, \text{ \&c.}}{\text{the body } b}$; that is, $SG . GO$ is always the same constant quantity, wherever the point of suspension S is placed; since the point G, and the bodies A, B, &c. are constant. Or GO is always reciprocally as SG, that is, GO is less, as SG is greater; and consequently the point rises upwards, and approaches towards the point G, as the

point S is removed to the greater distance; and they coincide when SC is infinite. But when S coincides with G, then GO is infinite, or O is at an infinite distance

PROP. XXIX.

59. *If a body A, at the distance SA from an axis passing through S, be made to revolve about that axis by any force f acting at P in the line SP, perpendicular to the axis of motion; it is required to determine the quantity or matter of another body Q, which, being placed at P, the point where the force acts, it shall be accelerated in the same manner, as when A revolved at the distance SA; and, consequently, that the angular velocity of A and Q about S, may be the same in both cases.*

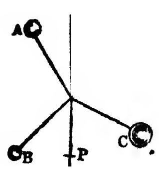
By the nature of the lever, $SA : SP :: f : \frac{SP}{SA} \cdot f$, the effect of the force f, acting at P, on the body at A; that is, the force f acting at P, will have the same effect on the body A, as the force $\frac{SP}{SA} f$, acting directly at the point A. But as ASP revolves altogether about the axis at S, the absolute velocities of the points A and S, or of the bodies A and Q, will be as the radii SA, SP of the circles described by them. Here then we have two bodies A and Q, which being urged directly by the forces f and $\frac{SP}{SA} f$, acquire velocities which are as SP and SA. But the motive forces of bodies are as their mass and velocity; therefore

$$\frac{SP}{SA} f : f :: A \cdot SA : Q \cdot SP, \text{ and } SP^2 : SA^2 :: A : Q = \frac{SA^2}{SP^2} A,$$

which therefore is the mass of matter which, being placed at P, would receive the same angular motion from the action of any force at P, as the body A receives. So that the resistance of any body A, to a force acting at any point P, is directly as the square of its distance SA from the axis of motion, and reciprocally as the square of the distance SP of the point where the force acts.

Corol. 1. Hence the force which accelerates the point P, is to the force of gravity, as $\frac{f \cdot SP^2}{A \cdot SA^2}$ to 1, or as $f \cdot SP^2$ to $A \cdot SA^2$.

Corol. 2. If any number of bodies, A, B, C, be put in motion, about a fixed axis passing through S, by a force acting at P; the point P will be accelerated in the same manner, and consequently the whole system will have the same angular velocity, if, instead of the bodies A, B, C, placed at the distances SA, SB, SC, there be substituted the bodies $\frac{SA^2}{SP^2} A$, $\frac{SB^2}{SP^2} B$, $\frac{SC^2}{SP^2} C$; these being collected into the point P. And hence, the moving force being f, and the matter moved being $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$; therefore the accelerating force

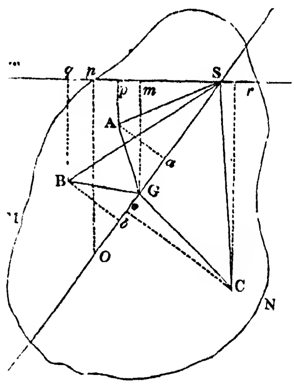


$\frac{f \cdot SP^2}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$; which is to the accelerating force of gravity as $f \cdot SP^2$ to $A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$

Corol. 3. The angular velocity of the whole system of bodies, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$. For the absolute velocity of the point P, is as the accelerating force, or directly as the motive force f, and inversely as the mass $\frac{A \cdot SA^2, \&c.}{SP^2}$: but the angular velocity is as the absolute velocity directly and the radius SP inversely; and therefore the angular velocity of P, or of the whole system, which is the same thing, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$.

PROP. XXX.

60. *To determine the centre of oscillation of any compound mass or body* MN, *or of any system of bodies* A, B, C.



Let MN be the plane of vibration, to which let all the matter be reduced, by letting fall perpendiculars from every particle, to this plane. Let G be the centre of gravity, and O the centre of oscillation; through the axis S draw SGO, and the horizontal line Sq; then from every particle A, B, C, &c. let fall perpendiculars Aa, Ap, Bb, Bq, Cc, Cr, to these two lines; and join SA, SB, SC; also, draw Gm, On perpendicular to Sq. Now, the forces of the weights A, B, C, to turn the body about the axis, are $A \cdot Sp$, $B \cdot Sq$, $- C \cdot Sr$; and therefore, by cor. 3, prop. 29, the angular motion generated by all these forces is $\frac{A \cdot Sp + B \cdot Sq - C \cdot Sr}{A \cdot SA^2 + SB^2 + C \cdot SC^2}$. Also, the angular velocity which any particle p, placed in O, generates, in the system, by its weight, is $\frac{p \cdot Sn}{p \cdot SO^2}$ or $\frac{Sn}{SO^2}$, or $\frac{Sm}{SG \cdot SO}$, because of the similar triangles SGm, SOn. But, by the problem, the vibrations are performed alike in both cases, and therefore these two expressions must be equal to each other, that is,

$$\frac{Sm}{SG \cdot SO} = \frac{A \cdot Sp + B \cdot Sq - C \cdot Sr}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$$

And hence $SO = \frac{Sm}{SG} \times \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A \cdot Sp + B \cdot Sq - C \cdot Sr}$.

But, by cor. 2, prop. 15, the sum $A \cdot Sp + B \cdot Sq - C \cdot Sr = (A + B + C)$ Sm; therefore the distance $SO = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SG \cdot (A + B + C)}$.

$= \frac{A . SA^2 + B . SB^2 + C . SC^2}{A . Sa + B . Sb + C . Sc}$ by prop. 16, the distance of the centre of oscillation O, below the axis of suspension; where any of the products A . Sa, B . Sb, must be negative, when a, b, &c. lie on the other side of S; which is the same expression as that for the distance of the centre of percussion, found in prop. 29.

Hence it appears, that the centres of percussion and of oscillation are in the very same point. And therefore the properties in all the corollaries there found for the former, are to be here understood of the latter.

Corol. 1. If m be any particle of a body and r its distance from the axis of motion S; also, G, O, the centres of gravity and oscillation. Then the distance of the centre of oscillation of the body, from the axis of motion, is $SO = \frac{\Sigma (mr^2)}{SG . \Sigma m} = \frac{\text{moment of inertia}}{SG . \text{mass}}$.

Corol. 2. If b denote the matter in any compound body, whose centres of gravity and oscillation are G and O; the body P, which being placed at P, where the force acts as in the last proposition, and which receives the same motion from that force as the compound body b, is $P = \frac{SG . SO}{SP^2} . b$.

For, by corol. 2, prop. 29, this body $P = \frac{A . SA^2 + B . SB^2 + C . SC^2}{SP^2}$.
But, by corol. 1, prop. 28, $SG . SO . b = A . SA^2 + B . SB^2 + C . SC^2$; therefore $P = \frac{SG . SO}{SP^2} . b$.

61. Scholium.—By the integral calculus, the centre of oscillation, for a regular body, will be found from cor. 1. But for an irregular one; suspend it at the given point; and hang up also a simple pendulum of such a length, that, making them both vibrate, they may keep time together. Then the length of the simple pendulum, is equal to the distance of the centre of oscillation of the body, below the point of suspension.

62. Or it will be still better found thus: suspend the body very freely by the given point, and make it vibrate in small arcs, counting the number of vibrations it makes in any time, as a minute, by a good stop watch; and let that number of vibrations made in a minute be called n: then shall the distance of the centre of oscillation, be $SO = \frac{140850}{nn}$ inches. For, the length of the pendulum vibrating seconds, or 60 times in a minute, being $39\frac{1}{8}$ inches, and the lengths of pendulums being reciprocally as the square of the number of vibrations made in the same time; therefore,

$$n^2 : 60^2 : : 39\tfrac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{nn} = \frac{140850}{nn},$$

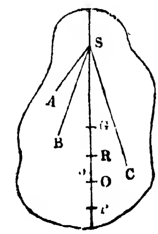
the length of the pendulum which vibrates n times in a minute, or the distance of the centre of oscillation below the axis of motion.

63. The foregoing determination of the point, into which all the matter of a body being collected, it shall oscillate in the same manner as before, only respects the case in which the body is put in motion by the gravity of its own particles, and the point is the centre of oscillation: but when the body is put in motion by some other extraneous force, instead of its gravity, then the point is

different from the former, and is called the Centre of Gyration; which is determined in the following manner:

PROP. XXXI.

64. *To determine the centre of gyration of a compound body, or of a system of bodies.*



Let R be the centre of gyration, or the point into which all the particles A, B, C, &c. being collected, it shall receive the same angular motion from a force f acting at P, as the whole system receives.

Now, by cor 3, prop. 29, the angular velocity generated in the system, by the force f, is as . . . $\frac{f . SP}{A . SA^2 + B . SB^2, \&c.}$; and, by the same, the angular velocity of the system placed in R, is $\frac{f . SP}{(A + B + C, \&c.) . SR^2}$; then, by making these two expressions equal to each other, we have

$$SR = \sqrt{\frac{A . SA^2 + B . SB^2 + C . SC^2}{A + B + C}},$$

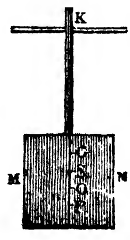
for the distance of the centre of gyration below the axis of motion.

Corol. 1. Because $A . SA^2 + B . SB^2, \&c. = SG . SO . b$, where G is the centre of gravity, O the centre of oscillation, and b the body $A + B + C$, &c.; therefore $SR^2 = SG . SO$; that is, the distance of the centre of gyration is a mean proportional between those of gravity and oscillation.

Cor. 2. If m denote any particle of a body at distance r from the axis of motion; then $SR^2 = \frac{\Sigma (mr^2)}{\Sigma m} = \frac{\text{moment of inertia}}{\text{mass}} = \frac{\int r^2 dm}{\int dm}$.

PROP XXXII

65. *To determine the velocity with which a ball moves, which being shot against a ballistic pendulum, causes it to vibrate through a given angle.*



The Ballistic Pendulum is a heavy block of wood MN, suspended vertically by a strong horizontal iron axis at K, to which it is connected by a firm iron stem. This problem is the application of the last proposition, or of prop. 29, and was invented by the very ingenious Mr Robins, to determine the initial velocities of military projectiles; a circumstance very useful in that science; and it is the only method yet known for determining them with any degree of accuracy.

Let G, S, O be the centres of gravity, gyration, and oscillation, as determined by the foregoing propositions; and let P be the point where the ball strikes the face of the pendulum, the

momentum of which, or the product of its weight and velocity, is expressed by the force f, acting at P, in the foregoing propositions.

Put $p =$ the whole weight of the pendulum,
$b =$ the weight of the ball,
$g =$ KG the dist. of the centre of gravity,
$o =$ KO the dist. of the centre of oscillation,
$r =$ KS $= \sqrt{go}$ the dist. of the centre of gyration,
$i =$ KP the distance of the point of impact,
$v =$ the velocity of the ball,
$u =$ the velocity of the point of impact P,
$c =$ chord of the arc described by the point O.

By Prop. 30, if the mass p be placed all at S, the pendulum will receive the same motion from the blow in the point P; and as $KP^2 : KS^2 :: p : \frac{KS^2}{KP^2} \cdot p$ or $\frac{r^2}{i^2} p$ or $\frac{go}{ii} p$, (prop. 29) the mass which being placed at P, the pendulum will still receive the same motion as before. Here then are two quantities of matter, namely, b and $\frac{go}{ii} p$, the former moving with the velocity v, and striking the latter at rest; to determine their common velocity u, with which they will jointly proceed forward together after the stroke. In which case, by the law of the impact of non-elastic bodies, we have $\frac{go}{ii} p + b : b :: v : u$, and therefore $v = \frac{bii + gop}{bii} u$ the velocity of the ball in terms of u, the velocity of the point P, and the known dimensions and weights of the bodies.

But now to determine the value of u, we must have recourse to the angle through which the pendulum vibrates; for when the pendulum descends down again to the vertical position, it will have acquired the same velocity with which it began to ascend, and, by the laws of falling bodies, the velocity of the centre of oscillation is such, as a heavy body would acquire by freely falling through the versed sine of the arc described by the same centre O. But the chord of that arc is c, and its radius is o; and, by the nature of the circle, the chord is a mean proportional between the versed sine and diameter, therefore $2o : c :: c : \frac{cc}{2o}$, the versed sine of the arc described by O. Then by the laws of falling bodies, $\sqrt{16\frac{1}{12}} : \sqrt{\frac{cc}{2o}} :: 32\frac{1}{6} : c\sqrt{\frac{2a}{o}}$, the velocity acquired by the point O in descending through the arc whose chord is c, where $a = 16\frac{1}{12}$ feet: and therefore $o : i :: c\sqrt{\frac{2a}{o}} : \frac{ci}{o}\sqrt{\frac{2a}{o}}$, which is the velocity u, of the point P.

Then, by substituting this value for u, the velocity of the ball, before found, becomes $v = \frac{bii + gop}{bio} \times c\sqrt{\frac{2a}{o}}$. So that the velocity of the ball, is directly as the chord of the arc described by the pendulum in its vibration.

66. Scholium.—In the foregoing solution, the change in the centre of oscillation is omitted, which is caused by the ball lodging in the point P. But the allowance for that small change, and that of some other small quantities, may be seen in my Tracts, where all the circumstances of this method are treated at full length.

67. For an example in numbers of this method, suppose the weights and dimensions to be as follow: namely,

$p = 570$ lb.
$b = 18$ oz. $1\frac{1}{2}$ dr.
$\quad = 1{\cdot}131$ lb.
$g = 78\frac{1}{2}$ inches.
$o = 84\frac{7}{9}$ inches.
$\quad 7{\cdot}065$ feet.
$i = 94\frac{3}{10}$ inches.
$c = 18{\cdot}73$ inches.

Then

$$\frac{bii + gop}{bio} \times c = \frac{1{\cdot}131 \times 94{\cdot}3^2 + 78\frac{1}{2} \times 84\frac{7}{9} \times 570}{1{\cdot}131 \times 94\frac{3}{10} \times 84\frac{7}{9}} \cdot \frac{18.73}{12} = 656{\cdot}56.$$

$$\text{And } \sqrt{\frac{2a}{o}} = \sqrt{\frac{32\frac{1}{6}}{7{\cdot}065}} = \sqrt{\frac{193}{42{\cdot}39}} = 2{\cdot}1337$$

Therefore $656{\cdot}56 \times 2{\cdot}1337$, or 1401 feet, is the velocity, per second, with which the ball moved, when it struck the pendulum.

OF HYDROSTATICS.

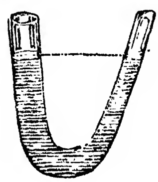
1. HYDROSTATICS is the science which treats of the pressure, or weight, and equilibrium of water, and other fluids, especially those that are non-elastic.

2. A fluid is elastic, when it can be reduced into a less bulk by compression, and which restores itself to its former bulk again when the pressure is removed; as, air. And it is non-elastic, when it is not compressible or expansible; as, water, &c.

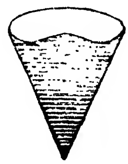
PROP. I

3. *If any part of a fluid be raised higher than the rest, by any force, and then left to itself; the higher parts will descend to the lower places, and the fluid will not rest, till its surface be quite even and level.*

FOR, the parts of a fluid being easily moveable every way, the higher parts will descend by their superior gravity, and raise the lower parts, till the whole come to rest in a level or horizontal plane.



Corol. 1. Hence, water which communicates with other water, by means of a close canal or pipe, will stand at the same height in both places. Like as water in the two legs of a syphon.



Corol. 2. For the same reason, if a fluid gravitate towards a centre; it will dispose itself into a spherical figure, the centre of which is the centre of force Like as the sea in respect of the earth.

PROP. II.

4. *When a fluid is at rest in a vessel, the base of which is parallel to the horizon; equal parts of the base are equally pressed by the fluids.*

For, upon every equal part of the base there is an equal column of the fluid supported by it. And, as all the columns are of equal height, by the last proposition, they are of equal weight, and therefore they press the base equally; that is, equal parts of the base sustain an equal pressure.

266. *Corol.* 1. All parts of the fluid press equally at the same depth.

For, if a plane parallel to the horizon be conceived to be drawn at that depth; then, the pressure being the same in any part of that plane, by the proposition, therefore the parts of the fluid, instead of the plane, sustain the same pressure at the same depth.

267. *Corol.* 2. The pressure of the fluid at any depth, is as the depth of the fluid.

For the pressure is as the weight, and the weight is as the height of the fluid.

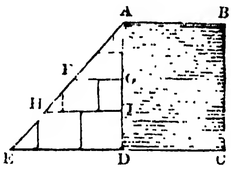
PROP. III.

5. *When a fluid is pressed by its own weight, or by any other force; at any point it presses equally, in all directions whatever.*

This arises from the nature of fluidity, by which it yields to any force in any direction. If it cannot recede from any force applied, it will press against other parts of the fluid in the direction of that force. And the pressure in all directions will be the same. For if it were less in any part, the fluid would move that way, till the pressure were equal every way.

Corol. 1. In a vessel containing a fluid; the pressure is the same against the bottom, as against the sides, or even upwards, at the same depth.

Corol. 2. Hence, and from the last proposition, if ABCD be a vessel of water, and there be taken, in the base produced, DE to represent the pressure at the bottom; joining AE, and drawing any parallels to the base, as FG, HI; then shall FG represent the pressure at the depth AG, and HI the pressure at the depth AI, and so on; because the parallels, FG, HI, ED, by sim. triangles, are as the depths, AG, AI, AD; which are as the pressures, by the proposition.



And hence the sum of all the FG, HI, &c. or area of the triangle ADE, is as the pressure against all the points G, I, &c. that is, against the line AD. But as every point in the line CD is pressed with a force as DE, and that thence the pressure on the whole line CD is as the rectangle ED . DC, while that against the side is as the triangle ADE or $\frac{1}{2}$AD . DE; therefore the pressure on the horizontal line DC, is to the pressure against the vertical line DA, as DC to $\frac{1}{2}$DA. And hence, if the vessel be an upright rectangular one, the pressure on the bottom, or whole weight of the fluid, is to the pressure against

one side, as the base is to half that side. And therefore the weight of the fluid is to the pressure against all the four upright sides, as the base is to half the upright surface. And the same holds true also in any upright vessel, whatever the sides be, or in a cylindrical vessel. Or, in the cylinder, the weight of the fluid, is to the pressure against the upright surface, as the radius of the base is to double the altitude.

Moreover, when the rectangular prism becomes a cube, it appears that the weight of the fluid on the base, is double the pressure against one of the upright sides, or half the pressure against the whole upright surface.

Corol. 3. The pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid whose base is the surface pressed, and its altitude the same as the altitude of that surface.

For the pressure on a horizontal base equal to the upright surface, is equal to that column; and the pressure on the upright surface is but half that on the base, of the same area.

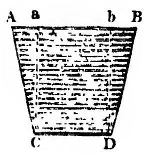
So that, if b be the breadth, and d the depth of such a gate or upright surface; then the pressure against it, is equal to the weight of the fluid whose magnitude is $\frac{1}{2}bd^2 = \frac{1}{2}AB \cdot AD^2$.

If the fluid be water, a cubic foot of which weighs 1000 ounces, or $62\frac{1}{2}$ pounds; and if the depth AD be 12 feet, the breadth AB 20 feet; then the content, or $\frac{1}{2}AB \cdot AD^2$ is 1440 feet; and the pressure is 1,440,000 ounces, or 90,000 pounds, or $40\frac{1}{5}$ tons weight nearly.

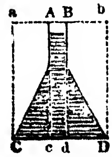
PROP. IV.

6. *The pressure of a fluid, on the base of the vessel in which it is contained, is as the base and perpendicular altitude; whatever be the figure of the vessel that contains it.*

If the sides of the base be upright, so that it be a prism of a uniform width throughout, then the case is evident; for then the base supports the whole fluid, and the pressure is just equal to the weight of the fluid.



But if the vessel be wider at top than bottom; then the bottom sustains, or is pressed by, only the part contained within the upright lines aC, bD; because the parts ACa, BDb are supported by the sides AC, BD; and those parts have no other effect on the part abDC than keeping it in its position, by the lateral pressure against aC and bD, which does not alter its perpendicular pressure downwards. And thus the pressure on the bottom is less than the weight of the contained fluid.



And if the vessel be widest at bottom; then the bottom is still pressed with a weight which is equal to that of the whole upright column ABDC. For, as the parts of the fluid are in equilibrio, all the parts have an equal pressure at the same depth; so that the parts within Cc and dD press equally as those in cd, and therefore equally the same

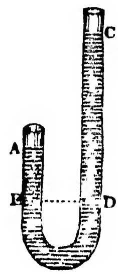
as if the sides of the vessel had gone upright to A and B, the defect of fluid in the parts ACa and BDb being exactly compensated by the downward pressure or resistance of the sides aC and bD against the contiguous fluid. And thus the pressure on the base may be made to exceed the weight of the contained fluid, in any proportion whatever.

So that, in general, be the vessels of any figure whatever, regular or irregular, upright or sloping, or variously wide and narrow, in different parts, if the bases and perpendicular altitudes be but equal, the bases always sustain the same pressure. And as that pressure, in the regular upright vessel, is the whole column of the fluid, which is as the base and altitude, therefore the pressure in all figures is in the same ratio.

Corol. 1. Hence, when the heights are equal, the pressures are as the bases. And when the bases are equal, the pressure is as the heights. But when both the heights and bases are equal, the pressures are equal in all, though their contents be ever so different.

Corol. 2. The pressure on the base of any vessel, is the same as on that of a cylinder, of an equal base and height.

Corol. 3. If there be an inverted syphon, or bent tube, ABC, containing two different fluids CD, ABD, that balance each other, or rest in equilibrio; then their heights in the two legs AE, CD, above the point of meeting, will be reciprocally as their densities.



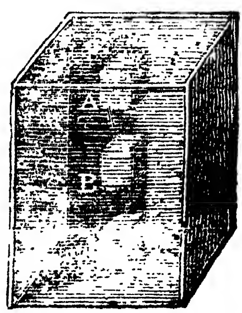
For, if they do not meet at the bottom, the part BD balances the part BE, and therefore the part CD balances the part AE; that is, the weight of CD is equal to the weight of AE. And as the surface at D is the same, where they act against each other, therefore AE : CD :: density of CD : density of AE.

So, if CD be water, and AE quicksilver, which is near 14 times heavier; then CD will be = 14AE; that is, if AE be 1 inch, CD will be 14 inches; if AE be 2 inches, CD will be 28 inches; and so on.

PROP. V.

7. *If a body be immersed in a fluid of the same density or specific gravity; it will rest in any place where it is put. But a body of a greater density will sink; and one of a less density will ascend to the top, and float.*

The body, being of the same density, or of the same weight with the like bulk of the fluid, will press the fluid under it, just as much as if its space was filled with the fluid itself. The pressure then all around it will be the same as if the fluid were in its place; consequently there is no force, neither upwards nor downwards, to put the body out of its place. And therefore it will remain wherever it is put.



But if the body be lighter, its pressure downwards will be less than before, and less than

the water upwards at the same depth; therefore the greater force will overcome the less, and push the body upwards to A.

And if the body be heavier, the pressure downwards will be greater than the fluid at the same depth; and therefore the greater force will prevail, and carry the body to the bottom at C.

Corol. 1. A body immersed in a fluid, loses as much weight, as an equal bulk of the fluid weighs. And the fluid gains the same weight.

Thus, if the body be of equal density with the fluid, it loses all its weight, and so requires no force but the fluid to sustain it. If it be heavier, its weight in the water will be only the difference between its own weight and the weight of the same bulk of water; and it requires a force to sustain it just equal to that difference. But if it be lighter, it requires a force equal to the same difference of weights to keep it from rising up in the fluid.

Corol. 2. The weights lost, by immerging the same body in different fluids, are as the specific gravities of the fluids. And bodies of equal weight, but different bulk, lose, in the same fluid, weights which are reciprocally as the specific gravities of the bodies, or directly as their bulks.

Corol. 3. The whole weight of a body, which will float in a fluid, is equal to as much of the fluid, as the immersed part of the body takes up, when it floats.

For the pressure under the floating body, is just the same as so much of the fluid as is equal to the immersed part; and therefore the weights are the same.

Corol. 4. Hence the magnitude of the whole body, is to the magnitude of the part immersed, as the specific gravity of the fluid, is to that of the body.

For, in bodies of equal weight, the densities, or specific gravities, are reciprocally as their magnitudes.

Corol. 5. And because, when the weight of a body taken in a fluid, is subtracted from its weight out of the fluid, the difference is the weight of an equal bulk of the fluid; this therefore is to its weight in the air, as the specific gravity of the fluid, is to that of the body.

Therefore, if W be the weight of a body in air,
w its weight in water, or any fluid,
S the specific gravity of the body, and
s the specific gravity of the fluid;

then $W - w : W :: s : S$, which proportion will give either of those specific gravities, the one from the other

Thus $S = \frac{W}{W - w} s$, the specific gravity of the body;

and $s = \frac{W - w}{W} S$, the specific gravity of the fluid.

So that the specific gravities of bodies, are as their weights in the air directly, and their loss in the same fluid inversely.

Corol. 6. And hence, for two bodies connected together, or mixed together into one compound, of different specific gravities, we have the following equations, denoting their weights and specific gravities, as below, viz.

H = weight of the heavier body in air, h = weight of the same in water, } S its specific gravity;
L = weight of the lighter body in air, l = weight of the same in water, } s its specific gravity;
C = weight of the compound in air, c = weight of the same in water, } f its specific gravity
w = the specific gravity of water. Then

1st, $(H - h)\,S = Hw$,
2d, $(L - l)\,s = Lw$,
3d, $(C - c)\,f = Cw$,
4th, $H + L = C$,
5th, $h + l = c$,
6th, $\frac{H}{S} + \frac{L}{s} = \frac{C}{f}$.

From which equations, may be found any of the above quantities, in terms of the rest.

Thus, from one of the first three equations, is found the specific gravity of any body, as $s = \frac{Lw}{L - l}$, by dividing the absolute weight of the body by its loss in water, and multiplying by the specific gravity of water.

But if the body L be lighter than water; then l will be negative, and we must divide by $L + l$ instead of $L - l$, and to find l we must have recourse to the compound mass C; and because, from the 4th and 5th equations, $L - l = \overline{C - c} - \overline{H - h}$, therefore $s = \frac{Lw}{(C - c) - (H - h)}$; that is, divide the absolute weight of the light body, by the difference between the losses in water, of the compound and heavier body, and multiply by the specific gravity of water. Or thus, $s = \frac{SfL}{CS - Hf}$, as found from the last equation.

Also, if it were required to find the quantities of two ingredients mixed in a compound, the 4th and 6th equations would give their values as follows, viz.

$$H = \frac{(f - s)\,S}{(S - s)\,f}\,C, \text{ and } L = \frac{(S - f)\,s}{(S - s)\,f}\,C,$$

the quantities of the two ingredients H and L, in the compound C. And so for any other demand.

PROP. VI.

To find the specific gravity of a body.

8. Case I.—*When the body is heavier than water;* weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then, by corol. 6, prop. 4, $s = \frac{Bw}{B - b}$, where B is the weight of the body out of water, b its weight in water, s its specific gravity, and w the specific gravity of water. That is,

As the weight lost in water,
Is to the whole or absolute weight,
So is the specific gravity of water,
To the specific gravity of the body.

Example. If a piece of stone weigh 10 lb. but in water only 6¾ lb., required its specific gravity, that of water being 1000? Ans. 3077.

9. Case II.—*When the body is lighter than water, so that it will not sink;* annex to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body, and the com-

pound mass, separately, both in water, and out of it; then find how much each loses in water, by subtracting its weight in water, from its weight in air; and subtract the less of these remainders from the greater. Then say,

As the last remainder,
Is to the weight of the light body in air,
So is the specific gravity in water,
To the specific gravity of the body.

That is, the specific gravity $s = \dfrac{Lw}{(C - c) - (H - h)}$ by cor. 6, prop. 4.

Example. Suppose a piece of elm weighs 15 lb. in air; and that a piece of copper, which weighs 18 lb. in air and 16 lb. in water, is affixed to it, and that the compound weighs 6 lb. in water; required the specific gravity of the elm? Ans. 600.

10. Case III.—*For a fluid of any sort.* Take a piece of a body of known specific gravity; weigh it both in and out of the fluid, finding the loss of weight by taking the difference of the two; then say,

As the whole or absolute weight,
Is to the loss of weight,
So is the specific gravity of the solid,
To the specific gravity of the fluid.

That is, the specific gravity $w = \dfrac{B - b}{B} s$, by cor. 6, prop. 4.

Example. A piece of cast iron weighed $34\frac{61}{100}$ ounces in a fluid, and 40 ounces out of it; of what specific gravity is that fluid? Ans. 1000.

PROP. VII.

11. *To find the quantities of two ingredients in a given compound.*

Take the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply each specific gravity by the difference of the other two. Then say,

As the greatest product,
Is to the whole weight of the compound,
So is each of the other two products,
To the weights of the two ingredients.

That is, the one $H = \dfrac{(f - s)\,S}{(S - s)\,f}\,C$; and the other $L = \dfrac{(S - f)\,s}{(S - s)\,f}\,C$, by cor. 6. prop. 4.

Example. A composition of 112 lb. being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and that of copper 9000?

Answer—There is 100 lb. of copper, and consequently 12 lb. of tin, } in the composition.

12. Scholium.—The specific gravities of several sorts of matter, as found from experiments, are expressed by the numbers annexed to their names in the following Table:

A Table of the Specific Gravities of bodies.

Platina (pure)	23400	Brick	2000
Fine gold	19640	Common earth	1984
Standard gold	18888	Nitre	1900
Quicksilver	13600	Ivory	1825
Lead	11325	Brimstone	1810
Fine silver	11091	Solid gunpowder	1745
Standard silver	10535	Sand	1520
Copper	9000	Coal	1250
Copper halfpence	8915	Box-wood	1030
Gun metal	8784	Sea-water	1030
Cast brass	8000	Common water	1000
Steel	7850	Oak	925
Iron	7645	Gunpowder, close shaken	937
Cast iron	7425	Ditto, in a loose heap	836
Tin	7320	Ash	800
Clear crystal glass	3150	Maple	755
Marble and hard stone	2700	Elm	600
Common green glass	2600	Fir	550
Flint	2570	Charcoal	
Common stone	2520	Cork	240
Clay	2160	Air at a mean state	$1\frac{1}{4}$

13. *Note.* The several sorts of wood are supposed to be dry. Also, as a cubic foot of water weighs just 1000 ounces avoirdupois, the numbers in this table express, not only the specific gravities of the several bodies, but also the weight of a cubic foot of each, in avoirdupois ounces; and therefore, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the next two propositions.

PROP. VIII.

14. *To find the magnitude of any body, from its weight.*

As the tabular specific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, respectively.

Example 1. Required the content of an irregular block of common stone, which weighs 1 cwt. or 112 lb.? Ans. $1228\frac{4}{5}$ cubic inches.

Example 2. How many cubic inches of gunpowder are there in 1 lb. weight? Ans. 30 cubic inches nearly.

Example 3. How many cubic feet are there in a ton weight of dry oak? Ans. $38\frac{138}{185}$ cubic feet.

PROP. IX.

15. *To find the weight of a body, from its magnitude.*

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is its tabular specific gravity,
To the weight of the body.

Example 1. Required the weight of a block of marble, whose length is 63 feet, and the breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbec?

Ans. $683\frac{4}{10}$ ton, which is nearly equal to the burthen of an East India ship.

Example 2. What is the weight of 1 pint, ale measure, of gunpowder?

Ans. 19 oz. nearly.

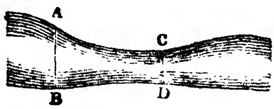
Example 3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep? Ans. $4335\frac{15}{16}$ lb.

OF HYDRAULICS.

16. HYDRAULICS is the science which treats of the motion of fluids, and the forces with which they act upon bodies.

PROP. X.

17. *If a fluid run through a canal or river, or pipe of various widths, always filling it; the velocity of the fluid in different parts of it,* AB, CD *will be reciprocally as the transverse sections in those parts.*



THAT is,

$$\text{Veloc. at A : veloc. at C} :: \frac{1}{AB} : \frac{1}{CD};$$
$$\text{or} :: CD : AB;$$

where AB and CD denote, not the diameters at A and B, but the areas, or sections, there.

For, as the channel is always equally full, the quantity of water running through AB is equal to the quantity running through CD, in the same time; that is, the column through AB is equal to the column through CD, in the same time; or AB $\times$ length of its column $=$ CD $\times$ length of its column; therefore AB : CD :: length of column through CD : length of column through AB. But the uniform velocity of the water, is as the space run over, or length of the columns; therefore AB : CD :: velocity through CD : velocity through AB.

Corol. Hence, by observing the velocity at any place AB, the quantity of water discharged in a second, or any other time, will be found, namely, by multiplying the section AB by the velocity there.

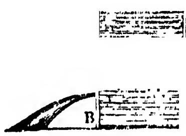
But if the channel be not a close pipe or tunnel, kept always full, but an open canal or river; then the velocity in all parts of the section will not be the same, because the velocity towards the bottom and sides will be diminished by the friction against the bed or channel; and therefore a medium among the three ought to be taken. So,

If the velocity at the top be	. .	100 feet per minute.
That at the bottom	. . . .	60
And that at the sides	. . .	50
	3)	210 sum;
Dividing their sum by 3 gives		70 the mean velocity,

which is to be multiplied by the section, to give the quantity discharged in a minute.

PROP. XI.

18. *The velocity with which a fluid runs out by a hole in the bottom or side of a vessel, kept always full, is equal to that which is generated by gravity through the height of the water above the hole; that is, the velocity of a heavy body acquired by falling freely through the height* AB.



Divide the altitude AB into a great number of very small parts, each being 1, their number a, or $a =$ the altitude AB.

Now, by prop. III. the pressure of the fluid against the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whose height is AB or a, and base the area of the hole B. Therefore the pressure on the hole, or small part of the fluid 1, is to its weight, or the natural force of gravity, as a, to 1. But since the velocities generated in the same body in any time, are as those forces; and because gravity generates the velocity 2 in descending through the small space 1, therefore $1 : a :: 2 : 2a$, the velocity generated by the pressure of the column of fluid in the same time. But $2a$, is also, (formerly shown,) the velocity generated by gravity in descending through a or AB. That is, the velocity of the issuing water, is equal to that which is acquired by a body in falling through the height AB.

Corol. 1. The velocity, and quantity run out, at different depths, are as the square roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{AB}$.

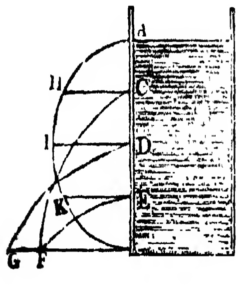
Corol. 2. The water spouts out with the same velocity, whether it be downwards or upwards, or sideways; because the pressure of fluids is the same in all directions, at the same depth. And therefore, if the adjutage be turned upwards, the jet will ascend to the height of the surface of the water in the vessel. And this is confirmed by experience, by which it is found that jets really ascend nearly to the height of the reservoir, abating a small quantity only, for the friction against the sides, and some resistance from the oblique motion of the water in the hole.

Corol. 3. The quantity run out in any time, is equal to a column or prism, whose base is the area of the hole, and its length the space described in

siderable, and the altitude may be estimated from the centre of the hole, to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next proposition.

20. It is not to be expected that experiments, as to the quantity of water run out, will exactly agree with this theory, both on account of the resistance of the air, the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water in entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion; but also particles from all the surrounding parts of the fluid, which is in a commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from whence it happens, that the real velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column of the fluid. And experiments on the quantity of water discharged through apertures, show that the velocity must be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity equal to that of a body falling through $\frac{1}{2}$ the height of the fluid above the orifice. Or else, that the orifice is not quite full of particles that spout out with the whole velocity, assigned in the proposition.

21. Experiments have also been made on the extent to which the spout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile discharged with the velocity acquired by descending through the height of the fluid. For, when the aperture is in the side of the vessel, the fluid spouts out horizontally with a uniform velocity, which, combined with the perpendicular velocity from the action of gravity, causes the jet to form the curve of a parabola. Then the distances to which the jet will spout on the horizontal plane BG, will be as the roots of the rectangles of the segments AC . CB, AD . DB, AE . EB. For the spaces BF, BG, are as the times and horizontal velocities; but the velocities are as $\sqrt{AC}$, and the time of the fall, which is the same as the time of moving, or as $\sqrt{CB}$; therefore the distance BF is as $\sqrt{AC \cdot CB}$ and the distance BG as $\sqrt{AD \cdot DB}$.



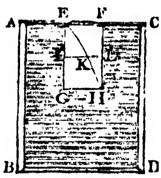
And hence, if two holes are made equidistant from the top and bottom, they will project the water to the same distance; for if AC = EB, then the rectangle AC . CB is equal the rectangle AE . EB; which makes BF then the same for both. Or, if on the diameter AB a semicircle be described; then, because the squares of the ordinates CH, DI, EK are equal to the rectangles AC . CB, &c; therefore the distances BF, BG are as the ordinates CH, DI. And hence also it follows, that the projection from the middle point D will be farthest, for DI is the greatest ordinate,

These are the *proportions* of the distances: but for the absolute distances, it will be thus. The velocity, through any hole, C, is such as will carry the water horizontally through a space equal to 2 AC in the time of falling through AC: but, after quitting the hole, it describes a parabola, and comes to F in the time a body will fall through CB; and to find this distance, since the times are as the roots of the spaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{AC . CB} = 2CH = BF$, the space ranged on the horizontal plane. And the greatest range BG = 2DI, or 2AD, or equal to AB.

And as these ranges answer very exactly to the experiments, this confirms the theory as to the velocity assigned.

PROP. XII.

22. *If a notch or slit* EH, *in form of a parallelogram, be cut in the side of a vessel, full of water,* AD; *the quantity of water flowing through it, will be* $\frac{2}{3}$ *of the quantity flowing through an equal orifice, placed at the whole depth* EG, *or at the base* GH, *in the same time; it being supposed that the vessel is always kept full.*



For the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$, that is, as GH or IL to IK, the ordinate of a parabola EKH, whose axis is EG. Therefore the sum of the velocities at all the points I, is to as many times the velocity at G, as the sum of all the ordinates IK to the sum of all the IL's, namely, as the area of the parabola EGH is to the area EGHF; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE to EGHF, or as 2 to 3; the area of a parabola being $\frac{2}{3}$ of its circumscribing parallelogram

Corol. 1. The mean velocity of the water in the notch, is equal to $\frac{2}{3}$ of that at GH.

Corol, 2. The quantity flowing through the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic frustum IGHK, is to the rectangle IGHL. As appears from the demonstration.

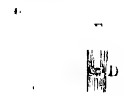
OF PNEUMATICS.

23. Pneumatics is the science which treats of the properties of air, or elastic fluids.

[redacted]

Also, as it is a fluid, it will spread itself all over on the earth; and, like other fluids, it will gravitate and press every where on the earth's surface.

25. The gravity and pressure of the air is also evident from many experiments. Thus for instance, if water or quicksilver, be poured into the tube ACE, and the air be suffered to press on it, in both ends of the tube, the fluid will rest at the same height in both legs of the tube: but if the air be drawn out of one end as E, by any means, then the air pressing on the other end A, will press down the fluid in this leg at B, and raise it up in the other to D, as much higher than at B, as the pressure of the air is equal to.



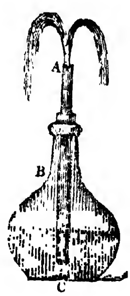
By which it appears, not only that the air does really press, but also what the quantity of that pressure is equal to. And this is the principle of the barometer.

PROP. XIV.

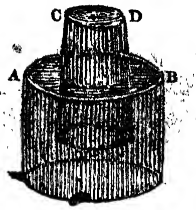
26. *The air is also an elastic fluid, being condensible and expansible. And the law it observes is this, that its density is proportional to the force which compresses it.*

This property of the air is proved by many experiments. Thus if the handle of a syringe be pushed inwards, it will condense the enclosed air into less space, thereby showing its condensibility. But the included air, thus condensed, will be felt to act strongly against the hand, resisting the force compressing it more and more; and, on withdrawing the hand, the handle is pushed back again where it was at first. Which shows that the air is elastic.

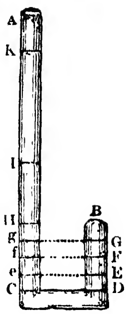
27. Again fill a strong bottle half full of water, and then insert a pipe into it, putting its lower end down near to the bottom, and cementing it very close round the mouth of the bottle. Then, if air be strongly injected through the pipe, as by blowing with the mouth or otherwise, it will pass through the water from the lower end, ascending into the parts before occupied with air at B, and the whole mass of air become there condensed, because the water is not compressible into a less space. But, on removing the force which injected the air at A, the water will begin to rise from thence in a jet, being pushed up the pipe by the increased



elasticity of the air B, by which it presses on the surface of the water, and forces it through the pipe, till as much be expelled as there was air forced in; when the air at B will be reduced to the same density as at first, and, the balance being restored, the jet will cease.

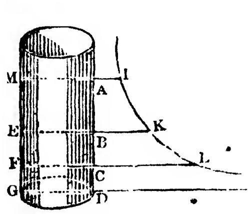


28. Likewise, if into a jar of water AB, be inverted an empty glass tumbler CD, or such like, the mouth downwards; the water will enter it, and partly fill it, but not near so high as the water in the jar, compressing and condensing the air into a less space in the upper parts CD, and causing the glass to make a sensible resistance to the hand in pushing it down. Then, on removing the hand, the elasticity of the internal condensed air throws the glass up again. All these showing that the air is condensible and elastic.



29. Again, to show the rate or proportion of the elasticity to the condensation; take a long crooked glass tube, equally wide throughout, or at least in the part BD, open at A, but close at the other end B. Pour in a little quicksilver at A, just to cover the bottom to the bend at CD, and to stop the communication between the external air and the air in BD. Then pour in more quicksilver, and mark the corresponding heights at which it stands in the two legs: so, when it rises to H in the open leg AC, let it rise to E in the close one, reducing its included air from the natural bulk BD to the contracted space BE, by the pressure of the column He; and when the quicksilver stands at I and K, in the open leg, let it rise to F and G in the other, reducing the air to the respective spaces BF, BG, by the weights of the columns If, Kg. Then it is always found that the condensations and elasticities are as the compressing weights, or columns, of the quicksilver, and the atmosphere together. So, if the natural bulk of the air DB be compressed into the spaces BE, BF, BG, or reduced by the spaces DE, DF, DG, which are $\frac{1}{4}$, $\frac{1}{2}$, [illegible] of DB, or as the numbers 1, 2, 3; then the atmosphere, together with the corresponding columns He, If, Kg, will also be found to be in the same proportion, or as the numbers 1, 2, 3. And then He $= \frac{1}{2}$A, If $=$ A, and Kg $=$ 3A; where A is the weight of the atmosphere. Which shows, that the condensations are directly as the compressing forces. And the elasticities are in the same ratio, since the columns in AC are sustained by the elasticities in BD.

From the foregoing principles may be deduced many useful remarks, as in the following corollaries, viz.



30. *Corol* 1. The space which any quantity of air is confined in, is reciprocally as the force that compresses it. So, the forces which confine a quantity of air in the cylindrical spaces AG, BG, CG, are reciprocally as the same, or reciprocally as the heights, AD, BD, CD. And therefore, if to the two perpendicular lines, DA, DH, as asymptotes, the hyperbola IKL be described, and the ordinates AI, BK, CL be drawn; then the forces which confine the air in the spaces AG, BG, CG, will be directly as the corresponding ordinates AI, BK, CL, since these are reciprocally as the abscisses AD, BD, CD, by the nature of the hyperbola.

PROP. XV.

31. *Heat increases the elasticity of the air, and cold diminishes it. Or, heat expands, and cold condenses the air.*

This property is also proved by experience.

32. Thus, tie a bladder very close with some air in it; and lay it before the fire: then as it warms, it will more and more distend the bladder, and at last burst it, if the heat be continued, and increased high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. And it was upon this principle, that the first air-balloons were made by Montgolfier: for, by heating the air within them, by a fire underneath, the hot air distends them to a size which occupies a space in the atmosphere, whose weight of common air exceeds that of a balloon.

33. Also, if a cup or glass, with a little air in it, be inverted into a vessel of water; and the whole be heated over the fire, or otherwise: the air in the top will expand till it fill the glass, and expel the water out of it; and part of the air itself will follow, by continuing or increasing the heat.

Many other experiments, to the same effect, might be adduced, all proving the properties mentioned in the proposition.

34. Scholium.—So that, when the force of the elasticity of air is considered regard must be had to its heat or temperature; the same quantity of air being more or less elastic, as its heat is more or less. And it has been found, by experiment, that the elasticity is increased by the 435th part, by each degree of heat, of which there are 180 between the freezing and boiling heat of water.

35. *N.B.* Water expands about the $\frac{3}{20000}$ part, with each degree of heat (Sir Geo. Shuckburgh, Philos. Trans. 1777, p. 560, &c).

Also, the

Spec. grav. of	air -	1	when the barom. is at 29·27,
	water	836	and the thermom. at 53°.
	mercury	11365	

Or thus,	air -	1	when the barom. is 29·27,
	water	832	and the thermom. at 55°.
	mercury	11315	

Or thus, air - 1 } when the barom. is 29·5,
water 826 } and the thermom. is 55°,
mercury 11227 } which are their mean heights in this country.

Or thus, air 1·201 or $1\frac{1}{5}$ }
water 1000 } in the last circumstances.
mercury 13592 }

Or thus, air 1·222 or $1\frac{2}{9}$ } nearly, when the barom. is 30,
water 1000 } and thermometer 55.
mercury 13600 }

PROP. XVI.

86. *The weight or pressure of the atmosphere, on any base at the earth's surface, is equal to the weight of a column of quicksilver, of the same base, and the height of which is between* 28 *and* 31 *inches.*

This is proved by the barometer, an instrument which measures the pressure of the air, and which is described below. For, at some seasons, and in some places, the air sustains and balances a column of mercury, of about 28 inches; but at other times it balances a column of 29 or 30, or near 31 inches high; seldom in the extremes 28 or 31, but commonly about the means 29 or 30. A variation which depends partly on the different degrees of heat in the air near the surface of the earth, and partly on the commotions and changes in the atmosphere, from winds and other causes, by which it is accumulated in some places, and depressed in others, being thereby rendered denser and heavier, or rarer and lighter; which changes in its state are almost continually happening in any one place. But the medium state is commonly about $29\frac{1}{2}$ or 30 inches.

Corol. 1. Hence the pressure of the atmosphere on every square inch of the earth's surface, at a medium, is very near 15 pounds avoirdupois.

For, a cubic foot of mercury weighing 13600 ounces nearly, an inch of it will weigh 7·866 or almost eight ounces, or near half a pound, which is the weight of the atmosphere for every inch of the barometer on a base of a square inch; and therefore 30 inches, or the medium height, weighs very near $14\frac{3}{4}$ pounds.

Corol. 2. Hence also, the weight or pressure of the atmosphere, is equal to that of a column of water from 32 to 35 feet high, or on a medium 33 or 34 feet high.

For, water and quicksilver are in weight nearly as 1 to 13·6; so that the atmosphere will balance a column of water 13·6 times as high as one of quicksilver; consequently

13·6 times 28 inches = 381 inches, or $31\frac{3}{4}$ feet,
13·6 times 29 inches = 394 inches, or $32\frac{5}{6}$ feet,
13·6 times 30 inches = 408 inches, or 34 feet,
13·6 times 31 inches = 422 inches, or $35\frac{1}{6}$ feet.

And hence a common sucking pump will not raise water higher than about 34 feet. And a syphon will not run, if the perpendicular height of the top of it be more than about 33 or 34 feet.

Corol. 3. If the air were of the same uniform density at every height up to the top of the atmosphere, as at the surface of the earth; its height would be about $5\frac{1}{4}$ miles at a medium.

reducing the factor 10592 to 10000, by changing the temperature proportionally from 55° : thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 5 5; therefore the corresponding change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the result is to be increased by so many times its 435th part.

43. EXAM. 1.—To find the height of a hill when the pressure of the atmosphere is equal to 29·68 inches of mercury at the bottom, and 25·28 at the top, the mean temperature being 50°? Ans. 4363 feet, or 727 fathoms.

EXAM. 2.—To find the height of a hill when the atmosphere weighs 29·45 inches of mercury at the bottom, and 26·82 at the top, the mean temperature being 33°? Ans. 2443 feet, or 408 fathoms.

EXAM. 3.—At what altitude is the density of the atmosphere only the 4th part of what it is at the earth's surface? Ans. 6020 fathoms.

By the weight and pressure of the atmosphere, the effect and operations of pneumatic engines may be accounted for, and explained; such as syphons, pumps, barometers, &c; of which it may not be improper here to give a brief description.

OF THE SIPHON.

44. THE Siphon, or Syphon, is any bent tube, having its two legs either of equal or of unequal length.

If it be filled with water, and then inverted, with the two open ends downward, and held level in that position; the water will remain suspended in it, if the two legs be equal. For the atmosphere will press equally on the surface of the water in each end, and support them, if they are not more than 34 feet high;

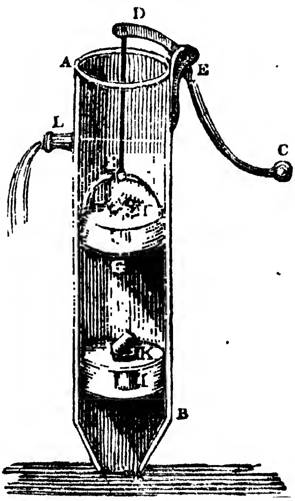
and the legs being equal, the water in them is an exact counterpoise by their equal weights; so that the one has no power to move more than the other; and they are both supported by the atmosphere.

But if now the syphon be a little inclined to one side, so that the orifice of one end be lower than that of the other; or if the legs be of unequal length, which is the same thing; then the equilibrium is destroyed, and the water will all descend out by the lower end, and rise up in the higher. For, the air pressing equally, but the two ends weighing unequally, a motion must commence where the power is greatest, and so continue till all the water has run out by the lower end. And if the shorter leg be immersed into a vessel of water, and the syphon be set a running as above, it will continue to run till all the water be exhausted out of the vessel, or at least as low as that end of the syphon. Or, it may be set a running without filling the syphon as above, by only inverting it, with its shorter leg into the vessel of water; then, with the mouth applied to the lower orifice A, suck the air out, and the water will presently follow, being forced up into the syphon by the pressure of the air on the water in the vessel.

OF THE PUMP.

45. There are three sorts of pumps; the sucking, the lifting, and the forcing pump. By the former, water can be raised only to about 34 feet, viz. by the pressure of the atmosphere; but by the others, to any height; but then they require more apparatus and power.

The annexed figure represents a common sucking pump. AB is the barrel of the pump, being a hollow cylinder, made of metal, and smooth within, or of wood for very common purposes. CD is the handle, moveable about the pin E, by moving the end C up and down. DF an iron rod turning about a pin D, which connects it to the end of the handle. This rod is fixed to the piston, bucket, or sucker, FG, by which this is moved up and down within the barrel, which it must fit very tight and close that no air or water may pass between the piston and the sides of the barrel; and for this purpose it is commonly armed with leather. The piston is made hollow, or it has a perforation through it, the orifice of which is covered by a valve H opening upwards. I is a plug firmly fixed in the lower part of the barrel, also perforated, and covered by a valve K opening upwards.



46. When the pump is first to be worked, and the water is below the plug I; raise the end C of the handle, and the piston descending, compresses the air in HI, which by its spring shuts fast the valve K, and pushes up the valve H, and so enters into the barrel above the piston. Then putting the end C of the handle down again, raises the piston or sucker, which lifts up with it the column of air

OF THE AIR PUMP.

47. Nearly on the same principles as the water pump, is the invention of the Air pump, by which the air is drawn out of any vessel, like as water is drawn out by the former. A brass barrel is bored and polished truly cylindrical, and exactly fitted with a turned piston, so that no air can pass by the sides of it, and furnished with a proper valve opening upwards. Then, by lifting up the piston, the air in the close vessel below it follows the piston, and fills the barrel; and being thus diffused through a larger space than before, when it occupied the vessel or receiver only, but not the barrel, it is made rarer than it was before, in proportion as the capacity of the barrel and receiver together exceeds the receiver alone. Another stroke of the piston exhausts another barrel of this now rarer air, which again rarefies it in the same proportion as before. And so on, for any number of strokes of the piston, still exhausting in the same geometrical progression, of which the ratio is that which the capacity of the receiver and barrel together exceeds the receiver, till this is exhausted to any proposed degree, or as far as the nature of the machine is capable of performing; which happens when the elasticity of the included air is so far diminished, by rarefying, that it is too feeble to push up the valve of the piston, and escape.

48. From the nature of this exhausting, in geometrical progression, we may easily find how much the air in the receiver is rarefied by any number of strokes of the piston; or what number of such strokes is necessary, to exhaust the receiver to any given degree. Thus, if the capacity of the receiver and barrel together, be to that of the receiver alone, as c to r, and 1 denote the natural density of the air at first; then,

$c : r :: 1 : \frac{r}{c}$, the density after 1 stroke of the piston,

$c : r :: \frac{r}{c} : \frac{r^2}{c^2}$, the density after 2 strokes,

$c : r :: \frac{r^2}{c^2} : \frac{r^3}{c^3}$, the density after 3 strokes,

&c., and $\frac{r^n}{c^n}$ the density after n strokes.

OF THE DIVING BELL, AND CONDENSING MACHINE.

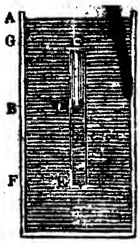
50. On the same principles, too, depend the operations and effect of the condensing engine, by which air may be condensed to any degree, instead of rarefied as in the air pump, And, like as the air pump rarefies the air, by extracting always one barrel of air after another: so, by this other machine, the air is condensed by throwing in or adding always one barrel of air after another; which it is evident may be done by only turning the valves of the piston and barrel, that is, making them to open the contrary way, and working the piston in the same manner: so that, as they both open upwards, or outwards, in the air-pump, or rarefier, they will both open downwards, or inwards, in the condenser.

51. And on the same principles, namely of the compression and elasticity of the air, depends the use of the Diving Bell, which is a large vessel, in which a person descends to the bottom of the sea, the open end of the vessel being downwards; only, in this case, the air is not condensed by forcing more of it into the same space, as in the condensing engine; but by compressing the same quantity of air into a less space in the bell, by increasing always the force which compresses it.

52. If a vessel of any sort be inverted into water, and pushed or let down to any depth in it; then by the pressure of the water some of it will ascend into the vessel, but not so high as the water without, and will compress the air into less space, according to the difference between the heights of the internal and external water; and the density and elastic force of the air will be increased in the same proportion, as its space in the vessel is diminished.

So, if the tube CE be inverted, and pushed down into water, till the external water exceed the internal, by the height AB, and the air of the tube be reduced to the space CD; then that air is pressed both by a column of water of the height AB, and by the whole atmosphere which presses on the upper surface of the water; consequently the space CD is to the whole space CE, as the weight of the atmosphere, is to the weights both of the atmosphere and the column of

water AB. So that, if AB be about 34 feet, which is equal to the force of the atmosphere, then CD will be equal to ½CE; but if AB be double of that, or 68 feet, then CD will … ⅓CE; and so on. And hence, by knowing the … h AF, to which the vessel is sunk, we … the point D, to which the water will rise within it at any time. For, let the weight of the atmosphere at that time be equal to that … water; also, let the depth AF be … feet, an… of the tube CE 4 feet; then putting the height of the internal water DE = …



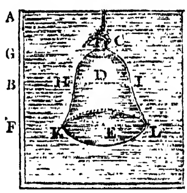
$$\ldots + AB : 34 :: CE : CD,$$

that is, $34 + AF - DE : 34 :: CE : CE - DE,$

or $54 - x : 34 :: 4 : 4 - x;$

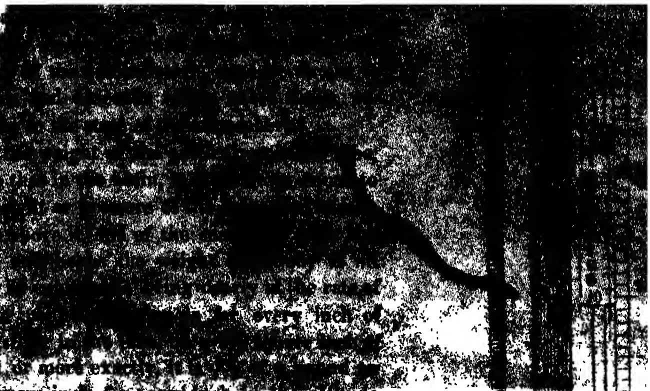
hence, multiplying the extremes and means, $216 - 58x + x^2 = 136$, and the root is $x = 1\cdot414$ of a foot, or 17 inches nearly; being the height DE to which the water will rise within the tube.

53. But if the vessel be not equally wide throughout, but of any other shape, as of a bell-like form, such as is used in diving; then the altitudes will not observe the proportion above, but the spaces or bulks only, will respect that proportion, namely, $34 + AB : 34 ::$ capacity CKL : capacity CHI, if it be common or fresh water; and $33 + AB : 33 ::$ capacity CKL : capacity CHI, if it be sea-water. From which proportion, the height DE may be found, when the nature or shape of the vessel or bell CKL is known.

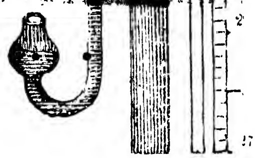


OF THE BAROMETER.

54. THE BAROMETER is an instrument for measuring the pressure of the atmosphere, and elasticity of the air, at any time. It is commonly made of a glass tube, of near 3 feet long, close at one end, and filled with mercury. When the tube is full, by stopping the open end with the finger, then inverting the tube, and immersing that end with the finger into a basin of quicksilver, on removing the finger from the orifice, the quicksilver in the tube will descend into the basin, till what remains in the tube be of the same weight with a column of the atmosphere; which is commonly between 28 and 31 inches of quicksilver; and leaving an entire vacuum in the upper end of the tube above the mercury. For, as the upper end of the tube is quite void of air, there is no pressure downwards but from the column of quicksilver, and therefore that will be an exact balance to the counter pressure of the whole column of atmosphere, acting on the orifice of the tube by the quicksilver in the basin. The upper three inches of the tube, namely, from 28 to 31 inches, have a scale attached to them,



the square inch, for every inch in the altitude of the quicksilver: for the cubic inch of quicksilver weighs just $\frac{58}{120}$ lb., or nearly ½ a pound, in the mean temperature of 55° of heat. And consequently, when the barometer stands at 30 inches, or 2½ feet high, which is the medium or standard height, the whole pressure of the atmosphere is equal to $14\frac{3}{4}$ pounds, on every square inch of the base. And so in proportion for other heights.

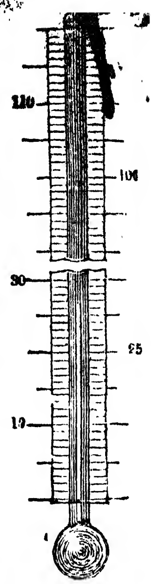


OF THE THERMOMETER.

55. THE THERMOMETER is an instrument for measuring the temperature of the air, as to heat and cold.

It is found by experience, that all bodies expand by heat, and contract by cold: and hence the degrees of expansion become the measure of the degrees of heat. Fluids are more convenient for this purpose, than solids: and quicksilver is now most commonly used for it. A very fine glass tube, having a pretty large hollow ball at the bottom, is filled about half way up with quicksilver: the whole being then heated very hot till the quicksilver rise quite to the top, the top is then hermetically sealed, so as perfectly to exclude all communication with the outward air. Then, in cooling, the quicksilver contracts, and consequently its surface descends in the tube, till it come to a certain point, correspondent to the temperature or heat of the air. And when the weather becomes warmer, the quicksilver expands, and its surface rises in the tube; and again contracts and descends when the weather becomes cooler. So that, by placing a scale of any divisions against the side of the tube, it will show the degrees of heat, by the expansion and contraction of the quicksilver in the tube; observing at what division of the scale the top of the quicksilver stands. And the method of preparing the scale, as used in England, is thus:—Bring the thermometer into a temperature of just freezing, by immersing the ball in water just freezing,

or in ice just thawing, and mark the scale where the mercury then stands, for the point of freezing. Next, immerge it in boiling water; and the quicksilver will rise to a certain height in the tube; which mark also on the scale, for the boiling point, or the heat of boiling water. Then the distance between those two points is divided into 180 equal divisions, or degrees; and the like equal degrees are also continued to any extent below the freezing point, and above the boiling point. These divisions are then numbered as follows, namely, at the freezing point is set the number 32, and consequently 212 at the boiling point; and all the other numbers in their order.



This division of the scale, is commonly called Fahrenheit's. According to this division, 55 is at the mean temperature of the air in this country; and it is in this temperature, and in an atmosphere which sustains a column of 30 inches of quicksilver in the barometer, that all measures and specific gravities are taken, unless when otherwise mentioned; and in this temperature and pressure, the relative weights, or specific gravities, of air, water, and quicksilver, are as $1\frac{2}{9}$ for air, 1000 for water, and 13600 for mercury; and these also are the weights of a cubic foot of each, in avoirdupois ounces, in that state of the barometer and thermometer. For other states of the thermometer, each of these bodies expands or contracts, according to the following rate, with each degree of heat; viz.

Air about $\frac{1}{435}$ part of its bulk,
Water about $\frac{1}{2500}$ part of its bulk,
Mercury about $\frac{1}{9600}$ part of its bulk.

OF THE MEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

56. From the principles laid down in the Scholium to prop. 17, concerning the measuring of altitudes by the barometer, and the foregoing descriptions of the barometer and thermometer, we may now collect together the precepts for the practice of such measurements, which are as follow:

First, Observe the height of the barometer at the bottom of any height, or depth, intended to be measured; with the temperature of the quicksilver by means of a thermometer attached to the barometer, and also the temperature of the air in the shade by a detached thermometer.

Second, Let the same thing be done also at the top of the said height or depth, and at the same time, or as near the same time as may be. And let those altitudes of barometer be reduced to the same temperature, if it be thought necessary, by correcting either the one or the other, that is, augment the height

[illegible] ... fathoms; or, being multiplied by 6, it will be the altitude in feet.

Example 1.—Let the state of the barometers and thermometers be as follows, to find the altitude, viz.

	Thermom.		
Barom.	attach.	detach.	Ans. the alt. is
Lower 29·68	57	57	720 fath.
Upper 25·28	43	42	

Example 2.—To find the altitude, when the state of the barometers and thermometers are as follows, viz.

	Thermom.		
Barom.	attach.	detach.	Ans. the alt. is
Lower 29·45	38	31	410 fath.
Upper 26·82	41	35	

OF THE RESISTANCE OF FLUIDS, WITH THEIR FORCES AND ACTION ON BODIES.

PROP. XVIII.

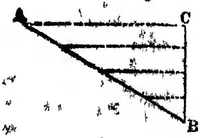
57. *If any body move through a fluid at rest, or the fluid move against the body at rest; the force or resistance of the fluid against the body, will be as the square of the velocity and the density of the fluid. That is,* $R \propto dv^2$.

For, the force or resistance is as the quantity of matter or particles struck, and the velocity with which they are struck. But the quantity or number of particles struck, in any time, are as the velocity and the density of the fluid. Therefore the resistance or force of the fluid, is as the density and square of the velocity.

Corol. 1. The resistance to any plane, is also more or less, as the plane is greater or less; and therefore the resistance on any plane, is as the area of the plane a, the density of the medium, and the square of the velocity. That is, $R \propto adv^2$.

Corol. 2. If the motion be not perpendicular, but oblique to the plane, or to the face of the body; then the resistance, in the direction of motion, will

be diminished in the triplicate ratio of radius to the sine of the angle of inclination of the plane to the direction of motion, or as the cube of radius to the cube of the sine of that angle. So that R $\propto$ ad s^3, putting $1 =$ radius, and $s =$ sine of the angle of inclination CAB.



For, if AB be the plane, AC the direction of motion, and BC perpendicular to AC; then no more particles meet the plane than what meet the perpendicular BC, and therefore their number is diminished as AB to BC, or as 1 to s. But the force of each particle, striking the plane obliquely in the direction CA, is also diminished as AB to BC, or as 1 to s; therefore the resistance, which is perpendicular to the face of the plane, by art. 8 is as 1^2 to s^2. But again, this resistance in the direction perpendicular to the face of the planes, is to that in the direction AC, by art. 8 as AB to BC, or as 1 to s. Consequently, on all these accounts, the resistance to the plane when moving perpendicular to its face, is to that when moving obliquely, as 1^3 to s^3, or 1 to s^3. That is the resistance in the direction of the motion, is diminished, as 1 to s^3, or in the triplicate ratio of radius to the sine of inclination.

PROP. XIX.

58. *The real resistance to a plane, by a fluid acting in a direction perpendicular to its face, is equal to the weight of a column of the fluid, whose base is the plane, and altitude equal to that which is due to the velocity of the motion, or through which a heavy body must fall to acquire that velocity.*

The resistance to the plane moving through a fluid, is the same as the force of the fluid in motion with the same velocity, on the plane at rest. But the force of the fluid in motion, is equal to the weight or pressure which generates that motion; and this is equal to the weight or pressure of a column of the fluid, whose base is the area of the plane, and its altitude that which is due to the velocity.

Corol. 1. If a denote the area of the plane, v the velocity, n the density or specific gravity of the fluid, and $\frac{1}{2}g = 16\frac{1}{12}$ feet, or 193 inches. Then, the altitude due to the velocity v being $\frac{v^2}{2g}$, the whole resistance, or motive force R, will be $a \times n \times \frac{v^2}{2g} = \frac{anv^2}{2g}$.

Corol. 2. If the direction of motion be not perpendicular to the face of the plane, but oblique to it, in an angle whose sine is s. Then the resistance to the plane will be $\frac{anv^2s^3}{2g}$.

Corol. 3. Also, if w denote the weight of the body, whose plane face is resisted by the absolute force R; then the retarding force f, or $\frac{R}{w}$, will be $\frac{anv^2s^3}{2gw}$.

Corol. 4. And if the body be a cylinder, whose face or end is a, and radius r moving in the direction of its axis; because then $s = 1$, and $a = pr^2$, where $p = 3{\cdot}1416$; the resisting force R will be $\frac{pnv^2r^2}{2g}$, and the retarding force $f = \frac{pnv^2r^2}{2gw}$.